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Research Article

Sufficient Dilated LMI Conditions for H_∞ Static Output Feedback Robust Stabilization of Linear Continuous-Time Systems

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New sufficient dilated linear matrix inequality (LMI) conditions for the H_∞ static output feedback control problem of linear continuous-time systems with no uncertainty are proposed. The used technique easily and successfully extends to systems with polytopic uncertainties, by means of parameter-dependent Lyapunov functions (PDLFs). In order to reduce the conservatism existing in early standard LMI methods, auxiliary slack variables with even more relaxed structure are employed. It is shown that these slack variables provide additional flexibility to the solution. It is also shown, in this paper, that the proposed dilated LMI-based conditions always encompass the standard LMI-based ones. Numerical examples are given to illustrate the merits of the proposed method.

1. Introduction

The interest in static output feedback controllers lies in their conceptual simplicity and ease in practical implementation. In addition, because in practice, only partial information through measurements is available, the most common, albeit often complex, dynamic output feedback design procedure can always be reformulated, via a simple plant augmentation, as a static output problem. Paradoxically, the design of static output feedback controllers remains an open question in control theory. For instance, it still cannot be entirely solved, for instance, through convex optimization. This tool is known to lead, in this case, to only bilinear matrix inequality (BMI) conditions that are necessary and sufficient, but nonconvex and NP-hard to solve [1]. Various approaches have been proposed to deal with this issue. While some authors presented Riccati-like conditions [2–4], others proposed rank-constrained LMI conditions [5, 6] or iterative linear matrix inequality (ILMI) conditions [7–12], which, albeit, can only be numerically solved with limited efficiency. Recently, other approaches offered a convex

LMI formulation of the problem providing, however, only sufficient LMI conditions, for both continuous-time and discrete-time systems [13–19]. Since then, most of the research efforts have been deployed into the direction of relaxing the conservatism of those sufficient conditions. For instance, in [14, 19], the Lyapunov variable was forced to satisfy a linear matrix equality constraint. In [17], the Lyapunov matrix was restricted to a block diagonal structure. All these approaches used the standard analysis or synthesis conditions as a basis, which are known, in general, to lead to quite conservative results. Many efforts are now being made in the direction of further reducing conservatism of standard methods through a dilatation procedure as in [11, 20–30]. More specifically, the authors in [31] derived sufficient dilated LMI condition for the existence of a robust H_2 static output feedback controller for continuous- as well as discrete-time systems. In their approach, the authors used auxiliary slack variables with structure that provided extra degree of freedom and offered more flexibility for the controller design. Their result turns out to be significantly less conservative than the results reported in [14, 17, 19]. To our knowledge, the robust H_∞ static output feedback stabilization problem for linear continuous-time system has not been addressed in the literature. The main objective of this paper is to derive sufficient dilated LMI conditions for this problem. It will be shown that the obtained conditions are naturally less conservative than early standard LMI methods. The structural restriction imposed on a Lyapunov variable is bypassed by means of the auxiliary slack variables with structure and a scalar. This extra degree of freedom has provided additional flexibility that greatly helps in solving the robust H_∞ static output feedback control problem via the parameter-dependent Lyapunov functions (PDLF). It is also shown, in this paper, that the proposed dilated LMI-based solution always encompasses the standard LMI-based one. This means that the conservatism of our method can never be worse than the one of existing in LMI standard methods.

The paper is organized as follows. In the next section, the H_∞ static output feedback stabilization problem is formulated. In Section 3, new sufficient dilated LMI-based conditions, for this problem, are proposed. These results are extended to the case of systems with polytopic uncertainties in Section 4. Section 5 gives numerical examples illustrating the advantages of the proposed method and compares it, when applicable, to the standard approaches.

Notation 1. The notation used in this paper is standard. In particular, $P > 0$ means that the matrix P is symmetric and positive definite. $\mathrm{Sym}\{A\}$ is used to denote the expression $A + A^T$, and $*$ is used to denote symmetric matrix blocks.

2. Problem Statement and Preliminaries

Let us consider the linear continuous-time stationary system described by the state-space equations

$$\begin{aligned} \dot{x} &= Ax + B_w w + B_u u, \\ z &= C_z x + D_{zw} w + D_{zu} u, \\ y &= C_y x, \end{aligned} \tag{2.1}$$

where $x \in R^n$ is the plant state vector, $w \in R^m$ is the exogenous input vector, $u \in R^q$ is the controller input vector, $z \in R^p$ is the controlled output vector, and $y \in R^r$ is the measured output vector. The static output feedback control problem is to find a constant matrix gain $K \in R^{q\times r}$ defined by the control law

$$u = Ky, \tag{2.2}$$

which stabilizes system (2.1) and guarantees that the H_∞ norm of the transfer matrix from w to z is less than a prescribed level γ. The feedback connection of the system and the controller gives a linear system whose state-space representation is

$$\begin{aligned} \dot{x} &= A_{\mathrm{cl}}x + B_{\mathrm{cl}}w, \\ z &= C_{\mathrm{cl}}x + D_{\mathrm{cl}}w, \end{aligned} \tag{2.3}$$

where the matrices A_{cl}, B_{cl}, C_{cl}, and D_{cl} are given by

$$\begin{aligned} A_{\mathrm{cl}} &= A + B_uKC_y, & B_{\mathrm{cl}} &= B_w, \\ C_{\mathrm{cl}} &= C_z + D_{zu}KC_y, & D_{\mathrm{cl}} &= D_{zw}. \end{aligned} \tag{2.4}$$

The closed-loop transfer matrix from w to z, $G_{wz}(s)$, is then given by

$$G_{wz}(s) = C_{\mathrm{cl}}(sI - A_{\mathrm{cl}})^{-1}B_{\mathrm{cl}} + D_{\mathrm{cl}}. \tag{2.5}$$

From the bounded real lemma [32, 33], system (2.1) is stabilizable via the static output feedback controller (2.2), and $\|G_{wz}(s)\|_\infty < \gamma$ if and only if there exist a symmetric and definite positive matrix $X \in R^{n\times n}$ and a general matrix $K \in R^{q\times r}$ such that

$$\begin{bmatrix} \mathrm{Sym}\{A_{\mathrm{cl}}X\} & B_{\mathrm{cl}} & XC_{\mathrm{cl}}^T \\ * & -\gamma I_{m\times m} & D_{\mathrm{cl}}^T \\ * & * & -\gamma I_{p\times p} \end{bmatrix} < 0. \tag{2.6}$$

Note that despite the fact that A_{cl} and C_{cl} are both affine in K, as it is shown in (2.4), the matrix inequality (2.6) is a BMI in the variables X and K, that is, non convex and NP-hard to solve [1]. In order to circumvent this difficulty, many authors searched, instead, LMI conditions which are only sufficient, hence gaining tractability at the expense of some more or less conservatism. For the incoming developments, a particular linear transformation is needed for system (2.1) by means of nonsingular state coordinate transformation matrix $T \in R^{n\times n}$ insuring that $\overline{C}_y = C_yT^{-1} = [I_{r\times r} \;\; 0_{r\times(n-r)}]$ in case C_y is a full row-rank matrix, and thus leading to the transformed generalized plant representation

$$\left[\begin{array}{c|c|c} \overline{A} & \overline{B}_w & \overline{B}_u \\ \hline \overline{C}_z & \overline{D}_{zw} & \overline{D}_{zu} \\ \hline \overline{C}_y & & \end{array}\right] = \left[\begin{array}{c|c|c} TAT^{-1} & TB_w & TB_u \\ \hline C_zT^{-1} & D_{zw} & D_{zu} \\ \hline C_yT^{-1} & & \end{array}\right] = \left[\begin{array}{c|c|c} TAT^{-1} & TB_w & TB_u \\ \hline C_zT^{-1} & D_{zw} & D_{zu} \\ \hline [I_{r\times r} \;\; 0_{r\times(n-r)}] & & \end{array}\right]. \tag{2.7}$$

The nonsingular state coordinate transformation matrix T is usually taken as in [31] to be $T = [C_y^T(C_yC_y^T)^{-1} \;\; C_y^{\perp}]^{-1}$, with $C_y^{\perp}$ denoting an orthogonal basis for the null space of C_y.

The following lemma [17], for instance, proposed the following sufficient standard LMI conditions for the H_∞ static output feedback control problem.

Lemma 2.1. *Assume that C_y is a full row-rank matrix. System* (2.1) *is stabilizable by the static output feedback controller* (2.2) *and $\|G_{wz}(s)\|_\infty < \gamma$ if there exist a symmetric, structured, and positive definite matrix $X = \mathrm{diag}(X_1, X_2)$ with $X_1 \in R^{r\times r}$and $X_2 \in R^{(n-r)\times(n-r)}$ and a structured matrix $U = \lfloor U_1 \;\; 0_{q\times(n-r)}\rfloor$ with $U_1 \in R^{q\times r}$ such that*

$$\begin{bmatrix} \mathrm{Sym}\left\{\overline{A}X + \overline{B}_u U\right\} & \overline{B}_w & X\overline{C}_z^T + U^T\overline{D}_{zu}^T \\ * & -\gamma I_{m\times m} & \overline{D}_{zw}^T \\ * & * & -\gamma I_{p\times p} \end{bmatrix} < 0. \tag{2.8}$$

Furthermore, a static output feedback controller gain is given by $K = U_1X_1^{-1}$.

Remarks 1. (i) Lemma 2.1 is only applicable when matrix C_y has a full row rank. A dual version of this lemma is readily obtained when matrix C_y is row-rank deficient, but instead, when matrix B_u has a full column rank.

(ii) Because of the structure imposed on the Lyapunov variable X, a great deal of conservatism is known to limit the usefulness of Lemma 2.1.

3. H_∞ Static Output Controller Synthesis

In order to reduce the conservatism encountered in the standard approach, in the following lemma, new sufficient dilated LMI-based conditions are now proposed.

Lemma 3.1. *Assume that C_y is a full row-rank matrix. System* (2.1) *is stabilizable by the static output feedback controller* (2.2) *and $\|G_{wz}(s)\|_\infty < \gamma$ if for some positive scalar α, there exist a symmetric definite positive matrix $Y \in R^{n\times n}$ and structured matrices $Z = \left[\begin{smallmatrix} Z_1 & 0_{r\times(n-r)} \\ Z_2 & Z_3 \end{smallmatrix}\right]$ with $Z_1 \in R^{r\times r}$, $Z_2 \in R^{(n-r)\times r}$, and $Z_3 \in R^{(n-r)\times(n-r)}$ and $L = \lfloor L_1 \;\; 0_{q\times(n-r)}\rfloor$ with $L_1 \in R^{q\times r}$ such that*

$$\begin{bmatrix} \alpha\mathrm{Sym}\left\{\overline{A}Z + \overline{B}_u L\right\} & \overline{B}_w & \alpha\left(Z^T\overline{C}_z^T + L^T\overline{D}_{zu}^T\right) & Y + \overline{A}Z + \overline{B}_u L - \alpha Z^T \\ * & -\gamma I_{m\times m} & \overline{D}_{zw}^T & 0_{m\times n} \\ * & * & -\gamma I_{p\times p} & \overline{C}_z Z + \overline{D}_{zu} L \\ * & * & * & -\mathrm{Sym}\{Z\} \end{bmatrix} < 0. \tag{3.1}$$

Furthermore, a static output feedback controller gain is given by $K = L_1Z_1^{-1}$.

Proof. Assume that, for some positive scalar α, a solution with variables Y, $Z = \begin{bmatrix} Z_1 & 0_{r\times(n-r)} \\ Z_2 & Z_3 \end{bmatrix}$ and $L = \lfloor L_1 \;\; 0_{q\times(n-r)} \rfloor$ of inequality (3.1) exists. Substituting the expressions for $\overline{A}$, $\overline{B}_w$, $\overline{B}_u$, $\overline{C}_z$, $\overline{D}_{zw}$, and $\overline{D}_{zu}$ into this inequality, defining $\overline{Z} = T^{-1}ZT^{-T}$ and $\overline{Y} = T^{-1}YT^{-T}$, it comes that

$$\begin{bmatrix} \alpha\mathrm{Sym}\left\{T\left(A\overline{Z} + B_uLT^{-T}\right)T^T\right\} & TB_w & \alpha T\left(\overline{Z}^T C_z^T + T^{-1}L^T D_{zu}^T\right) & \mathcal{A} \\ * & -\gamma I_{m\times m} & D_{zw}^T & 0_{m\times n} \\ * & * & -\gamma I_{p\times p} & \left(C_z\overline{Z} + D_{zu}LT^{-T}\right)T^T \\ * & * & * & -\mathrm{Sym}\left\{T\overline{Z}T^T\right\} \end{bmatrix} < 0, \tag{3.2}$$

where $\mathcal{A}$ denotes $T(\overline{Y} + A\overline{Z} + B_uLT^{-T} - \alpha\overline{Z}^T)T^T$. In view of the expressions of L and K in Lemma 3.1, the term LT^{-T}, in this inequality, can be developed into

$$\begin{aligned} LT^{-T} &= \left[L_1 \;\; 0_{q\times(n-r)}\right]T^{-T} = K\left[I_{r\times r} \;\; 0_{r\times(n-r)}\right]\begin{bmatrix} Z_1 & 0_{r\times(n-r)} \\ Z_2 & Z_3 \end{bmatrix}T^{-T} \\ &= KC_yT^{-1}\begin{bmatrix} Z_1 & 0_{r\times(n-r)} \\ Z_2 & Z_3 \end{bmatrix}T^{-T} = KC_y\overline{Z}. \end{aligned} \tag{3.3}$$

Now, using (2.4) leads to

$$\begin{bmatrix} \alpha\,\mathrm{Sym}\left\{TA_{\mathrm{cl}}\overline{Z}T^T\right\} & TB_{\mathrm{cl}} & \alpha T\overline{Z}^T C_{\mathrm{cl}}^T & T\left(\overline{Y} + A_{\mathrm{cl}}\overline{Z} - \alpha\overline{Z}^T\right)T^T \\ * & -\gamma I_{m\times m} & D_{\mathrm{cl}}^T & 0_{m\times n} \\ * & * & -\gamma I_{p\times p} & C_{\mathrm{cl}}\overline{Z}T^T \\ * & * & * & -\mathrm{Sym}\left\{T\overline{Z}T^T\right\} \end{bmatrix} < 0. \tag{3.4}$$

Now, pre- and postmultiplying this inequality by $\begin{bmatrix} T^{-1} & 0_{n\times m} & 0_{n\times p} & A_{\mathrm{cl}}T^{-1} \\ 0_{m\times n} & I_{m\times m} & 0_{m\times p} & 0_{m\times n} \\ 0_{p\times n} & 0_{p\times m} & I_{p\times p} & C_{\mathrm{cl}}T^{-1} \end{bmatrix}$ and its transposed, respectively, lead to Inequality (2.6). This completes the proof. □

In this lemma, the Lyapunov variable Y has been separated from the controller variable K. Hence, there is no need to impose any structure upon the Lyapunov variable as in Lemma 2.1. The structural restriction is, instead, imposed on the introduced auxiliary slack variables Z. These slack variables, along the scalar α, also provide additional degrees of freedom, hence, possibly reducing conservativeness. The following theorem proves that the degree of conservatism can, indeed, be almost always reduced.

Theorem 3.2. *If the standard LMI conditions of Lemma 2.1 are satisfied and achieve an upper bound γ^S, then the dilated inequality conditions of Lemma 3.1 are satisfied with an upper bound $\gamma^D \leq \gamma^S$.*

Proof. Suppose that the matrix inequality conditions of Lemma 2.1 are satisfied and achieve an upper bound γ^S with the variables X and U. In the right-hand side of the matrix inequality conditions of Lemma 3.1, if we let $Y = X$, $Z = \alpha^{-1}X$, and $L = \alpha^{-1}U$, with $\alpha > 0$, this right-hand side becomes

$$\begin{bmatrix} \mathrm{Sym}\left\{\overline{A}X + \overline{B}_u U\right\} & \overline{B}_w & \left(X\overline{C}_z^T + U^T\overline{D}_{zu}^T\right) & \alpha^{-1}\left(\overline{A}X + \overline{B}_u U\right) \\ * & -\gamma I_{m\times m} & \overline{D}_{zw}^T & 0_{m\times n} \\ * & * & -\gamma I_{p\times p} & \alpha^{-1}\left(\overline{C}_z X + \overline{D}_{zu} U\right) \\ * & * & * & -2\alpha^{-1}X \end{bmatrix}. \tag{3.5}$$

By virtue of the Schur complement, this matrix will be negative definite if and only if $X > 0$ and

$$\begin{aligned} &\begin{bmatrix} \mathrm{Sym}\left\{\overline{A}X + \overline{B}_u U\right\} & \overline{B}_w & X\overline{C}_z^T + U^T\overline{D}_{zu}^T \\ * & -\gamma I_{m\times m} & \overline{D}_{zw}^T \\ * & * & -\gamma I_{p\times p} \end{bmatrix} \\ &\quad + 0.5\times\alpha^{-1}\begin{bmatrix} \overline{A}X + \overline{B}_u U \\ 0_{m\times n} \\ \overline{C}_z X + \overline{D}_{zu} U \end{bmatrix} X^{-1} \begin{bmatrix} \overline{A}X + \overline{B}_u U \\ 0_{m\times n} \\ \overline{C}_z X + \overline{D}_{zu} U \end{bmatrix}^T < 0. \end{aligned} \tag{3.6}$$

As the first term in this matrix inequality is no other than the standard H_∞ conditions in Lemma 2.1 and is therefore negative definite, there always exists a sufficiently large scalar $\alpha > 0$ which achieves this condition. This proves that the dilated LMI H_∞ conditions in Lemma 3.1 always encompass the standard ones. Clearly, this means that the dilated-based approach yields upper bounds that are always such that $\gamma^D \leq \gamma^S$. □

This theorem proves that the new dilated LMI conditions of Lemma 3.1 for the H_∞ static output controller synthesis do encompass the standard LMI-based ones in Lemma 1 in [15] when $Y = X$, $Z = \alpha^{-1}X$, $L = \alpha^{-1}U$, and $\alpha \in [\alpha_{\min}, +\infty]$ ($\alpha_{\min}$ being the minimum value which satisfies the inequality condition (3.6)). When these parameters are set free in the new dilated LMI conditions of Lemma 3.1, it is likely that they actually reduce conservatism when compared to the standard LMI "undilated" counterparts. The best reduction in conservatism can be easily obtained through a simple line search of the scalar α.

4. Extension to Uncertain Systems

In this section, an extension to uncertain systems with polytopic uncertainties is made. To this end, let us consider the linear continuous-time uncertain system described by the following state space equations:

$$\begin{aligned} \dot{x} &= \mathbf{A}x + \mathbf{B_w}w + \mathbf{B_u}u, \\ z &= \mathbf{C_z}x + \mathbf{D_{zw}}w + \mathbf{D_{zu}}u, \\ y &= \mathbf{C_y}x, \end{aligned} \tag{4.1}$$

where $\mathbf{A}$, $\mathbf{B_w}$, $\mathbf{B_u}$, $\mathbf{C_z}$, $\mathbf{C_y}$, $\mathbf{D_{zw}}$, and $\mathbf{D_{zu}}$ are not precisely known but belong to a polytopic uncertainty domain Ω defined by

$$\Omega = \left\{ \left[\begin{array}{c|c|c} \mathbf{A} & \mathbf{B_w} & \mathbf{B_u} \\ \hline \mathbf{C_z} & \mathbf{D_{zw}} & \mathbf{D_{zu}} \\ \hline \mathbf{C_y} & & \end{array} \right] = \sum_{i=1}^{N} \theta_i \left[\begin{array}{c|c|c} A_i & B_{wi} & B_{ui} \\ \hline C_{zi} & D_{zwi} & D_{zui} \\ \hline C_{yi} & & \end{array} \right] \right\}, \tag{4.2}$$

where $\theta_i \geq 0$ and $\sum_{i=1}^{N} \theta_i = 1$.

With the static output feedback controller given by (2.2), the closed-loop transfer matrix from w to z, $\mathbf{G_{wz}}(s,\theta)$, is given by

$$\mathbf{G_{wz}}(s,\theta) = \mathbf{C_{cl}}(sI - \mathbf{A_{cl}})^{-1}\mathbf{B_{cl}} + \mathbf{D_{cl}}, \tag{4.3}$$

where the matrices $\mathbf{A_{cl}}$, $\mathbf{B_{cl}}$, $\mathbf{C_{cl}}$, and $\mathbf{D_{cl}}$ are given by

$$\begin{aligned} \mathbf{A_{cl}} &= \mathbf{A} + \mathbf{B_u}\mathbf{K}\mathbf{C_y}, \qquad & \mathbf{B_{cl}} &= \mathbf{B_w}, \\ \mathbf{C_{cl}} &= \mathbf{C_z} + \mathbf{D_{zu}}\mathbf{K}\mathbf{C_y}, \qquad & \mathbf{D_{cl}} &= \mathbf{D_{zw}}. \end{aligned} \tag{4.4}$$

In view of Lemma 2.1, it is straightforward to derive sufficient conditions for the existence of a static output feedback controller as in (2.2) which robustly stabilizes system (4.1) solely in case of $\mathbf{C}_y$ is certain and, in the same time, insures that a $\|\mathbf{G_{wz}}(s,\theta)\|_\infty < \gamma$.

Lemma 4.1. *Assume that $\mathbf{C}_y = C_y$ is a full-row rank fixed matrix. The system* (4.1) *is robustly stabilizable by the static output feedback controller* (2.2) *and $\|\mathbf{G_{wz}}(s,\theta)\|_\infty < \gamma$ if there exist a symmetric structured matrix $X = \operatorname{diag}(X_1, X_2)$ with $X_1 \in R^{r\times r}$and $X_2 \in R^{(n-r)\times(n-r)}$ and a structured matrix $U = [U_1 \;\; 0_{q\times(n-r)}]$ with $U_1 \in R^{q\times r}$ such that, for $i = 1,\ldots,N$,*

$$\begin{bmatrix} \operatorname{Sym}\left\{\overline{A}_i X + \overline{B}_{ui} U\right\} & \overline{B}_{wi} & X\overline{C}_{zi}^{T} + U^T D_{zui}^T \\ * & -\gamma I_{m\times m} & D_{zwi}^T \\ * & * & -\gamma I_{p\times p} \end{bmatrix} < 0, \tag{4.5}$$

where a single nonsingular state coordinate transformation matrix T is used for all the system vertices satisfying $\overline{C}_y = C_y T^{-1} = [I_{r\times r} \;\; 0_{r\times(n-r)}]$, as $\mathbf{C}_y$ is certain. Furthermore, the static output feedback controller gain is given by $K = U_1 X_1^{-1}$.

This lemma provides sufficient standard LMI-based conditions that are known to be, in general, overly conservative. This conservatism is due to the fact that not only common Lyapunov matrix is used for all the vertices, but, in addition, some structure was also imposed on this matrix. The following lemma provides, by means of parameter-dependent Lyapunov functions, new sufficient conditions for the existence of a robust static output feedback controller, as in (2.2), which robustly stabilizes system (4.1) with the matrix $\mathbf{C}_y$ allowed to be uncertain and, at the same time, insures that $\|\mathbf{G_{wz}}(s,\theta)\|_\infty < \gamma$.

Lemma 4.2. *Assume that, for $i = 1, \ldots, N$, matrix C_{yi} has a full-row rank. The uncertain system* (4.1) *is robustly stable by a static output feedback controller and $\|\mathbf{G}_{\mathbf{wz}}(s,\theta)\|_\infty < \gamma$ if, for some positive scalar α and for $1 \le i, j \le N$, there exist symmetric definite positive matrices $\overline{Y}_j \in R^{n\times n}$ and structured matrices $Z_i = \left[\begin{smallmatrix} Z_1 & 0_{r\times(n-r)} \\ Z_{2i} & Z_{3i}\end{smallmatrix}\right]$ with $Z_1 \in R^{r\times r}$, $Z_{2i} \in R^{(n-r)\times r}$, and $Z_{3i} \in R^{(n-r)\times(n-r)}$ and $L = \lfloor L_1 \;\; 0_{q\times(n-r)} \rfloor$ with $L_1 \in R^{q\times r}$ such that*

$$\begin{bmatrix} \alpha \mathrm{Sym}\left\{\overline{\mathrm{A}}_\mathrm{i} \mathrm{Z}_\mathrm{i} + \overline{\mathrm{B}}_\mathrm{uij}\mathrm{L}\right\} & \overline{B}_{wi} & \alpha\left(Z_i^T\overline{C}_{zi}^T + L^T\overline{D}_{zuj}^T\right) & T_i\overline{Y}_jT_i^T + \overline{A}_iZ_i + \overline{B}_{uij}L - \alpha Z_i^T \\ * & -\gamma I_{m\times m} & \overline{D}_{zwi}^T & 0_{m\times n} \\ * & * & -\gamma I_{p\times p} & \overline{C}_{zi}Z_i + \overline{D}_{zuj}L \\ * & * & * & -\mathrm{Sym}\{Z_i\} \end{bmatrix} < 0, \tag{4.6}$$

where a different nonsingular state coordinate transformation matrix T_i is used for each of the system vertices satisfying $\overline{C}_{yi} = C_{yi}T_i^{-1} = [I_{r\times r} \quad 0_{r\times(n-r)}]$ and where $\overline{B}_{uij} = T_iB_{uj}$. Furthermore, a static output feedback controller gain is given by $K = L_1Z_1^{-1}$.

Proof. Suppose that, for some scalar $\alpha > 0$, a solution to inequalities (4.6) exists with variables $Z_i = \left[\begin{smallmatrix} Z_1 & 0_{r\times(n-r)} \\ Z_{2i} & Z_{3i}\end{smallmatrix}\right]$, $\overline{Y}_j$, and $L = \lfloor L_1 \;\; 0_{q\times(n-r)}\rfloor$. Substituting $\overline{A}_i$, $\overline{B}_{wi}$, $\overline{B}_{uij}$, $\overline{C}_{zi}$, $\overline{D}_{zwi}$, and $\overline{D}_{zuj}$ into these inequalities, defining $\overline{Z}_i = T_i^{-1}Z_iT_i^{-T}$ and using the fact that the term LT_i^{-T}, in this inequality, can be developed into

$$\begin{aligned} LT_i^{-T} &= \left[L_1 \;\; 0_{q\times(n-r)}\right]T_i^{-T} = K\left[I_{r\times r} \;\; 0_{r\times(n-r)}\right]\begin{bmatrix} Z_1 & 0_{r\times(n-r)} \\ Z_{2j} & Z_{3j}\end{bmatrix}T_i^{-T} \\ &= KC_{yi}T_i^{-1}\begin{bmatrix} Z_1 & 0_{r\times(n-r)} \\ Z_{2j} & Z_{3j}\end{bmatrix}T_i^{-T} = KC_{yi}\overline{Z}_i, \end{aligned} \tag{4.7}$$

it comes that

$$\begin{bmatrix} \alpha\,\mathrm{Sym}\left\{T_i\left(A_i + B_{uj}KC_{yi}\right)\overline{Z}_iT_i^T\right\} & T_iB_{wi} & \alpha T_i\overline{Z}_i^T\left(C_{zi}^T + C_{yi}^TK^TD_{zuj}^T\right) & \mathcal{B} \\ * & -\gamma I_{m\times m} & D_{zwi}^T & 0_{m\times n} \\ * & * & -\gamma I_{p\times p} & \mathcal{C} \\ * & * & * & -\mathrm{Sym}\left\{T_i\overline{Z}_iT_i^T\right\} \end{bmatrix} < 0, \tag{4.8}$$

where $\mathcal{B}$ denotes $T_i(\overline{Y}_j + (A_i + B_{uj}KC_{yi})\overline{Z}_i - \alpha\overline{Z}_i^T)T_i^T$ and $\mathcal{C}$ denotes $(C_{zi} + D_{zuj}KC_{yi})\overline{Z}_iT_i^T$.

Now, pre- and postmultiplying this inequality by $\begin{bmatrix} T_i^{-1} & 0_{n\times m} & 0_{n\times p} & -\alpha T_i^{-1} \\ 0_{m\times n} & I_{m\times m} & 0_{m\times p} & 0_{m\times n} \\ 0_{p\times n} & 0_{p\times m} & I_{p\times p} & 0_{p\times n} \\ 0_{n\times n} & 0_{n\times m} & 0_{n\times p} & \overline{Z}_i^{-T} T_i^{-1} \end{bmatrix}$ and its transposed, respectively, it becomes equivalent to

$$\begin{bmatrix} -2\alpha\overline{Y}_j & B_{wi} & 0_{n\times p} & \overline{Y}_j\overline{Z}_i^{-1} + \left(A_i + B_{uj}KC_{yi}\right) + \alpha I_{n\times n} \\ * & -\gamma I_{m\times m} & D_{zwi}^T & 0_{m\times n} \\ * & * & -\gamma I_{p\times p} & C_{zi} + D_{zuj}KC_{yi} \\ * & * & * & -\mathrm{Sym}\left\{\overline{Z}_i^{-1}\right\} \end{bmatrix} < 0. \tag{4.9}$$

Multiplying this inequality by $\theta_i\theta_j$ and summing up lead to

$$\begin{bmatrix} -2\alpha\overline{\mathbf{Y}} & \mathbf{B_w} & 0_{n\times p} & \overline{\mathbf{Y}\mathbf{Z}}^{-1} + \left(\mathbf{A} + \mathbf{B_u}K\mathbf{C}_y\right) + \alpha I_{n\times n} \\ * & -\gamma I_{m\times m} & \mathbf{D}_{\mathbf{zw}}^T & 0_{m\times n} \\ * & * & -\gamma I_{p\times p} & \mathbf{C_z} + \mathbf{D_{zu}}K\mathbf{C}_y \\ * & * & * & -\mathrm{Sym}\left\{\overline{\mathbf{Z}}^{-1}\right\} \end{bmatrix} < 0, \tag{4.10}$$

where $\overline{\mathbf{Y}} = \sum_{j=1}^{N}\theta_j\overline{Y}_j$ and $\overline{\mathbf{Z}}^{-1} = \sum_{i=1}^{N}\theta_i\overline{Z}_i^{-1}$. This obviously implies that $\overline{\mathbf{Y}} > 0$. On the other hand, pre- and postmultiplying this inequality by $\begin{bmatrix} I_{n\times n} & 0_{n\times m} & 0_{n\times p} & (\mathbf{A}_{\mathrm{cl}}+\alpha I)\overline{\mathbf{Z}}^T \\ 0_{m\times n} & I_{m\times m} & 0_{m\times p} & 0_{m\times n} \\ 0_{p\times n} & 0_{p\times m} & I_{p\times p} & \mathbf{C}_{\mathrm{cl}}\overline{\mathbf{Z}}^T \end{bmatrix}$ and its transposed, respectively, lead to $\begin{bmatrix} \mathrm{Sym}\{\mathbf{A}_{\mathrm{cl}}\overline{\mathbf{Y}}\} & \mathbf{B}_{\mathrm{cl}} & \overline{\mathbf{Y}}\mathbf{C}_{\mathrm{cl}}^{\mathbf{T}} \\ * & -\gamma I_{m\times m} & \mathbf{D}_{\mathrm{cl}}^{\mathbf{T}} \\ * & * & -\gamma I_{p\times p} \end{bmatrix}$, which completes the proof. □

Similarly to the certain case, in this lemma, the Lyapunov variable has been separated from the controller parameter. This has permitted the use of parameter-dependent Lyapunov functions (PDLFs) and, hence, provided the opportunity to possibly lessen conservativeness. In addition, contrarily to many results reported in the literature (as in, for instance, [14, 17]), the output matrix $\mathbf{C}_y$ is allowed to be uncertain. This constitutes an important contribution of this paper. It is also important to note that when the output matrix remains certain (fixed), our proposed LMI conditions always recover the standard LMI conditions. A numerical example is given to support this claim.

5. Numerical Examples

In this section, numerical examples are presented to illustrate the merit of the proposed H_∞ static output feedback synthesis method, both in the cases of with or without uncertainties. When applicable, a comparison with the standard conditions, for instance the ones reported in [17], is then made. In all the following examples, we need to design an H_∞ static output feedback controller which not only stabilizes the considered system (certain or uncertain), but also minimizes the H_∞ norm of the closed-loop transfer function.

Table 1: Controller gain and the optimal H_∞ guaranteed cost.

Design methods	γ	K	$\|G_{wz}(s)\|_\infty$
Standard [17]	6.04	$\begin{bmatrix} -0.0029 \\ -0.1664 \end{bmatrix}$	2.07
Our dilated method (Lemma 3.1)	4.17 ($\alpha = 1.1$)	$\begin{bmatrix} 0.1244 \\ -0.2151 \end{bmatrix}$	2.49

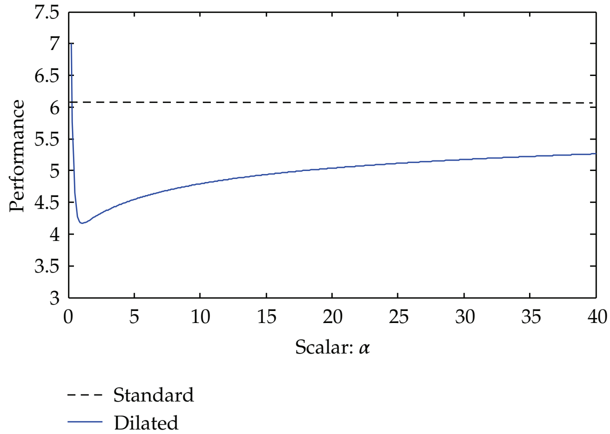

Figure 1: The relation between the performance level γ and the scalar α.

Example 5.1. Take the continuous-time, unstable, and certain system

$$\left[\begin{array}{c|c|c} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ \hline C_y & & \end{array}\right] = \left[\begin{array}{ccc|c|cc} 0.1 & 0 & 2 & 1 & 1 & 2 \\ 0 & -0.2 & 1 & 2 & 0 & 0 \\ 0 & 0.3 & -1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 & & & \end{array}\right]. \tag{5.1}$$

The simulation results for an H_∞ static output feedback controller synthesis are given in Table 1. Clearly, for this example, the proposed dilated LMI conditions of Lemma 3.1 give a very significant improvement (around 30%), when compared with the classical standard conditions of Lemma 1 in [17].

Figure 1 depicts the effect of varying the scalar parameter α, involved in Lemma 3.1, upon the performance level γ. Clearly, from this figure, it appears that for small values of this parameter α, no recovery is achieved, and the performance level remains above the standard level. When α is greater than a certain threshold, a better than standard performance level is achieved culminating at a unique minimum, before tending to the standard level for greater values of α. This seems to be a general trend as the coming examples will show.

Table 2: Controller gain and the optimal H_∞ guaranteed cost.

Design methods	γ	K	$\Vert\mathbf{G}_{\mathbf{wz}}(s,\theta)\Vert_\infty(\theta_1,\theta_2)=(0.2,0.8)$
Standard [17] (Lemma 4.1)	6.17	-6.314×10^5	2.3
Our dilated method (Lemma 4.2)	4.54 ($\alpha=4.8$)	-2.2786	1.6

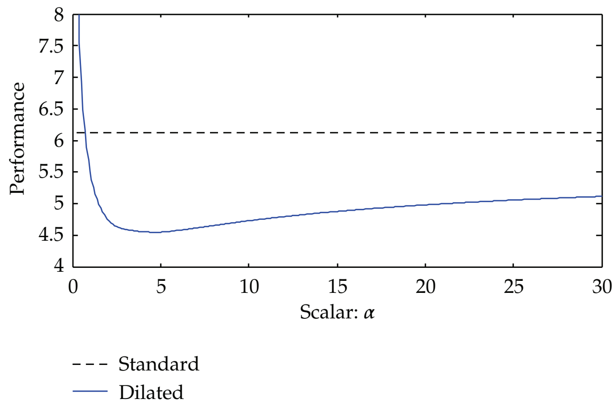

Figure 2: The relation between the performance level γ and the scalar α.

Example 5.2. Consider the continuous-time system with polytopic uncertainties of two vertices defined by

$$\left[\begin{array}{c|c|c} A_1 & B_{w1} & B_{u1} \\ \hline C_{z1} & D_{zw1} & D_{zu1} \\ \hline C_{y1} & & \end{array}\right]=\left[\begin{array}{cc|c|c} 2 & 1 & 1 & 1 \\ 0 & -4 & 1 & 0 \\ \hline 1 & 1 & 4 & 1 \\ \hline 2 & 1 & & \end{array}\right],\qquad \left[\begin{array}{c|c|c} A_2 & B_{w2} & B_{u2} \\ \hline C_{z2} & D_{zw2} & D_{zu2} \\ \hline C_{y2} & & \end{array}\right]=\left[\begin{array}{cc|c|c} 1 & -1 & 1 & 1 \\ 0 & -5 & 1 & 0 \\ \hline 1 & 2 & 1 & 3 \\ \hline 2 & 1 & & \end{array}\right]. \tag{5.2}$$

In this case, the output matrix is fixed, that is, supposed to be certain. The simulation results are given in Table 2. Clearly, for this example, the proposed dilated LMI condition gives, for a given realization, a very significant improvement (around 25%), when compared with the classical standard conditions of Lemma 3 in [17].

Figure 2 depicts the effect of varying the scalar parameter α upon the performance level γ. The general trend suspected in Example 5.1 (Figure 1) seems to be confirmed.

Example 5.3. Consider the continuous-time system with polytopic uncertainties defined by

$$\left[\begin{array}{c|c|c} A_1 & B_{w1} & B_{u1} \\ \hline C_{z1} & D_{zw1} & D_{zu1} \\ \hline C_{y1} & & \end{array}\right]=\left[\begin{array}{cc|c|c} 2 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ \hline 1 & 2 & 2 & 1 \\ \hline 1 & 0 & & \end{array}\right],\qquad \left[\begin{array}{c|c|c} A_2 & B_{w2} & B_{u2} \\ \hline C_{z2} & D_{zw2} & D_{zu2} \\ \hline C_{y2} & & \end{array}\right]=\left[\begin{array}{cc|c|c} 2 & -1 & 1 & 1 \\ 0 & -3 & 1 & 0 \\ \hline 0.5 & 1 & 1 & 2 \\ \hline 2 & 1 & & \end{array}\right], \tag{5.3}$$

in which the output matrix is uncertain. The application of Lemma 4.2 and a line search of the scalar α yielded, for the realization $(\theta_1, \theta_2) = (0.4, 0.6)$, the guaranteed H_∞ performance $\gamma = 5.08$, for $\alpha = 4.2$, a controller gain $K = -4.4266$ and an actual performance level $\|\mathbf{G_{wz}}(s, \theta)\|_\infty = 1.52$.

6. Conclusion

In this paper, we have presented new sufficient dilated LMI conditions for solving H_∞ static output feedback control problems for continuous-time systems. Auxiliary slack variables with even more relaxed structure are employed in order to provide additional flexibility in the design. The method easily and successfully extends to the case of systems with polytopic uncertainties by means of parameter-dependent Lyapunov functions. It is shown that the proposed dilated LMI-based conditions always encompass the standard LMI-based ones. The consequence is a significant reduction of conservatism. The numerical examples have supported all these claims.

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