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Research Article

Hopf Bifurcation of a Differential-Algebraic Bioeconomic Model with Time Delay

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We investigate the dynamics of a differential-algebraic bioeconomic model with two time delays. Regarding time delay as a bifurcation parameter, we show that a sequence of Hopf bifurcations occur at the positive equilibrium as the delay increases. Using the theories of normal form and center manifold, we also give the explicit algorithm for determining the direction of the Hopf bifurcations and the stability of the bifurcating periodic solutions. Numerical tests are provided to verify our theoretical analysis.

1. Introduction

Predator-prey systems with delay play an important role in population dynamics since Vito Volterra and James Lotka proposed the seminal models of predator-prey in the mid of 1920s. In recent years, predator-prey systems with delay have been studied extensively due to their theoretical and practical significance.

In 1973, May [1] first proposed the delayed predator-prey system

$$\begin{aligned} \dot{x} &= x\left(r_1 - a_{11}x(t-\tau) - a_{12}y\right), \\ \dot{y} &= y\left(-r_2 + a_{21}x - a_{22}y\right), \end{aligned} \tag{1.1}$$

where x and y can be interpreted as the population densities of prey and predator at time t, respectively. $\tau > 0$ is the feedback time delay of prey species to the growth of species itself. $r_1 > 0$ denotes intrinsic growth rate of prey, and $r_2 > 0$ denotes the death rate of predator. The parameters a_{ij} $(i, j = 1, 2)$ are all positive constants.

Recently, Song and Wei [2] further considered the existence of local Hopf bifurcations of system (1.1). They regarded the feedback time delay τ as a bifurcation parameter and investigated the stability and the direction of periodic solutions bifurcating from Hopf bifurcations, by applying the normal form theory and the center manifold reduction developed by Hassard et al. [3].

Considering the feedback time delay of predator species to the growth of species itself and also with the delay τ, Yan and Li [4] studied Hopf bifurcation and global periodic solutions of the following modified delayed predator-prey system:

$$\begin{aligned} \dot{x} &= x\left(r_1 - a_{11}x(t-\tau) - a_{12}y\right), \\ \dot{y} &= y\left(-r_2 + a_{21}x - a_{22}y(t-\tau)\right). \end{aligned} \tag{1.2}$$

They established the global existence results of periodic solutions bifurcating from Hopf bifurcations using a global Hopf bifurcation result due to Wu [5], which was different from that used in Song and Wei [2]. In [6], Yuan and Zhang also investigated the stability of the positive equilibrium and existence of Hopf bifurcation of the model (1.2).

Noting that, in real situations, the feedback time delay of the prey to the growth of the species itself and the feedback time delay of the predator to the growth of the species itself are different. If the time delay of the predator to the growth of the species itself is zero, (1.2) becomes (1.1). Meanwhile, if the time delay of the prey to the growth of the species itself is zero, (1.2) is simplified as

$$\begin{aligned} \dot{x} &= x(t)\left(r_1 - a_{11}x(t) - a_{12}y(t)\right), \\ \dot{y} &= y(t)\left(-r_2 + a_{21}x(t) - a_{22}y(t-\tau)\right). \end{aligned} \tag{1.3}$$

In 1954, Gordon [7] studied the effect of harvest effort on ecosystem from an economic perspective and proposed the following economic theory:

$$\text{Net Economic Revenue (NER)} = \text{Total Revenue (TR)} - \text{Total Cost (TC)}. \tag{1.4}$$

This provides theoretical fundament for the establishment of bioeconomic systems by differential-algebraic equations (DAEs).

Let E and y represent the harvest effort and the density of harvested population, respectively. Then TC $= cE$ and TR $= \alpha E y$, where α is unit price of harvested population and c is the cost of harvest effort. Considering the economic interest m of the harvest effort on the predator, we can establish the following algebraic equation:

$$E(\alpha y - c) = m. \tag{1.5}$$

Recently, a class of differential-algebraic bioeconomic models were proposed and analyzed in [8–14]. For example, [8] discussed the problems of chaos and chaotic control for a differential-algebraic system. The existence and stability of equilibrium points, the sufficient conditions of existence for various bifurcations (transcritical bifurcation, singular induced bifurcation, and Hopf bifurcation) are invested in [9–14]. However, the stability and the

direction of periodic solutions bifurcating from Hopf bifurcations have not been discussed there. Very recently, Zhang et al. [15] analyzed the stability and the Hopf bifurcation of a differential-algebraic bioeconomic system with a single time delay and Holling II type functional response and first investigated the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions.

Based on the above economic theory and system (1.3), in this paper, we study the Hopf bifurcation of the following differential-algebraic bioeconomic system:

$$\begin{aligned} \dot{x} &= x\left(r_1 - a_{11}x(t) - a_{12}y\right), \\ \dot{y} &= y\left(-r_2 + a_{21}x - a_{22}y(t-\tau)\right) - Ey, \\ 0 &= E\left(\alpha y - c\right) - m. \end{aligned} \tag{1.6}$$

The rest of this paper is organized as follows. In Section 2, regarding time delay τ as bifurcation parameter, first, we study the stability of the equilibrium point of system based on the new normal form approach proposed by Chen et al. [16]. Then, following the normal form approach theory and the center manifold theory introduced by Hassard et al. [3], we compute the normal form for the Hopf bifurcation of system (1.6) and analyze the direction and stability of the bifurcating periodic orbits of the system. Numerical examples are given in Section 3 to illustrate our theoretical results.

2. Hopf Bifurcation

In this section, we analyze the stability and the Hopf bifurcation of the differential-algebraic bioeconomic system (1.6). Furthermore, we will derive the formula for determining the properties of the Hopf bifurcation.

2.1. Existence of Hopf Bifurcation

For $m > 0$, the system (1.6) has positive equipment point $P_* = (x_*, y_*, E_*)$, where $x_* = (r_1 - a_{12}y_*)/a_{11}$, $E_* = m/(\alpha y_* - c)$ and y_* is the positive root of the following equation:

$$\begin{aligned} &\alpha(a_{11}a_{22} + a_{12}a_{21})y^2 - (ca_{11}a_{22} + ca_{12}a_{21} + \alpha r_1 a_{21} - \alpha a_{11}r_2)y \\ &\quad + (ma_{11} + cr_1a_{21} - ca_{11}r_2) = 0. \end{aligned} \tag{2.1}$$

In order to guarantee the existence of the positive equipment point P_*, some restrictions must be satisfied also:

$$\frac{c}{\alpha} < y_* < \frac{r_1}{a_{12}}, \qquad ca_{11}a_{22} + ca_{12}a_{21} + \alpha r_1 a_{21} > \alpha a_{11}r_2, \tag{2.2}$$

and $ma_{11} + cr_1a_{21} - ca_{11}r_2 < 0$.

In order to analyze the local stability of the positive equilibrium point, we first use the linear transformation $(x, y, E)^T = Q \cdot (u, v, \overline{E})^T$, where

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\dfrac{\alpha E_*}{\alpha y_* - c} & 1 \end{bmatrix}. \tag{2.3}$$

Then the system (1.6) yields

$$\begin{aligned} \dot{u} &= u(r_1 - a_{11}u - a_{12}v), \\ \dot{v} &= v(-r_2 + a_{21}u - a_{22}v(t-\tau)) - \overline{E}v + \frac{\alpha E_* v^2}{\alpha y_* - c}, \\ 0 &= \left(\overline{E} - \frac{\alpha E_* v}{\alpha y_* - c}\right)(\alpha v - c) - m. \end{aligned} \tag{2.4}$$

According to the literature [16], considering the local parametrization of the third equation of the system (2.4), and we can obtain the following parametric system of it:

$$\begin{aligned} \dot{x}_1 &= (x_1 + u_*)(r_1 - a_{11}(x_1 + u_*) - a_{12}(x_2 + v_*)), \\ \dot{x}_2 &= (x_2 + v_*)(-r_2 + a_{21}(x_1 + u_*) - a_{22}(x_2(t-\tau) + v_*)) - \left(\overline{E}_* + h\right)(x_2 + v_*) + \frac{\alpha E_*(x_2 + v_*)^2}{\alpha y_* - c}, \end{aligned} \tag{2.5}$$

where

$$u = u_* + x_1, \qquad v = v_* + x_2, \qquad \overline{E} = \overline{E}_* + h(x_1, x_2), \tag{2.6}$$

$h : R^2 \to R$ is a smooth mapping, and $(u_*, v_*, \overline{E}_*)^T$ satisfies

$$(x_*, y_*, E_*)^T = Q \cdot \left(u_*, v_*, \overline{E}_*\right)^T. \tag{2.7}$$

Thus the linearized system of parametric system (2.5) at (0,0) is given as below:

$$\begin{aligned} \dot{x}_1 &= -a_{11}x_*x_1 - a_{12}x_*x_2, \\ \dot{x}_2 &= a_{21}y_*x_1 + \frac{\alpha E_* y_*}{\alpha y_* - c}x_2 - a_{22}y_*x_2(t-\tau). \end{aligned} \tag{2.8}$$

The characteristic equation of the linearized system of parametric system (2.8) at (0,0) is given by

$$\lambda^2 + p\lambda + r + (s\lambda + q)e^{-\lambda\tau} = 0, \tag{2.9}$$

where $p = a_{11}x_* - \alpha y_* E_*/(\alpha y_* - c)$, $r = x_* y_*(a_{21}a_{12} - \alpha a_{11}E_*/(\alpha y_* - c))$, $s = a_{22}y_*$, and $q = a_{11}a_{22}x_*y_*$.

The second-degree transcendental polynomial equation (2.9) has been extensively studied and applied by many researchers [2, 17–20].

Let

(H1) $r_1 > \alpha y_* E_* / (\alpha y_* - c)$;

(H2) $a_{12}a_{21} + a_{11}a_{22} > \alpha a_{11} E_* / (\alpha y_* - c)$;

(H3) either $2x_* y_* a_{21} a_{12} + a_{22}^2 y_*^2 < a_{11}^2 x_*^2 + (y_* E_* \alpha / (\alpha y_* - c))^2$ and $(a_{21}a_{12} - a_{11}E_*\alpha/(\alpha y_* - c))^2 > (a_{11}a_{22})^2$ or

$$\left(2x_* y_* a_{21} a_{12} + a_{22}^2 y_*^2 - a_{11}^2 x_*^2 - \left(\frac{y_* E_* \alpha}{\alpha y_* - c}\right)^2\right)^2 < 4x_*^2 y_*^2 \left(\left(a_{21}a_{12} - \frac{a_{11}E_*\alpha}{\alpha y_* - c}\right)^2 - a_{11}^2 a_{22}^2\right); \tag{2.10}$$

(H4) either $(a_{21}a_{12} - a_{11}E_*\alpha/(\alpha y_* - c))^2 < (a_{11}a_{22})^2$ or $2x_* y_* a_{21} a_{12} + a_{22}^2 y_*^2 > a_{11}^2 x_*^2 + (y_* E_* \alpha / (\alpha y_* - c))^2$, and

$$\left(2x_* y_* a_{21} a_{12} + a_{22}^2 y_*^2 - a_{11}^2 x_*^2 - \left(\frac{y_* E_* \alpha}{\alpha y_* - c}\right)^2\right)^2 = 4x_*^2 y_*^2 \left(\left(a_{21}a_{12} - \frac{a_{11}E_*\alpha}{\alpha y_* - c}\right)^2 - a_{11}^2 a_{22}^2\right); \tag{2.11}$$

(H5) $(a_{21}a_{12} - a_{11}E_*\alpha/(\alpha y_* - c))^2 > (a_{11}a_{22})^2$, $2x_* y_* a_{21} a_{12} + a_{22}^2 y_*^2 > a_{11}^2 x_*^2 + (y_* E_* \alpha / (\alpha y_* - c))^2$, and

$$\left(2x_* y_* a_{21} a_{12} + a_{22}^2 y_*^2 - a_{11}^2 x_*^2 - \left(\frac{y_* E_* \alpha}{\alpha y_* - c}\right)^2\right)^2 > 4x_*^2 y_*^2 \left(\left(a_{21}a_{12} - \frac{a_{11}E_*\alpha}{\alpha y_* - c}\right)^2 - a_{11}^2 a_{22}^2\right). \tag{2.12}$$

From Lemma 2.1 in [2], we easily obtain the following results about the stability of the positive equilibrium and the Hopf bifurcation of (1.6).

Theorem 2.1. *For system* (1.6). *One has the following.*

(i) *If (H1)–(H3) hold, then the equilibrium E_* of the system* (1.6) *is asymptotically stable for all $\tau \geq 0$.*

(ii) *If (H1), (H2), and (H4) hold, then the equilibrium E_* of the system* (1.6) *is asymptotically stable when $0 \leq \tau < \tau_0^+$ and unstable when $\tau > \tau_0^+$. System* (1.6) *undergoes a Hopf bifurcation at E_* when $\tau = \tau_j^+$.*

(iii) *If (H1), (H2), and (H5) hold, then there is a positive integer k, such that the equilibrium E_* switches k times from stability to instability to stability; that is, E_* is asymptotically stable when*

$$\tau \in [0, \tau_0^+) \cup (\tau_0^-, \tau_1^+) \cup \cdots \cup (\tau_{k-1}^-, \tau_k^+) \tag{2.13}$$

and unstable when

$$\tau \in (\tau_0^+, \tau_0^-) \cup (\tau_1^+, \tau_1^-) \cup \cdots \cup (\tau_{k-1}^+, \tau_{k-1}^-) \cup (\tau_{k-1}^+, +\infty). \tag{2.14}$$

Here,

$$\omega_{\pm} = \frac{\sqrt{2}}{2}\left[2x_*y_*a_{21}a_{12} + a_{22}^2y_*^2 - a_{11}^2x_*^2 - \left(\frac{y_*E_*\alpha}{\alpha y_* - c}\right)^2 \pm \sqrt{\begin{array}{c}\left(2x_*y_*a_{21}a_{12} + a_{22}^2y_*^2 - a_{11}^2x_*^2 - \left(\frac{y_*E_*\alpha}{\alpha y_* - c}\right)^2\right)^2 \\ +4x_*^2y_*^2\left(\left(a_{21}a_{12} - \frac{a_{11}E_*\alpha}{\alpha y_* - c}\right)^2 - (a_{11}a_{22})^2\right)\end{array}}\right]^{1/2},$$

$$\tau_j^{\pm} = \frac{1}{\omega_{\pm}}arccos\left(\frac{a_{22}(E_*\alpha/(\alpha y_* - c))\omega_{\pm}^2 - a_{11}a_{22}x_*^2(a_{21}a_{12} - a_{11}E_*\alpha/(\alpha y_* - c))}{a_{22}^2(\omega_{\pm}^2 + a_{11}^2x_*^2)}\right) + \frac{2j\pi}{\omega_{\pm}}, \quad j = 0,1,2,\ldots. \tag{2.15}$$

Denote $\lambda_j(\tau) = \alpha_j(\tau) + i\omega_j(\tau)$, $j = 0,1,2,\ldots$; then the following transversality conditions hold:

$$\frac{d}{d\tau}\operatorname{Re}\lambda_j\left(\tau_j^+\right) > 0, \qquad \frac{d}{d\tau}\operatorname{Re}\lambda_j\left(\tau_j^-\right) < 0. \tag{2.16}$$

2.2. Direction and Stability of the Hopf Bifurcation

In the previous section, we obtain the conditions for Hopf bifurcations at the positive equilibrium P_* when $\tau = \tau_n^{\pm}$, $n = 0,1,2,\ldots$. In this section, we will derive the formulaes determining the direction, stability, and period of these periodic solutions bifurcating from P_* at τ. The basic techniques used here are the normal form and the center manifold theory developed by Hassard et al. [3].

Assuming the system (1.6) undergoes Hopf bifurcations at the positive equilibrium P_* at $\tau = \tau_n$, we let $i\omega$ be the corresponding purely imaginary root of the characteristic equation at the positive equilibrium. In order to investigate the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions of system (1.6), we consider the parametric system (2.5) of system (2.4).

Let $t = t\tau, \tau = \tau_n + \mu$; then the parametric system (2.5) is equivalent to the following functional differential equation system:

$$\begin{aligned}\dot{z}_1 &= (\tau_n + \mu)(-a_{11}x_*z_1 - a_{12}x_*z_2),\\ \dot{z}_2 &= (\tau_n + \mu)\left(a_{21}y_*z_1 + \frac{\alpha E_*y_*}{\alpha y_* - c}z_2 - a_{22}y_*z_2(t-1)\right),\end{aligned} \tag{2.17}$$

where $z_1 = x_1(t\tau), z_2 = x_2(t\tau)$.

For $\Phi \in C([-1,0], R^2)$, define a family of operators

$$L_\mu \Phi := (\tau_n + \mu) \begin{bmatrix} -a_{11}x_* & -a_{12}x_* \\ a_{21}y_* & \dfrac{\alpha E_* y_*}{\alpha y_* - c} \end{bmatrix} \Phi(0) + (\tau_n + \mu) \begin{bmatrix} 0 & 0 \\ 0 & -a_{22}y_* \end{bmatrix} \Phi(-1),$$
$$f(\mu, \Phi) := (\tau_n + \mu) \times \begin{bmatrix} -a_{11}\Phi_1^2(0) - a_{12}\Phi_1(0)\Phi_2(0) + \cdots \\ a_{21}\Phi_1(0)\Phi_2(0) - a_{22}\Phi_2(0)\Phi_2(-1) - \dfrac{\alpha E_* c}{(\alpha y_* - c)^2}\Phi_2^2(0) + \cdots \end{bmatrix}, \tag{2.18}$$

where $\Phi = (\Phi_1, \Phi_2) \in C$. By the Riesz representation theorem, there exists a function of bounded variation for $\theta \in [-1,0]$, such that

$$L_\mu \Phi = \int_{-1}^{0} d\eta(\theta, \mu)\Psi(\theta), \quad \Psi \in C. \tag{2.19}$$

In fact, we can choose

$$\eta(\theta, \mu) = (\tau_n + \mu) \begin{bmatrix} -a_{11}x_* & -a_{12}x_* \\ a_{21}y_* & \dfrac{\alpha E_* y_*}{\alpha y_* - c} \end{bmatrix} \delta(\theta) + (\tau_n + \mu) \begin{bmatrix} 0 & 0 \\ 0 & a_{22}y_* \end{bmatrix} \delta(\theta + 1). \tag{2.20}$$

For $\Phi \in C$, define

$$A(\mu)\Phi := \begin{cases} \dfrac{d\Phi(\theta)}{d\theta}, & \theta \in [-1,0), \\ \displaystyle\int_{-1}^{0} d\eta(t,\mu)\Phi, & \theta = 0, \end{cases}$$
$$R(\mu) := \begin{cases} 0, & \theta \in [-1,0), \\ f(t,\Phi), & \theta = 0. \end{cases} \tag{2.21}$$

Hence system (2.17) can be rewritten as

$$\dot{Z}_t = A(\mu)Z_t + R(\mu)Z_t. \tag{2.22}$$

For $\Psi \in C([-1,0], (R^2)^*)$, define

$$A^*\Psi(s) := \begin{cases} \dfrac{d\Psi(s)}{ds}, & s \in [-1,0), \\ \displaystyle\int_{-1}^{0} d\eta^T(s,0)\Psi(-s), & s = 0, \end{cases} \tag{2.23}$$

and a bilinear inner product

$$\langle \Psi, \Phi \rangle = \overline{\Psi}^T(0)\Phi(0) - \int_{-1}^{0}\int_{0}^{\theta} \overline{\Psi}^T(\xi-\theta)d\eta(\theta)\Phi(\xi)d\xi, \tag{2.24}$$

where $\eta(\theta) = \eta(\theta, 0)$. Then A^* and $A(0)$ are adjoint operators. By the discussion in Section 2.1, we know that $\pm i\omega$ are eigenvalues of $A(0)$. Thus they are also eigenvalues of A^*. Suppose that $q(\theta) = (1, q_1)^T e^{i\omega\tau\theta}$ and $q^*(s) = D(q_1^*, 1)e^{i\omega\tau s}$ are eigenvectors of $A(0)$ and A^* corresponding to $i\omega\tau$ and $-i\omega\tau$, respectively. It is easy to obtain

$$\begin{aligned} q_1 &= -\frac{i\omega + a_{11}x_*}{a_{12}x_*}, \\ q_1^* &= \frac{i\omega - a_{22}y_* e^{i\omega\tau}}{a_{12}x_*}, \\ \overline{D} &= \left(q_1 + \overline{q}_1^* + \tau q_1 a_{22} y_* e^{-i\omega\tau}\right)^{-1}. \end{aligned} \tag{2.25}$$

Moreover, $\langle q^*(s), q(\theta)\rangle = 1$, $\langle q^*(s), \overline{q}(\theta)\rangle = 0$.

Now, we calculate the coordinates to describe the center manifold C_0 at $\mu = 0$. Using the same notations as in [3], we define

$$z = \langle q^*, Z_t \rangle, \qquad W(t,\theta) = Z_t(\theta) - 2\,\mathrm{Re}\{zq(\theta)\}. \tag{2.26}$$

On the center manifold C_0, we have $W(t,\theta) = W(z,\overline{z},\theta)$, where

$$W(z,\overline{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots, \tag{2.27}$$

and z and $\overline{z}$ are local coordinates for C_0 in the direction of q^* and $\overline{q}^*$. Note that W is real if Z_t is real; we only consider real solutions. For $Z_t \in C_0$, we have

$$\dot{z} = i\omega\tau z + \overline{q}^*(0)f_0(z,\overline{z}) := i\omega\tau z + g(z,\overline{z}), \tag{2.28}$$

where

$$g(z,\overline{z}) = \overline{q}^*(0)f_0(z,\overline{z}) = \overline{q}^*(0)f(0,Z_t) = g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\overline{z} + g_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots. \tag{2.29}$$

Moreover, we have

$$
\begin{aligned}
g_{20} &= 2\overline{D}\tau\left(a_{21}q_1 - a_{11}\overline{q}_1^* - a_{12}\overline{q}_1^* q_1 - \frac{\alpha E_* c}{(\alpha y_* - c)^2} q_1^2 - a_{22} q_1^2 e^{-i\omega\tau\theta}\right),\\
g_{11} &= 2\overline{D}\tau\left(a_{21}\operatorname{Re}(q_1) - a_{11}\overline{q}_1^* - a_{12}\overline{q}_1^*\operatorname{Re}(q_1) - a_{22}\operatorname{Re}\left(q_1\overline{q}_1 e^{i\omega\tau\theta}\right) - \frac{\alpha E_* c}{(\alpha y_* - c)^2} q_1\overline{q}_1\right),\\
g_{02} &= 2\overline{D}\tau\left(a_{21}\overline{q}_1 - a_{11}\overline{q}_1^* - a_{12}\overline{q}_1^*\overline{q}_1 - \frac{\alpha E_* c}{(\alpha y_* - c)^2}\overline{q}_1^2 - a_{22}\overline{q}_1^2 e^{i\omega\tau\theta}\right),\\
g_{21} &= 2\overline{D}\tau\Big(\left(a_{21}q_1 - 2a_{11}\overline{q}_1^* - a_{12}\overline{q}_1^* q_1\right)W_{11}^{(1)}(0)\\
&\quad + \frac{1}{2}\left(a_{21}\overline{q}_1 - a_{12}\overline{q}_1^*\overline{q}_1 - 2a_{11}\overline{q}_1^*\right)W_{20}^{(1)}(0)\\
&\quad + \left(a_{21} - a_{22}q_1 e^{-i\omega\tau\theta} - a_{12}\overline{q}_1^* - 2\frac{\alpha E_* c}{(\alpha y_* - c)^2} q_1\right)W_{11}^{(2)}(0)\\
&\quad + \frac{1}{2}\left(a_{21} - a_{22}\overline{q}_1 e^{i\omega\tau\theta} - a_{12}\overline{q}_1^* - 2\frac{\alpha E_* c}{(\alpha y_* - c)^2}\overline{q}_1\right)W_{20}^{(2)}(0)\\
&\quad - a_{22}q_1 W_{11}^{(2)}(-1) - \frac{1}{2}a_{22}\overline{q}_1 W_{20}^{(2)}(-1)\Big).
\end{aligned}
\tag{2.30}
$$

Next, we will calculate $W_{11}(\theta)$ and $W_{20}(\theta)$. From (2.22) and (2.28), we have

$$
\begin{aligned}
\dot{W} &= \dot{Z}_t - \dot{z}q - \dot{\overline{z}}\overline{q} = \begin{cases} \mathrm{AW} - 2\operatorname{Re}\{\overline{q}^*(0)f(z,\overline{z})q(\theta)\}, & \theta \in [-1,0),\\ \mathrm{AW} - 2\operatorname{Re}\{\overline{q}^*(0)f(z,\overline{z})q(\theta)\} + f, & \theta = 0,\end{cases}\\
&:= \mathrm{AW} + H(z,\overline{z},\theta),
\end{aligned}
\tag{2.31}
$$

where $H(z,\overline{z},\theta) = H_{20}(\theta)(z^2/2) + H_{11}(\theta)z\overline{z} + H_{02}(\theta)(\overline{z}^2/2) + \cdots$.

For $\theta \in [-1,0)$, we can get

$$
(A - 2i\omega\tau)W_{20}(\theta) = -H_{20}(\theta), \qquad \mathrm{AW}_{11}(\theta) = -H_{11}(\theta). \tag{2.32}
$$

On the other hand, $H(z,\overline{z},\theta) = \mathrm{AW} - 2\operatorname{Re}\{\overline{q}*(0)f(z,\overline{z})q(\theta)\}$; hence we can obtain

$$
H_{20}(\theta) = -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta), \qquad H_{11}(\theta) = -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta). \tag{2.33}
$$

It follows from (2.32) that

$$\begin{aligned} \dot{W}_{20}(\theta) &= 2i\omega W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta), \\ \dot{W}_{11}(\theta) &= g_{11}q(\theta) + \overline{g}_{11}\overline{q}(\theta). \end{aligned} \tag{2.34}$$

Solving it, we have

$$\begin{aligned} W_{20}(\theta) &= \frac{ig_{20}}{\omega\tau}q(0)e^{i\omega\tau\theta} + \frac{i\overline{g}_{02}}{3\omega\tau}\overline{q}(0)e^{-i\omega\tau\theta} + Me^{2i\omega\tau\theta}, \\ W_{11}(\theta) &= -\frac{ig_{11}}{\omega\tau}q(0)e^{i\omega\tau\theta} + \frac{i\overline{g}_{11}}{\omega\tau}\overline{q}(0)e^{-i\omega\tau\theta} + N. \end{aligned} \tag{2.35}$$

Now we will seek appropriate M and N. From (2.29) and (2.31), we have

$$\begin{aligned} H_{20}(\theta) &= -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) + 2\tau H_1, \\ H_{11}(\theta) &= -g_{11}q(0) - \overline{g}_{11}\overline{q}(0) + 2\tau H_2, \end{aligned} \tag{2.36}$$

where

$$\begin{aligned} H_1 &= \begin{bmatrix} -a_{11} - a_{12}q_1 \\ a_{21}q_1 - \dfrac{\alpha E_* c}{(\alpha y_* - c)^2}q_1^2 - a_{22}q_1^2 e^{-i\omega\tau\theta} \end{bmatrix}, \\ H_2 &= \begin{bmatrix} -a_{11} - a_{12}\operatorname{Re}(q_1) \\ a_{21}\operatorname{Re}(q_1) - a_{22}\operatorname{Re}(q_1\overline{q}_1 e^{i\omega\tau\theta}) - \dfrac{\alpha E_* c}{(\alpha y_* - c)^2}q_1\overline{q}_1 \end{bmatrix}. \end{aligned} \tag{2.37}$$

Noting that

$$\begin{aligned} \left(i\omega\tau I - \int_{-1}^{0} e^{i\omega\tau\theta}d\eta(\theta)\right)q(0) &= 0, \\ \left(-i\omega\tau I - \int_{-1}^{0} e^{-i\omega\tau\theta}d\eta(\theta)\right)q(0) &= 0, \end{aligned} \tag{2.38}$$

we have the following linear equations:

$$\begin{bmatrix} 2i\omega + a_{11}x_* & a_{12}x_* \\ -a_{21}y_* & 2i\omega - a_{22}y_* - \dfrac{\alpha E_* y_*}{\alpha y_* - c} \end{bmatrix} M = 2H_1,$$
$$\begin{bmatrix} a_{11}x_* & a_{12}x_* \\ -a_{21}y_* & -a_{22}y_* - \dfrac{\alpha E_* y_*}{\alpha y_* - c} \end{bmatrix} N = 2H_2. \tag{2.39}$$

It is easy to get M and N. Furthermore, we can get the following values:

$$C_1(0) = \frac{i}{2\omega\tau}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},$$
$$\mu_2 = -\frac{\operatorname{Re}(C_1(0))}{\operatorname{Re}(\lambda'(\tau))},$$
$$\beta_2 = 2\operatorname{Re}(C_1(0)),$$
$$T_2 = -\frac{\operatorname{Im}(C_1(0)) + \mu_2 \operatorname{Im}(\lambda'(\tau))}{\omega\tau}. \tag{2.40}$$

It is well known that, at the critical value τ, the sign of μ_2 determines the direction of the Hopf bifurcation, β_2 determines the stability of bifurcated periodic solutions, while T_2 determines the period of bifurcated periodic solutions, respectively. In fact, we have the theorem as follows.

Theorem 2.2. (i) *The Hopf bifurcation is supercritical if $\mu_2 > 0$ and subcritical if $\mu_2 < 0$.*
(ii) *The bifurcated periodic solutions are unstable if $\beta_2 > 0$ and stable if $\beta_2 < 0$.*
(iii) *The period of bifurcated periodic solutions increases if $T_2 > 0$ and decreases if $T_2 < 0$.*

3. Numerical Examples

We now perform some simulations for better understanding of our analytical treatment.

Example 3.1. Consider the following differential-algebraic system:

$$\dot{x} = x(6 - 2x(t) - 1.5y),$$
$$\dot{y} = y(-0.5 + 2x - y(t-\tau)) - Ey, \tag{3.1}$$
$$0 = E(2y - 1) - 1.5.$$

The only positive equilibrium point of the system (3.1) is $P^+ = (1.5, 2.0, 0.5)$. It is easy to get $p = 2.3333, q = 6, r = 7, s = 2$.

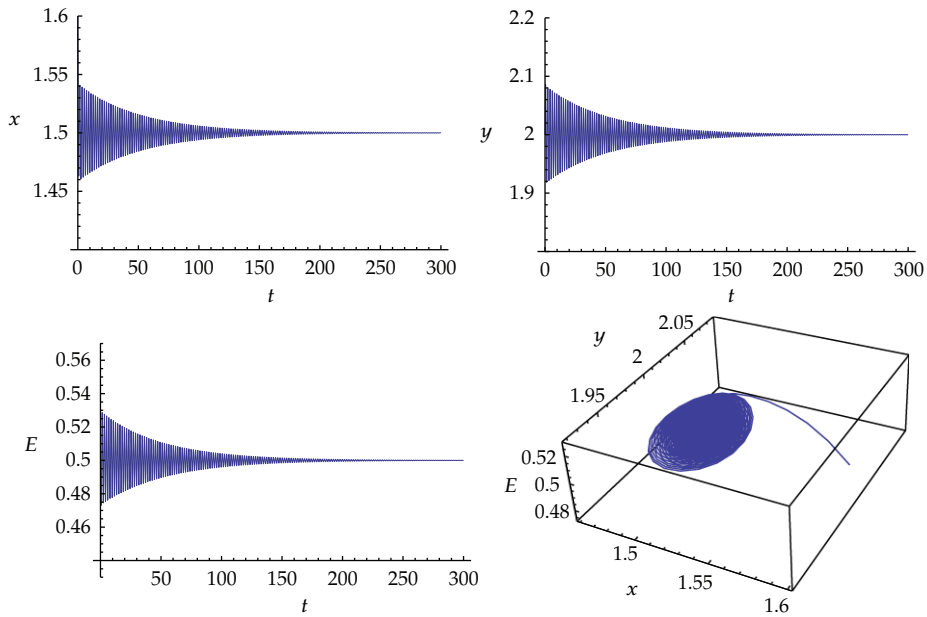

Figure 1: : Dynamic behavior of the differential-algebraic system (3.1) with $\tau = 0.55 < \tau_0^+$, and the positive equilibrium point P^+ is stable.

When $\tau = 0$, from (2.9), the characteristic equation of the linearized system of parametric system of (3.1) at P^+ is given by

$$\lambda^2 + (p+s)\lambda + r + q = 0. \tag{3.2}$$

By simple calculating, $p + s = 4.3333 > 0, q + r = 13 > 0$. According to Routh-Hurwitz stability criterion, P^+ is stable.

When $\tau > 0$, we can further get $r^2 - q^2 = 13 > 0, s^2 - p^2 + 2r = 12.5556 > 0$, and $(s^2 - p^2 + 2r)^2 - 4(r^2 - q^2) = 105.6420 > 0$ which mean (H1), (H2), and (H5) hold. Furthermore, we have $\omega_- = 1.0671, \omega_+ = 3.3789$, and hence $\tau_0^+ = 0.5638 < \tau_1^+ = 2.4233 < \tau_0^- = 2.8880 < \tau_2^+ = 4.2829 < \tau_3^+ = 6.1424 < \tau_1^- = 8.7762 < \tau_2^- = 14.6645 < \tau_3^- = 20.5527 < \cdots$. According to conclusion (iii) of Theorem 2.1, the equilibrium point P^+ is stable only for $\tau < \tau_0^+$, unstable for any $\tau > \tau_0^+$, and the Hopf bifurcation occurs at $\tau = \tau_0^+$.

By the aid of Mathematica, we can obtain the following values according to equalities in (2.40):

$$C_1(0) = 3.9725 - 4.7012i, \qquad \lambda'(\tau_0^+) = 1.3427 - 2.9776i. \tag{3.3}$$

So we have $\mu_2 = -2.9586 < 0, \beta_2 = 7.9451 > 0$, and $T_2 = -2.1567 < 0$. Thus from Theorem 2.2, we can conclude that the Hopf bifurcation of system (3.1) is subcritical, the bifurcating periodic solution exists when τ crosses τ_0^+ to the left, and the bifurcating periodic solution is unstable and decreases.

Figure 1 shows the positive equilibrium point P^+ of system (3.1) is locally asymptotically stable when $\tau = 0.55 < \tau_0^+ = 0.5638$. The periodic solutions occur from P^+

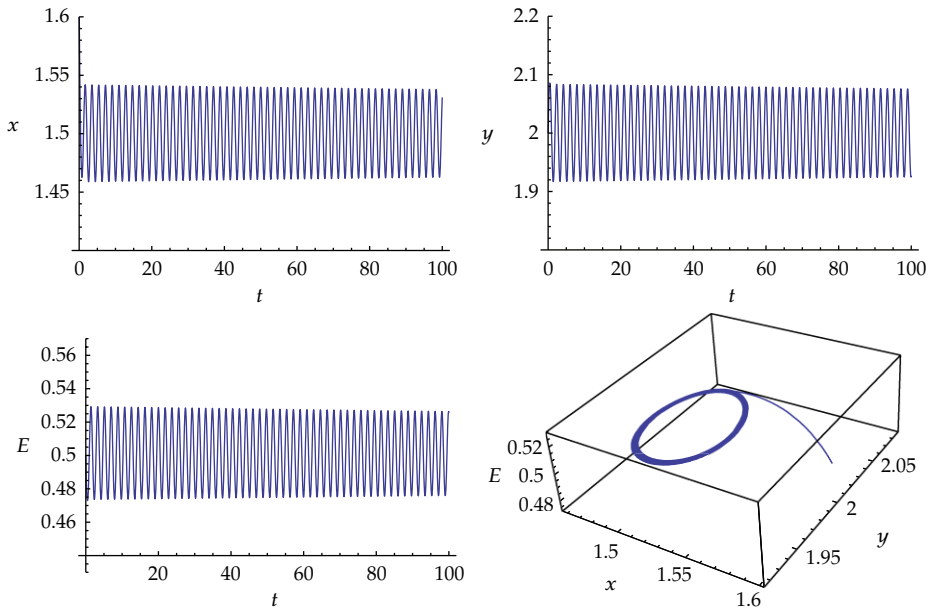

Figure 2: Dynamic behavior of the differential-algebraic system (3.1) with $\tau = 0.563 < \tau_0^+$, periodic solutions bifurcating from the positive equilibrium point P^+.

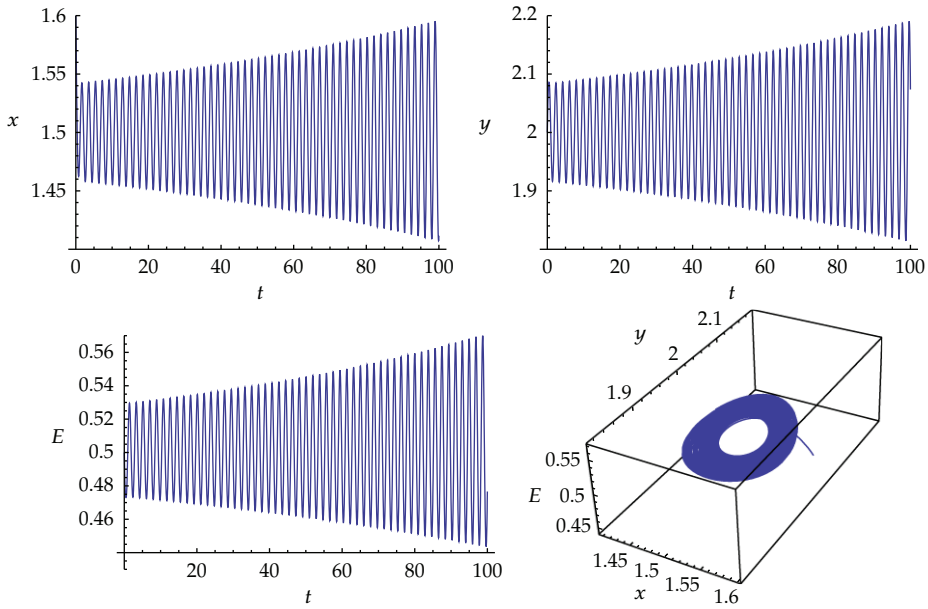

Figure 3: Dynamic behavior of the differential-algebraic system (3.1) with $\tau = 0.57 > \tau_0^+$, and the positive equilibrium point P^+ is unstable.

when $\tau = 0.563 < \tau_0^+ = 0.5638$ as is illustrated in Figure 2. When $\tau = 0.57 > \tau_0^+ = 0.5638$, the positive equilibrium point P^+ becomes unstable as shown in Figure 3.

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