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Research Article

Low-Order Nonconforming Mixed Finite Element Methods for Stationary Incompressible Magnetohydrodynamics Equations

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The nonconforming mixed finite element methods (NMFEMs) are introduced and analyzed for the numerical discretization of a nonlinear, fully coupled stationary incompressible magnetohydrodynamics (MHD) problem in 3D. A family of the low-order elements on tetrahedra or hexahedra are chosen to approximate the pressure, the velocity field, and the magnetic field. The existence and uniqueness of the approximate solutions are shown, and the optimal error estimates for the corresponding unknown variables in L^2-norm are established, as well as those in a broken H^1-norm for the velocity and the magnetic fields. Furthermore, a new approach is adopted to prove the discrete Poincaré-Friedrichs inequality, which is easier than that of the previous literature.

1. Introduction

This work deals with the numerical discretization of a nonlinear, fully coupled stationary incompressible MHD problem by a family of the low-order NMFEMs. This requires discretizing a system of partial differential equations that couples the incompressible Navier-Stokes equations with Maxwell's equations.

The MHD problem has a number of applications such as liquid-metal cooling of nuclear reactors, electromagnetic casting of metals, MHD power generation, and MHD ion propulsion (cf. [1, 2]). Thus, many studies have already been devoted to the MHD problem. For theoretical results, let us just mention those by [3–5]. It is important to employ effective numerical methods to approximate the exact solutions of the MHD problem because the exact solutions can be obtained only for some special cases [2]. Compared with the finite difference methods [6–8], most studies are performed by the finite element methods (FEMs) [9–24].

Precisely speaking, the work started with [9], where inf-sup stable mixed elements were used to discretize the velocity field and the pressure, and H^1-conforming elements for the magnetic field, and the existence and uniqueness of the discrete solutions with inhomogeneous boundary condition satisfying certain assumptions were proved and the convergence analysis was presented. In contrast to the results of [9], [10] derived the same results without any restrictions on the boundary data of the velocity field. Reference [11] examined the long-term dissipativity and unconditional nonlinear stability of time integration algorithms for an incompressible MHD problem. Reference [12] dealt with a decoupled linear MHD problem involving electrically conducting and insulating regions by Lagrange finite elements and gave error estimates for a fully discrete scheme. For convex polyhedral domains, or domains with a boundary $C^{1,1}$, the convergence analysis of a stabilized FEM, the optimal control method, and two-level FEMs were investigated in [13–15], [16], and [17], respectively.

On the other hand, some different approaches to achieve convergence results in general Lipschitz polyhedral domains were realized. For example, a mixed discrete formulation about the problem based on $H(\mathrm{curl})$-conforming (edge) elements to approximate the magnetic field was proposed in [18, 19]. This observation motivated the works such as the least-squares mixed FEM used in [20], the mixed discontinuous Galerkin method employed in [21, 22], and the splitting method presented in [23, 24]. However, all the analyses in [9–24] are about the conforming FEMs except [22].

As we know, nonconforming FEMs have certain advantages over conforming FEMs in some aspects. Firstly, the nonconforming elements are much easier to be constructed to satisfy the discrete inf-sup condition. Secondly, nonconforming elements have been used effectively especially in fluid and solid mechanics due to their stability. We refer to [25–34] for more details on the properties of nonconforming elements applied to incompressible flow problems.

For the Stokes equations, [25, 26] considered the approximations of nonconforming P_1/P_0 element and the rotated Q_1/Q_0 element and got first-order accuracy, respectively. Reference [27] modified the rotated Q_1/Q_0 element used in [26] and derived the same convergence order as [26]. For the Navier-Stokes equations, [28–30] obtained maximum norm estimates of P_1/P_0 element and the optimal error estimates of EQ_1^{rot}/Q_0 element both in broken H^1-norm for the velocity field and in L^2-norm for the pressure with moving grids and anisotropic meshes. Furthermore, NMFEMs also have been applied to other problems such as the Darcy-Stokes equations [31], the conduction-convection problem [32, 33], and the diffusion-convection-reaction equation [34].

Especially, [22] firstly presented a NMFEM with exactly divergence-free velocities for a incompressible MHD problem where the velocity and the magnetic fields were approximated by divergence-conforming elements and curl-conforming Nédélec elements, respectively, and derived nearly optimal error estimates. Motivated by the ideas of [22, 32, 34–36], in this paper, we are interested in discretizations for the MHD problem that are based on NMFEMs; a family of the low order elements will be adopted as approximation spaces for the velocity field, the piecewise constant element for the pressure, and the lowest order H^1-conforming element for the magnetic field on hexahedra or tetrahedra. The existence and uniqueness of the approximate solutions are shown, and the optimal error estimates for the corresponding unknown variables in L^2-norm are established, as well as those in a broken H^1-norm for the velocity, and the magnetic fields. Furthermore, a new approach is adopted to prove the discrete Poincaré-Friedrichs inequality, which is easier than that of the previous literature [37, 38].

The organization of this paper is as follows. In Section 2, we introduce the mixed variational formulation for the MHD problem. Section 3 will give the nonconforming mixed finite element schemes. In Section 4, we state some important lemmas and prove the existence and uniqueness of the approximate solutions. In Section 5, the optimal error estimates for the pressure, the velocity and the magnetic fields in L^2-norm are established, as well as ones in broken H^1-norm for the velocity and the magnetic fields.

Throughout the paper, C indicates a positive constant, possibly differs at different occurrences, which is independent of the mesh parameter h, but may depend on Ω and other parameters that appeared in this paper. Notations that are not especially explained are used with their usual meanings.

2. Equations and the Mixed Variational Formulation

In this section, we will consider a nonlinear, fully coupled stationary incompressible MHD problem in 3D as follows (see [9, 15]).

Problem (**I**). Find the velocity field $u = (u_1, u_2, u_3)$, the pressure p, the magnetic field $B = (B_1, B_2, B_3)$ such that

$$\begin{gathered}
-M^{-2}\Delta u + N^{-1}u\cdot\nabla u + \nabla p - R_m^{-1}(\nabla\times B)\times B = f \quad \text{in } \Omega,\\
R_m^{-1}\nabla\times(\nabla\times B) - \nabla\times(u\times B) = 0 \quad \text{in } \Omega,\\
\nabla\cdot u = 0 \quad \text{in } \Omega,\\
\nabla\cdot B = 0 \quad \text{in } \Omega,\\
u = 0 \quad \text{on } \partial\Omega,\\
B\cdot n = 0 \quad \text{on } \partial\Omega,\\
(\nabla\times B)\times n = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{2.1}$$

where Ω is a simply connected, bounded domain with unit outward normal $n = (n_1, n_2, n_3)$ on $\partial\Omega$. M, N, and R_m are the Hartman number, interaction parameter, and magnetic Reynolds number, respectively. The symbols Δ, ∇, and $\nabla\cdot$ denote the Laplace, gradient, and divergence operators, respectively. $\nabla\times(\nabla\times B) = \nabla(\nabla\cdot B) - \Delta B$. $f \in H^{-1}(\Omega)^3$ is the body force.

Set

$$\begin{gathered}
H_0^1(\Omega)^3 = \left\{v \in H^1(\Omega)^3; v|_{\partial\Omega} = 0\right\},\\
L_0^2(\Omega) = \left\{q \in L^2(\Omega); \int_\Omega q\,d\mathbf{x} = 0\right\},\\
H_n^1(\Omega)^3 = \left\{v \in H^1(\Omega)^3; (v\cdot n)|_{\partial\Omega} = 0\right\},
\end{gathered} \tag{2.2}$$

here and later, $\mathbf{x} = (x, y, z)$.

The mixed variational formulation for Problem (**I**) is written as follows.

Problem (**I**$_1$). Find $(u,B)\in W(\Omega)$, $p\in L_0^2(\Omega)$ such that

$$\begin{gathered} a((u,B),(u,B),(v,\Psi))+b\big((v,\Psi),p\big)=F((v,\Psi)),\quad \forall (v,\Psi)\in W(\Omega),\\ b\big((u,B),\chi\big)=0,\quad \forall \chi\in L_0^2(\Omega), \end{gathered} \tag{2.3}$$

where

$$\begin{gathered} W(\Omega)=H_0^1(\Omega)^3\times H_n^1(\Omega)^3,\\ a((u,B),(v,\Psi),(w,\Phi)):=a_0((v,\Psi),(w,\Phi))+a_1((u,B),(v,\Psi),(w,\Phi)),\\ a_0((v,\Psi),(w,\Phi)):=M^{-2}\int_\Omega \nabla v:\nabla w d\mathbf{x}+R_m^{-2}\int_\Omega[(\nabla\times\Psi)\cdot(\nabla\times\Phi)+(\nabla\cdot\Psi)(\nabla\cdot\Phi)]d\mathbf{x},\\ a_1((u,B),(v,\Psi),(w,\Phi)):=c_0(u;v,w)-c_1(B;w,\Psi)+c_2(B;v,\Phi),\\ c_0(u;v,w):=\int_\Omega (2N)^{-1}(u\cdot\nabla v\cdot w-u\cdot\nabla w\cdot v)d\mathbf{x},\\ c_1(B;w,\Psi):=\int_\Omega R_m^{-1}(\nabla\times\Psi)\times B\cdot w\,d\mathbf{x},\\ c_2(B;v,\Phi):=\int_\Omega R_m^{-1}(\nabla\times\Phi)\times B\cdot v\,d\mathbf{x},\\ b\big((v,\Psi),\chi\big):=-\int_\Omega \chi\nabla\cdot v\,d\mathbf{x},\qquad F((v,\Psi)):=\int_\Omega fv\,d\mathbf{x}. \end{gathered} \tag{2.4}$$

It has been shown in [9, 37, 38] that for $u,v,w\in H_0^1(\Omega)^3$, $B,\Psi,\Phi\in H_n^1(\Omega)^3$, there hold

$$\begin{gathered} c_0(u;v,w)=-c_0(u;w,v),\qquad c_0(w;v,v)=0,\\ a_1((u,B),(v,\Psi),(w,\Phi))=-a_1((u,B),(w,\Phi),(v,\Psi)),\\ a_1((u,B),(v,\Psi),(v,\Psi))=0. \end{gathered} \tag{2.5}$$

Let $Z(\Omega)=\{v\in H_0^1(\Omega)^3,\nabla\cdot v=0\}$. For $v\in H_0^1(\Omega)^3$ and $\Psi\in H_n^1(\Omega)^3$, we will equip $W(\Omega)$ with the norm

$$\begin{gathered} \|(v,\Psi)\|_W:=\Big(\|v\|_1^2+\|\Psi\|_1^2\Big)^{1/2},\\ \|f\|_{-1}:=\sup_{(0,0)\neq(v,\Psi)\in W(\Omega)}\frac{f((v,\Psi))}{\|(v,\Psi)\|_W}, \end{gathered} \tag{2.6}$$

respectively, where $\|\cdot\|_1$ is the H^1-norm.

The following result can be found in [9].

Theorem 2.1. *If $f \in H^{-1}(\Omega)^3$, then Problem $(\mathbf{I}_1)$ has at least a solution, in addition, that is unique provided that*

$$C_2\gamma_3(C_1\gamma_1)^{-2}\|f\|_{-1} < 1 \tag{2.7}$$

and satisfying the stability bound

$$\|(u,B)\|_W \le (C_1\gamma_1)^{-1}\|f\|_{-1}, \tag{2.8}$$

where $\gamma_1 = \min\{M^{-2}, R_m^{-2}\}$, $\gamma_2 = \max\{M^{-2}, R_m^{-2}\}$, $\gamma_3 = \max\{N^{-1}, R_m^{-1}\}$ and C_1, C_2 are positive constants only depending on the domain Ω.

3. Nonconforming Mixed Finite Element Schemes

Let $\Gamma^h = \{K\}$ be regular and quasi-uniform tetrahedra or hexahedra partition of Ω with mesh size h. We use the finite element spaces $X_{1h} \not\subset H_0^1(\Omega)^3$, $M_h \subset L_0^2(\Omega)$ and $X_{2h} \subset H_n^1(\Omega)^3$ to approximate the unknown variables u, p, and B. The following assumptions about the space pair (X_{1h}, M_h) are provided:

(A) for all $K \in \Gamma^h$, $P_1(K)^3 \subset X_{1h}$;

(B) $M_h = \{\chi^h \in L_0^2(\Omega); \chi^h|_K$ a constant, $\forall K \in \Gamma^h\}$;

(C) $\|\cdot\|_{1h} = (\sum_{K\in\Gamma^h} |\cdot|_{1,K}^2)^{1/2}$ is a norm X_{1h};

(D) for all $v^h \in X_{1h}$, $\int_F [v^h]ds = 0$, $F \subset \partial K$;

(E) for all $v \in H_0^1(\Omega)^3$, $q^h \in M_h$, $b_{1h}(v - \Pi^1 v, q^h) = 0$, $\|\Pi^1 v\|_{1h} \le C|v|_1$, where $[v^h]$ stands for the jump of v^h across the face F if F is an internal face, and it is equal to v^h itself if $F \subset \partial\Omega$, Π^1 is the interpolation operator associated with X_{1h} satisfying $\Pi_K = \Pi^1|_K$ for $K \in \Gamma^h$, and $P_1(K)$ is the polynomial space of degree less than or equal to one on K.

Introduce the finite element space

$$R_1(K) = \begin{cases} P_1(K) & \text{if } K \text{ is tetrahedra,} \\ Q_1(K) & \text{if } K \text{ is hexahedra.} \end{cases} \tag{3.1}$$

The finite element space X_{2h} is defined by

$$X_{2h} = \left\{\Psi^h \in H_n^1(\Omega)^3; q^h\Big|_K \in (R_1(K))^3, \left(\Psi^h \cdot n\right)\Big|_{\partial\Omega} = 0, \ \forall K \in \Gamma^h\right\}, \tag{3.2}$$

where $Q_1(K)$ is a space of polynomials whose degrees for x, y, z are equal to one. So these are the nonconforming mixed finite element schemes.

Remark 3.1. It can be checked that the nonconforming finite elements studied in [25–33, 39–45] satisfy the above assumptions (A)–(E).

4. The Existence and Uniqueness of the Approximate Solutions and Some Lemmas

In this section, we will prove some lemmas and the existence and uniqueness of the discrete solutions of nonconforming mixed finite element approximations for MHD equations.

Let $W_h = X_{1h} \times X_{2h}$ and the trilinear forms a_h, a_{1h}, c_{ih} $(i = 0, 1, 2)$ and the bilinear forms a_{0h} and b_h be defined as follows:

for $(u^h, B^h), (v^h, \Psi^h), (w^h, \Phi^h) \in W_h$ and $\chi^h \in M_h$,

$$\begin{aligned} &a_h\left(\left(u^h, B^h\right), \left(v^h, \Psi^h\right), \left(w^h, \Phi^h\right)\right) \\ &\quad := a_{0h}\left(\left(v^h, \Psi^h\right), \left(w^h, \Phi^h\right)\right) + a_{1h}\left(\left(u^h, B^h\right), \left(v^h, \Psi^h\right), \left(w^h, \Phi^h\right)\right), \end{aligned} \tag{4.1}$$

$$\begin{aligned} &a_{0h}\left(\left(v^h, \Psi^h\right), \left(w^h, \Phi^h\right)\right) \\ &\quad = \sum_{K \in \Gamma^h} \left\{ M^{-2} \int_K \nabla v^h \cdot \nabla w^h + R_m^{-2} \int_K \left[\left(\nabla \times \Psi^h\right) \cdot \left(\nabla \times \Phi^h\right) + \left(\nabla \cdot \Psi^h\right)\left(\nabla \cdot \Phi^h\right)\right] \right\} d\mathbf{x}, \end{aligned} \tag{4.2}$$

$$a_{1h}\left(\left(u^h, B^h\right), \left(v^h, \Psi^h\right), \left(w^h, \Phi^h\right)\right) := c_{0h}\left(u^h; v^h, w^h\right) - c_{1h}\left(B^h; w^h, \Psi^h\right) + c_{2h}\left(B^h; v^h, \Phi^h\right), \tag{4.3}$$

$$c_{0h}\left(u^h; v^h, w^h\right) := \sum_{K \in \Gamma^h} \int_K (2N)^{-1}\left(u^h \cdot \nabla v^h \cdot w^h - u^h \cdot \nabla w^h \cdot v^h\right) d\mathbf{x}, \tag{4.4}$$

$$c_{1h}\left(B^h; w^h, \Psi^h\right) =: \sum_{K \in \Gamma^h} \int_K R_m^{-1}\left(\nabla \times \Psi^h\right) \times B^h \cdot w^h \, d\mathbf{x}, \tag{4.5}$$

$$c_{2h}\left(B^h; v^h, \Phi^h\right) := \sum_{K \in \Gamma^h} \int_K R_m^{-1}\left(\nabla \times \Phi^h\right) \times B^h \cdot v^h \, d\mathbf{x} \tag{4.6}$$

$$b_h\left(\left(v^h, \Psi^h\right), \chi^h\right) := -\sum_{K \in \Gamma^h} \int_K \chi^h \nabla \cdot v^h \, d\mathbf{x}, \tag{4.7}$$

respectively.

Then the approximate formulation of Problem $(\mathbf{I}_1)$ reads as follows.

Problem $(\mathbf{I}_2)$. Find $(u^h, B^h) \in W_h$, $p^h \in M_h$ such that for all $(v^h, \Psi^h) \in W_h$, $\chi^h \in M_h$,

$$\begin{gathered} a_h\left(\left(u^h, B^h\right), \left(u^h, B^h\right), \left(v^h, \Psi^h\right)\right) + b_h\left(\left(v^h, \Psi^h\right), p^h\right) = F\left(\left(v^h, \Psi^h\right)\right), \\ b_h\left(\left(u^h, B^h\right), \chi^h\right) = 0. \end{gathered} \tag{4.8}$$

From the definition of (4.3), a_{1h} satisfies the following antisymmetric properties [9]:

$$\begin{gathered} a_{1h}\left(\left(u^h, B^h\right), \left(v^h, \Psi^h\right), \left(v^h, \Psi^h\right)\right) = 0, \\ a_{1h}\left(\left(u^h, B^h\right), \left(v^h, \Psi^h\right), \left(w^h, \Phi^h\right)\right) = -a_{1h}\left(\left(u^h, B^h\right), \left(w^h, \Phi^h\right), \left(v^h, \Psi^h\right)\right). \end{gathered} \tag{4.9}$$

Let $Z_h = \{v^h \in X_{1h},\ b((v^h, \Psi^h), \chi^h) = 0\}$. For all $v^h = (v_1^h, v_2^h, v_3^h) \in X_{1h}, \Psi^h = (\Psi_1^h, \Psi_2^h, \Psi_3^h) \in X_{2h}$, we define

$$\begin{gathered}\left\|v^h\right\|_{0h} = \left(\sum_{K\in\Gamma^h}\left\|v^h\right\|_{0,K}^2\right)^{1/2}, \qquad \left\|v^h\right\|_{1h} = \left(\sum_{K\in\Gamma^h}\left|v^h\right|_{1,K}^2\right)^{1/2}, \\ \left\|(v^h, \Psi^h)\right\|_h = \left(\sum_{K\in\Gamma^h}\left\|v^h\right\|_{1h}^2 + \left\|\Psi^h\right\|_1^2\right)^{1/2}, \\ \|f\|_h := \sup_{(0,0)\neq(v^h,\Psi^h)\in X_{1h}\times X_{2h}} \frac{F((v^h, \Psi^h))}{\left\|(v^h, \Psi^h)\right\|_h},\end{gathered} \tag{4.10}$$

respectively. Then it is easy to see that $\|\cdot\|_{0h}$ and $\|\cdot\|_{1h}$ are the norms over X_{1h} and $\|(\cdot,\cdot)\|_h$ is the norm over W_h.

Lemma 4.1. *The following discrete Poincaré-Friedrichs inequality holds:*

$$\left\|\Psi^h\right\|_0 \le C\left\|\nabla\times\Psi^h\right\|_0, \quad \forall\Psi^h \in X_{2h}. \tag{4.11}$$

Proof. We consider the following problem:

$$\begin{gathered}\nabla\times\left(\nabla\times\widetilde{B}\right) = f \quad \text{in } \Omega, \\ \nabla\cdot\widetilde{B} = 0 \quad \text{in } \Omega, \\ \widetilde{B}\cdot n = 0 \quad \text{on } \partial\Omega, \\ \left(\nabla\times\widetilde{B}\right)\times n = 0 \quad \text{on } \partial\Omega.\end{gathered} \tag{4.12}$$

Then by [3], the solution $\widetilde{B}$ of (4.12) satisfies

$$\left\|\nabla\times\widetilde{B}\right\|_0 \le C\|f\|_0. \tag{4.13}$$

On the one hand, by Green's formula and Hölder's inequality, we deduce that

$$\begin{aligned}\left|\int_\Omega f\Psi^h d\mathbf{x}\right| &= \left|\sum_{K\in\Gamma^h}\int_K\left(\nabla\times\widetilde{B}\right)\left(\nabla\times\Psi^h\right)d\mathbf{x}\right| \\ &\le \left\|\nabla\times\widetilde{B}\right\|_0\left\|\nabla\times\Psi^h\right\|_0.\end{aligned} \tag{4.14}$$

Using (4.13)-(4.14) and choosing $f = \Psi^h$, we may get the desired result. □

Remark 4.2. The method used in this lemma is different from and easier than that of [37, 38].

Lemma 4.3. *For (u^h, B^h), (v^h, Ψ^h), and $(w^h, \Phi^h) \in W_h$, we have*

(1) $|c_{0h}(u^h; v^h, w^h)| \le C\|u^h\|_{1h}\|v^h\|_{1h}\|w^h\|_{1h}$,

(2) $|c_{1h}(B^h; w^h, \Psi^h)| \le C\|\nabla \times \Psi^h\|_0\|B^h\|_1\|w^h\|_{1h}$,

(3) $|c_{2h}(B^h; v^h, \Phi^h)| \le C\|\nabla \times \Phi^h\|_0\|B^h\|_1\|v^h\|_{1h}$.

Proof. The first result is wellknown [30, 37, 38]. To prove the second result, we need the imbedding properties $H_n^1(\Omega)^3 \hookrightarrow H^1(\Omega)^3 \hookrightarrow L^4(\Omega)^3$ and the discrete imbedding inequality showed in [30, 32]:

$$\left\|v^h\right\|_{0,2k,\Omega} \le C\left\|v^h\right\|_{1h}, \quad \forall v^h \in X_{1h},\ k = 1, 2. \tag{4.15}$$

Thus,

$$\begin{aligned}\left|c_{1h}\left(B^h; w^h, \Psi^h\right)\right| &\le \sum_{K\in\Gamma^h}\int_K \frac{1}{R_m}\left|\left(\nabla \times \Psi^h\right) \times B^h \cdot w^h\right| d\mathbf{x} \\ &\le \frac{1}{R_m}\left\|(\nabla \times \Psi^h)\right\|_0\left\|B^h\right\|_{0,4}\left\|w^h\right\|_{0,4} \\ &\le C\left\|(\nabla \times \Psi^h)\right\|_0\left\|B^h\right\|_1\left\|w^h\right\|_{1h},\end{aligned} \tag{4.16}$$

the assertion for c_{1h} is proved. The proof for c_{2h} is analogous. □

Lemma 4.4. *Let (u^h, B^h), (v^h, Ψ^h), and $(w^h, \Phi^h) \in W_h$; then the following results hold:*

(1) $|a_{1h}((u^h, B^h), (v^h, \Psi^h), (w^h, \Phi^h))| \le C_c\gamma_3\|(u^h, B^h)\|_h\|(v^h, \Psi^h)\|_h\|(w^h, \Phi^h)\|_h$,

(2) $|a_{0h}((u^h, B^h), (u^h, B^h))| \ge C_a\gamma_1\|(u^h, B^h)\|_h^2$,

(3) $|a_{0h}((u^h, B^h), (v^h, \Psi^h))| \le C\gamma_2\|(u^h, B^h)\|_h\|(v^h, \Psi^h)\|_h$,

where C_c, C_a are positive constants, independent of h.

Proof. Firstly, using the triangle inequality and Lemma 4.3 yields

$$\begin{aligned}&\left|a_{1h}\left(\left(u^h, B^h\right), \left(v^h, \Psi^h\right), \left(w^h, \Phi^h\right)\right)\right| \\ &\quad\le \left|c_{0h}\left(u^h; v^h, w^h\right)\right| + \left|c_{1h}\left(B^h; w^h, \Psi^h\right)\right| + \left|c_{2h}\left(B^h; v^h, \Phi^h\right)\right| \\ &\quad\le C_c\gamma_3\left\|(u^h, B^h)\right\|_h\left\|(v^h, \Psi^h)\right\|_h\left\|(w^h, \Phi^h)\right\|_h.\end{aligned} \tag{4.17}$$

Applying $H_n^1(\Omega)^3 \hookrightarrow H^1(\Omega)^3$ and the following inequality [9, 37, 38]

$$\|v\|_0 \le C(\|\nabla \times v\|_0 + \|\nabla \cdot v\|_0), \quad \forall v \in H_n^1(\Omega)^3 \tag{4.18}$$

leads to

$$\begin{aligned}
&a_{0h}\left(\left(u^h, B^h\right), \left(u^h, B^h\right)\right) \\
&\quad = \sum_{K\in\Gamma^h}\left\{M^{-2}\int_K \nabla u^h\cdot\nabla u^h + R_m^{-2}\int_K\left[\left(\nabla\times B^h\right)\cdot\left(\nabla\times B^h\right) + \left(\nabla\cdot B^h\right)\left(\nabla\cdot B^h\right)\right]\right\}d\mathbf{x} \\
&\quad = M^{-2}\left\|\nabla u^h\right\|_{0h}^2 + R_m^{-2}\left(\left\|\nabla\times B^h\right\|_0^2 + \left\|\nabla\cdot B^h\right\|_0^2\right) \\
&\quad \geq C_a\min\left\{M^{-2}, R_m^{-2}\right\}\left(\left\|u^h\right\|_{1h}^2 + \left\|B^h\right\|_1^2\right) \\
&\quad = C_a\gamma_1\left\|(u^h, B^h)\right\|_h^2.
\end{aligned} \tag{4.19}$$

With the help of Hölder's inequality, we find

$$\begin{aligned}
&\left|a_{0h}\left(\left(u^h, B^h\right), \left(v^h, \Psi^h\right)\right)\right| \\
&\quad \leq \sum_{K\in\Gamma^h}\left\{M^{-2}\int_K\left|\nabla u^h\cdot\nabla v^h\right| + R_m^{-2}\int_K\left[\left|\left(\nabla\times B^h\right)\cdot\left(\nabla\times\Psi^h\right)\right| + \left|\left(\nabla\cdot B^h\right)\left(\nabla\cdot\Psi^h\right)\right|\right]\right\}d\mathbf{x} \\
&\quad \leq \left\{M^{-2}\left\|\nabla u^h\right\|_{0h}\left\|\nabla v^h\right\|_{0h} + CR_m^{-2}\left(\left\|\nabla\times B^h\right\|_0\left\|\nabla\times\Psi^h\right\|_0 + \left\|\nabla\cdot B^h\right\|_0\left\|\nabla\cdot\Psi^h\right\|_0\right)\right\} \\
&\quad \leq C\max\left\{M^{-2}, R_m^{-2}\right\}\left\|(u^h, B^h)\right\|_h\left\|(v^h, \Psi^h)\right\|_h \\
&\quad = C\gamma_2\left\|(u^h, B^h)\right\|_h\left\|(v^h, \Psi^h)\right\|_h.
\end{aligned} \tag{4.20}$$

The proof is completed. □

Lemma 4.5. *The spaces X_{1h} and M_h satisfy the discrete inf-sup condition [37, 38]; that is, there exists $\beta^\star > 0$ such that*

$$\inf_{\chi^h\in M_h}\sup_{(v^h,\Psi^h)\in W_h}\frac{b_h\left((v^h, \Psi^h), \chi^h\right)}{\left\|(v^h, \Psi^h)\right\|_h\left\|\chi^h\right\|_0} \geq \beta^\star. \tag{4.21}$$

Proof. On the one hand, by [37, 38], there exists a constant $\beta > 0$ such that

$$\inf_{\chi\in L_0^2(\Omega)}\sup_{(v,\Psi)\in W(\Omega)}\frac{b\left((v, \Psi), \chi\right)}{\left\|(v, \Psi)\right\|_W\left\|\chi\right\|_0} \geq \beta. \tag{4.22}$$

Therefore, by the assumption (E) and (4.22), we obtain

$$\begin{aligned}\sup_{(v^h,\Psi^h)\in W_h}\frac{b_h((v^h,\Psi^h),\chi^h)}{\|(v^h,\Psi^h)\|_h} &\geq \sup_{(v,\Psi^h)\in H_0^1(\Omega)^3\times X_{2h}}\frac{b_h((\Pi^1 v,\Psi^h),\chi^h)}{\|(\Pi^1 v,\Psi^h)\|_h}\\ &= \sup_{(v,\Psi^h)\in H_0^1(\Omega)^3\times X_{2h}}\frac{b_h((v,\Psi^h),\chi^h)}{\|(\Pi^1 v,\Psi^h)\|_h}\\ &\geq \frac{1}{C}\sup_{(v,\Psi^h)\in H_0^1(\Omega)^3\times X_{2h}}\frac{b((v,\Psi^h),\chi^h)}{\|(v,\Psi^h)\|_h}\\ &\geq \beta^*\left\|\chi^h\right\|_0,\end{aligned} \tag{4.23}$$

where $\beta^\star = \beta/C > 0$. The proof is completed. □

From Lemmas 4.4-4.5, we have the following.

Theorem 4.6. *For $f \in H^{-1}(\Omega)^3$, Problem $(\mathbf{I}_2)$ has at least one solution $((u^h,B^h),p^h) \in W_h \times M_h$ satisfying the stability bound $\|(u^h,B^h)\|_h \leq (C_a\gamma_1)^{-1}\|f\|_h$. Moreover, Problem $(\mathbf{I}_2)$ has a unique solution provided that $C_c\gamma_3(C_a\gamma_1)^{-2}\|f\|_h < 1$.*

5. The Convergence Analysis

In this section, we will state the main results of this paper, that is, the error estimates for the velocity and the magnetic fields in H^1-norm.

Theorem 5.1. *Assume that*

$$\frac{C_c\gamma_3\|f\|_{-1}}{C_aC_1\gamma_1^2} < \frac{1}{2}. \tag{5.1}$$

Let $((u,B),p) \in W(\Omega)\times L_0^2(\Omega)$ and $((u^h,B^h),p^h) \in W_h \times M_h$ be the solutions of Problems $(\mathbf{I}_1)$ and $(\mathbf{I}_2)$, respectively. Then there hold

(1)

$$\begin{aligned}&\left\|(u,B)-(u^h,B^h)\right\|_h\\ &\quad\leq C\left\{\inf_{(v^h,\Psi^h)\in W_h}\left\|(u,B)-\left(v^h,\Psi^h\right)\right\|_h+\inf_{s^h\in M_h}\left\|p-s^h\right\|_0+\sup_{(v^h,\Psi^h)\in Z_h\times X_{2h}}\frac{|E((v^h,\Psi^h))|}{\|(v^h,\Psi^h)\|_h}\right\},\end{aligned} \tag{5.2}$$

(2)

$$\begin{aligned}&\left\|p-p^h\right\|_0\\ &\quad\leq C\left\{\inf_{(v^h,\Psi^h)\in W_h}\left\|(u,B)-\left(v^h,\Psi^h\right)\right\|_h+\inf_{s^h\in M_h}\left\|p-s^h\right\|_0+\sup_{(v^h,\Psi^h)\in W_h}\frac{|E((v^h,\Psi^h))|}{\|(v^h,\Psi^h)\|_h}\right\},\end{aligned} \tag{5.3}$$

where

$$E\left(\left(v^h,\Psi^h\right)\right)=\sum_{K\in\Gamma^h}\int_{\partial K}\left[M^{-2}\frac{\partial u}{\partial n}v^h-pv^h\cdot n-(2N)^{-1}(u\cdot n)\left(u\cdot v^h\right)\right]ds. \tag{5.4}$$

Proof. We proceed in two steps.

Step 1. For $(v^h,\Psi^h)\in W_h$, by Green's formula, we have

$$\begin{aligned}
&a_{0h}\left((u,B),\left(v^h,\Psi^h\right)\right)+a_{1h}\left((u,B),(u,B),\left(v^h,\Psi^h\right)\right)+b_h\left(\left(v^h,\Psi^h\right),p\right)-F\left(\left(v^h,\Psi^h\right)\right)\\
&\quad=\sum_{K\in\Gamma^h}\left\{\int_K M^{-2}\nabla u\cdot\nabla v^h d\mathbf{x}+R_m^{-2}\int_K(\nabla\times B)\cdot\left(\nabla\times\Psi^h\right)d\mathbf{x}\right.\\
&\qquad+(2N)^{-1}\int_K\left(u\cdot\nabla u\cdot v^h-u\cdot\nabla v^h\cdot u\right)d\mathbf{x}\\
&\qquad-R_m^{-1}\int_K\left[(\nabla\times B)\times B\cdot v^h-\left(\nabla\times\Psi^h\right)\times B\cdot u\right]d\mathbf{x}\\
&\qquad\left.-\int_K p\cdot\nabla v^h\,d\mathbf{x}-\int_K fv^h\,d\mathbf{x}\right\}\\
&\quad=\sum_{K\in\Gamma^h}\left\{\int_K -M^{-2}\Delta u\cdot v^h\,d\mathbf{x}+\int_{\partial K}M^{-2}\frac{\partial u}{\partial n}v^h\,ds\right.\\
&\qquad+\int_K R_m^{-2}\nabla\times(\nabla\times B)\cdot\Psi^h\,d\mathbf{x}+\int_{\partial K}R_m^{-2}(\nabla\times B\times n)\cdot\Psi^h\,ds\\
&\qquad+\int_K N^{-1}u\cdot\nabla u\cdot v^h\,d\mathbf{x}-\int_{\partial K}(2N)^{-1}(u\cdot n)\left(u\cdot v^h\right)ds\\
&\qquad-R_m^{-1}\int_K(\nabla\times B)\times B\cdot v^h\,d\mathbf{x}-R_m^{-1}\int_K(\nabla\times u\times B)\cdot\Psi^h\,d\mathbf{x}\\
&\qquad\left.+\int_{\partial K}(u\times B\times n)\cdot\Psi^h\,ds+\int_K\nabla p\cdot v^h\,d\mathbf{x}-\int_{\partial K}pv^h\cdot n\,ds-\int_K fv^h\,d\mathbf{x}\right\}\\
&\quad=\sum_{K\in\Gamma^h}\left\{\int_K\left(-M^{-2}\Delta u+N^{-1}u\cdot\nabla u+\nabla p-R_m^{-1}(\nabla\times B)\times B-f\right)\cdot v^h\right.\\
&\qquad+R_m^{-1}\int_K\left[R_m^{-1}\nabla\times(\nabla\times B)-\nabla\times(u\times B)\right]\cdot\Psi^h\,d\mathbf{x}\\
&\qquad\left.+\int_{\partial K}\left[M^{-2}\frac{\partial u}{\partial n}v^h-(2N)^{-1}(u\cdot n)\left(u\cdot v^h\right)-pv^h\cdot n\right]ds\right\}\\
&\quad=E\left(\left(v^h,\Psi^h\right)\right).
\end{aligned}\tag{5.5}$$

Thus,

$$\begin{aligned}
&a_{0h}\left((u,B),\left(v^h,\Psi^h\right)\right)+a_{1h}\left((u,B),(u,B),\left(v^h,\Psi^h\right)\right)+b_h\left(\left(v^h,\Psi^h\right),p\right)\\
&\quad=F\left(\left(v^h,\Psi^h\right)\right)+E\left(\left(v^h,\Psi^h\right)\right).
\end{aligned}\tag{5.6}$$

Here, we have used the following equality:

$$\int_{\Omega} (\nabla \times \Phi) \cdot \Psi \, d\mathbf{x} = -\int_{\partial\Omega} (\Phi \times n) \cdot \Psi \, ds + \int_{\Omega} \Phi \cdot (\nabla \times \Psi) d\mathbf{x}. \tag{5.7}$$

On the other hand, we have from (4.8)

$$\begin{aligned} &a_{0h}\left(\left(u^h, B^h\right), \left(v^h, \Psi^h\right)\right) + a_{1h}\left(\left(u^h, B^h\right), \left(u^h, B^h\right), \left(v^h, \Psi^h\right)\right) + b_h\left(\left(v^h, \Psi^h\right), p^h\right) \\ &\quad = F\left(\left(v^h, \Psi^h\right)\right). \end{aligned} \tag{5.8}$$

Subtraction of (4.8) from (5.6) yields

$$\begin{aligned} &a_{0h}\left((u, B) - \left(u^h, B^h\right), \left(v^h, \Psi^h\right)\right) + a_{1h}\left((u, B) - \left(u^h, B^h\right), (u, B), \left(v^h, \Psi^h\right)\right) \\ &\quad + a_{1h}\left(\left(u^h, B^h\right), (u, B) - \left(u^h, B^h\right), \left(v^h, \Psi^h\right)\right) \\ &\quad + b_h\left(\left(v^h, \Psi^h\right), p - p^h\right) = E\left(\left(v^h, \Psi^h\right)\right). \end{aligned} \tag{5.9}$$

Let (w^h, Φ^h) be an arbitrary element of $Z_h \times X_{2h}$, that is:

$$b_h\left(\left(w^h, \Phi^h\right), \chi^h\right) = 0, \quad \forall \chi^h \in M_h. \tag{5.10}$$

Then,

$$b_h\left(\left(u^h - w^h, B^h - \Phi^h\right), \chi^h\right) = b_h\left(\left(u^h, B^h\right), \chi^h\right) - b_h\left(\left(w^h, \Phi^h\right), \chi^h\right) = 0. \tag{5.11}$$

For all $(v^h, \Psi^h) \in W_h, s^h \in M_h$, by virtue of $(u^h - w^h, B^h - \Phi^h) \in Z_h \times X_{2h}$ and (5.9), we get

$$\begin{aligned} &a_{0h}\left(\left(w^h, \Phi^h\right) - \left(u^h, B^h\right), \left(v^h, \Psi^h\right)\right) + a_{1h}\left(\left(w^h, \Phi^h\right) - \left(u^h, B^h\right), (u, B), \left(v^h, \Psi^h\right)\right) \\ &\quad + a_{1h}\left(\left(u^h, B^h\right), \left(w^h, \Phi^h\right) - \left(u^h, B^h\right), \left(v^h, \Psi^h\right)\right) + b_h\left(\left(v^h, \Psi^h\right), s^h - p^h\right) \\ &\quad = a_{0h}\left(\left(w^h, \Phi^h\right) - (u, B), \left(v^h, \Psi^h\right)\right) + a_{1h}\left(\left(w^h, \Phi^h\right) - (u, B), (u, B), \left(v^h, \Psi^h\right)\right) \\ &\quad + a_{1h}\left(\left(u^h, B^h\right), \left(w^h, \Phi^h\right) - (u, B), \left(v^h, \Psi^h\right)\right) + b_h\left(\left(v^h, \Psi^h\right), s^h - p\right) \\ &\quad + E\left(\left(v^h, \Psi^h\right)\right). \end{aligned} \tag{5.12}$$

Notice that

$$\begin{gathered} a_{1h}\left(\left(u^h, B^h\right), \left(w^h, \Phi^h\right) - \left(u^h, B^h\right), \left(w^h, \Phi^h\right) - \left(u^h, B^h\right)\right) = 0, \\ b_h\left(\left(u^h - w^h, B^h - \Phi^h\right), s^h - p^h\right) = 0. \end{gathered} \tag{5.13}$$

Let $(v^h, \Psi^h) = (w^h, \Phi^h) - (u^h, B^h)$, and by (5.12), we obtain

$$\begin{aligned}
&a_{0h}\left(\left(w^h,\Phi^h\right)-\left(u^h,B^h\right),\left(w^h,\Phi^h\right)-\left(u^h,B^h\right)\right)\\
&\quad+a_{1h}\left(\left(w^h,\Phi^h\right)-\left(u^h,B^h\right),(u,B),\left(w^h,\Phi^h\right)-\left(u^h,B^h\right)\right)\\
&\quad=a_{0h}\left(\left(w^h,\Phi^h\right)-(u,B),\left(w^h,\Phi^h\right)-\left(u^h,B^h\right)\right)\\
&\qquad+a_{1h}\left(\left(w^h,\Phi^h\right)-(u,B),(u,B),\left(w^h,\Phi^h\right)-\left(u^h,B^h\right)\right)\\
&\qquad+a_{1h}\left(\left(u^h,B^h\right),\left(w^h,\Phi^h\right)-(u,B),\left(w^h,\Phi^h\right)-\left(u^h,B^h\right)\right)\\
&\qquad+b_h\left(\left(w^h,\Phi^h\right)-\left(u^h,B^h\right),s^h-p\right)+E\left(\left(w^h,\Phi^h\right)-\left(u^h,B^h\right)\right).
\end{aligned} \tag{5.14}$$

Using the continuity properties of a_{0h}, a_{1h} and the stability bounds for $\|(u,B)\|_W$ and $\|(u^h,B^h)\|_h$ in Theorems 2.1 and 4.6, respectively, the right-hand side of (5.14) can be bounded by

$$\begin{aligned}
\text{r.h.s.} &\le \left\|(w^h,\Phi^h)-(u^h,B^h)\right\|_h\\
&\quad\times\left[C\gamma_2\left\|(w^h,\Phi^h)-(u,B)\right\|_h+C_c\left\|(w^h,\Phi^h)-(u,B)\right\|_h\|(u,B)\|_W\right.\\
&\qquad\left.+C_c\left\|(w^h,\Phi^h)-(u,B)\right\|_h\left\|(u^h,B^h)\right\|_h+C\left\|s^h-p\right\|_0+\frac{E\left((w^h,\Phi^h)-(u^h,B^h)\right)}{\left\|(w^h,\Phi^h)-(u^h,B^h)\right\|_h}\right]\\
&\le C\left\|(w^h,\Phi^h)-(u^h,B^h)\right\|_h\left[\left\|(w^h,\Phi^h)-(u,B)\right\|_h+\left\|s^h-p\right\|_0+\frac{E\left((w^h,\Phi^h)-(u^h,B^h)\right)}{\left\|(w^h,\Phi^h)-(u^h,B^h)\right\|_h}\right].
\end{aligned} \tag{5.15}$$

Next, the coercivity property of the form a_{0h}, continuity of a_{1h} in Lemma 4.4, stability bound for $\|(u,B)\|_W$ in Theorem 2.1, and the assumption $C_c\gamma_3\|f\|_{-1}/C_aC_1\gamma_1^2 < 1/2$ allow us to bound the left-hand side of (5.14) as

$$\begin{aligned}
\text{l.h.s.} &\ge C_a\gamma_1\left\|(w^h,\Phi^h)-(u^h,B^h)\right\|_h^2-C_c\gamma_3\left\|(w^h,\Phi^h)-(u^h,B^h)\right\|_h^2\|(u,B)\|_W\\
&\ge\frac{1}{2}C_a\gamma_1\left\|(w^h,\Phi^h)-(u^h,B^h)\right\|_h^2.
\end{aligned} \tag{5.16}$$

Combining these bounds, we have

$$\left\|(w^h,\Phi^h)-(u^h,B^h)\right\|_h\le C\left[\left\|\left(w^h,\Phi^h\right)-(u,B)\right\|_h+\left\|s^h-p\right\|_0+\frac{E\left((w^h,\Phi^h)-(u^h,B^h)\right)}{\left\|(w^h,\Phi^h)-(u^h,B^h)\right\|_h}\right]. \tag{5.17}$$

Then, applying the triangle inequality, we get

$$\left\|(u,B)-(u^h,B^h)\right\|_h \le C\left[\left\|\left(w^h,\Phi^h\right)-(u,B)\right\|_h+\left\|s^h-p\right\|_0+\frac{E\left(\left(w^h,\Phi^h\right)-\left(u^h,B^h\right)\right)}{\left\|\left(w^h,\Phi^h\right)-\left(u^h,B^h\right)\right\|_h}\right]. \tag{5.18}$$

Now, for $(w^h,\Phi^h)\in Z_h\times X_{2h}$, $s^h\in M_h$, taking the infimum of (5.18) yields

$$\begin{aligned}&\left\|(u,B)-(u^h,B^h)\right\|_h\\&\quad\le C\left[\inf_{\left(w^h,\Phi^h\right)\in Z_h\times X_{2h}}\left\|\left(w^h,\Phi^h\right)-(u,B)\right\|_h+\inf_{s^h\in M_h}\left\|s^h-p\right\|_0+\sup_{\left(v^h,\Psi^h\right)\in Z_h\times X_{2h}}\frac{E\left(\left(v^h,\Psi^h\right)\right)}{\left\|\left(v^h,\Psi^h\right)\right\|_h}\right].\end{aligned} \tag{5.19}$$

With the argument as [37], we know that

$$\inf_{\left(w^h,\Phi^h\right)\in Z_h\times X_{2h}}\left\|\left(w^h,\Phi^h\right)-(u,B)\right\|_h\le C\inf_{\left(v^h,\Psi^h\right)\in W_h}\left\|\left(w^h,\Phi^h\right)-(u,B)\right\|_h. \tag{5.20}$$

Substituting (5.20) into (5.19) implies (5.2).

Step 2. For $(v^h,\Psi^h)\in W_h$, $s^h\in M_h$, we have from (5.9) that

$$\begin{aligned}b_h\left(\left(v^h,\Psi^h\right),s^h-p^h\right)&=b_h\left(\left(v^h,\Psi^h\right),s^h-p\right)+b_h\left(\left(v^h,\Psi^h\right),p-p^h\right)\\&=b_h\left(\left(v^h,\Psi^h\right),s^h-p\right)-a_{0h}\left((u,B)-\left(u^h,B^h\right),\left(v^h,\Psi^h\right)\right)\\&\quad-a_{1h}\left((u,B)-\left(u^h,B^h\right),(u,B),\left(v^h,\Psi^h\right)\right)\\&\quad-a_{1h}\left(\left(u^h,B^h\right),(u,B)-\left(u^h,B^h\right),\left(v^h,\Psi^h\right)\right)\\&\quad+E\left(\left(v^h,\Psi^h\right)\right).\end{aligned} \tag{5.21}$$

Using the continuity properties of a_{0h} and a_{1h} and the discrete inf-sup condition (4.21) of Lemma 4.5, it follows that

$$\begin{aligned}&\left\|s^h-p^h\right\|_0\\&\le\frac{1}{\beta^\star}\left\{C\left\|s^h-p\right\|_0+\left[C\gamma_2+C_c\left(\|(u,B)\|_W+\left\|\left(u^h,B^h\right)\right\|_h\right)\right]\left\|(u,B)-\left(u^h,B^h\right)\right\|_h+\frac{E\left(\left(v^h,\Psi^h\right)\right)}{\left\|\left(v^h,\Psi^h\right)\right\|_h}\right\}.\end{aligned} \tag{5.22}$$

Then, with the help of the triangle inequality and (5.2), we complete the proof. □

Theorem 5.2. *Let $u \in (H_0^1(\Omega)^3 \cap H^2(\Omega)^3)$, $B \in (H^2(\Omega)^3 \cap H_n^1(\Omega)^3)$, $p \in (L_0^2(\Omega) \cap H^1(\Omega))$, and $((u^h, B^h), p^h) \in W_h \times M_h$ be the solutions of Problems $(\mathbf{I}_1)$ and $(\mathbf{I}_2)$, respectively. Then there holds*

$$\left\|(u,B)-(u^h,B^h)\right\|_h + \left\|p-p^h\right\|_0 \le Ch\left(|u|_2 + \|B\|_2 + \|p\|_1\right). \tag{5.23}$$

Proof. On the one hand, the interpolation theory gives

$$\begin{gathered}\inf_{v^h\in X_{1h}}\left\|u-v^h\right\|_{1h}^2 \le \left\|u-\Pi^1 u\right\|_{1h}^2 \le Ch^2|u|_2^2,\\ \inf_{\Psi^h\in X_{2h}}\left\|B-\Psi^h\right\|_1^2 \le Ch^2\|B\|_2^2.\end{gathered} \tag{5.24}$$

Therefore, by (5.24), we obtain

$$\inf_{(v^h,\Psi^h)\in W_h}\left\|(u,B)-(v^h,\Psi^h)\right\|_h \le Ch(|u|_2 + \|B\|_2). \tag{5.25}$$

At the same time, for $p \in L_0^2(\Omega)$, we define the interpolation $R_0^h p \in M_h$ on each element K as

$$\int_K \left(p - R_0^h p\right) d\mathbf{x} = 0. \tag{5.26}$$

Then there holds

$$\inf_{s^h\in M_h}\left\|p-s^h\right\|_0 \le \left\|p-R_0^h p\right\|_0 \le Ch\|p\|_1. \tag{5.27}$$

On the other hand, by the similar techniques to [25–27, 29, 30, 32], we have

$$\left|E\left(\left(v^h,\Psi^h\right)\right)\right| \le Ch\left(|u|_2 + \|p\|_1\right)\left\|(v^h,\Psi^h)\right\|_h. \tag{5.28}$$

Substituting (5.24)–(5.28) into (5.2) and (5.3) yields the desired result.

Next, we will establish the error estimates in L^2-norm for the velocity and the magnetic fields by use of the duality argument introduced in [46].

We consider the following dual problem. Find (w, Φ) and s such that.

$$\begin{aligned}
&-M^{-2}\Delta w + N^{-1}[w\cdot\nabla u - u\cdot\nabla w] + \nabla s + R_m^{-1}(\nabla\times\Phi)\times B = u - u^h, \quad \text{in } \Omega,\\
&R_m^{-2}[\nabla\times(\nabla\times\Phi) - \nabla(\nabla\cdot\Phi)] + R_m^{-1}[(\nabla\times B)\times w - (\nabla\times\Phi)\times u - \nabla\times(B\times w)]\\
&\qquad = B - B^h, \quad \text{in } \Omega,\\
&\nabla\cdot w = 0, \quad \text{in } \Omega,\\
&w = 0, \quad \text{on } \partial\Omega,\\
&B\cdot n = 0, \qquad R_m^{-1}(\nabla\times\Phi)\times n + w\times B\times n = 0, \quad \text{on } \partial\Omega.
\end{aligned} \tag{5.29}$$

□

The variational formulation of (5.29) is written as follows.

Problem ($\mathbf{I}_3$). Find $(w,\Phi)\in W(\Omega)$ and $s\in L_0^2(\Omega)$ such that for all $(v,\Psi)\in W(\Omega),\psi\in L_0^2(\Omega)$

$$\begin{aligned}
&a_0((v,\Psi),(w,\Phi))+a_1((u,B),(v,\Psi),(w,\Phi))+a_1((v,\Psi),(u,B),(w,\Phi))+b((v,\Psi),s)\\
&\quad=\left((u,B)-\left(u^h,B^h\right),(v,\Psi)\right),\\
&b\big((w,\Phi),\psi\big)=0.
\end{aligned}\tag{5.30}$$

Under the same hypotheses as Theorem 2.1, we may easily know that Problem ($\mathbf{I}_3$) has a unique solution $((w,\Phi),s)\in W(\Omega)\times L_0^2(\Omega)$.

We require that (5.29) be H^2-regular, that is:

$$\|(w,\Phi)\|_2+\|s\|_1\le C\left\|(u,B)-(u^h,B^h)\right\|_0.\tag{5.31}$$

Let $((w^h,\Phi^h),s^h)\in W_h\times M_h$ satisfy

$$\left\|(w,\Phi)-(w^h,\Phi^h)\right\|_h+\left\|s-s^h\right\|_0\le Ch(|w|_2+\|\Phi\|_2+\|s\|_1).\tag{5.32}$$

Theorem 5.3. *Under the hypothesis of Theorem 5.2, let* $((w,\Phi),s)$ *be the solution of Problem* ($\mathbf{I}_3$), *and assume that* (5.31) *holds. Then we have*

$$\left\|(u,B)-(u^h,B^h)\right\|_0\le Ch^2\big(|u|_2+\|B\|_2+\|p\|_1\big).\tag{5.33}$$

Proof. By (5.31) and (5.32), we deduce that

$$\left\|(w,\Phi)-\left(w^h,\Phi^h\right)\right\|_h+\left\|s-s^h\right\|_0\le Ch\left\|(u,B)-(u^h,B^h)\right\|_0.\tag{5.34}$$

Multiplying $(u-u^h)$ and $(B-B^h)$ both sides of the first and the second equation of (5.29), respectively, and integrating by parts on each element, we see that

$$\begin{aligned}
\left\|(u,B)-(u^h,B^h)\right\|_0^2&=a_{0h}\left((u,B)-\left(u^h,B^h\right),(w,\Phi)\right)+a_{1h}\left((u,B),(u,B)-\left(u^h,B^h\right),(w,\Phi)\right)\\
&\quad+a_{1h}\left((u,B)-\left(u^h,B^h\right),(u,B),(w,\Phi)\right)+b_h\left((u,B)-\left(\mathrm{u}^h,B^h\right),s\right)\\
&\quad-(2N)^{-1}\sum_{K\in\Gamma^h}\int_K\operatorname{div}\left(u-u^h\right)(u\cdot w)d\mathbf{x}+F\left(\left(u-u^h\right)\right),
\end{aligned}\tag{5.35}$$

where

$$F\left(\left(u-u^h\right)\right) = -M^{-2}\sum_{K\in\Gamma^h}\int_{\partial K}\frac{\partial w}{\partial n}\left(u-u^h\right)ds - (2N)^{-1}\sum_{K\in\Gamma^h}\int_{\partial K}(u\cdot n)\left(w\cdot\left(u-u^h\right)\right)ds$$
$$+ (2N)^{-1}\sum_{K\in\Gamma^h}\int_{\partial K}\left(\left(u-u^h\right)\cdot n\right)(u\cdot w)ds + \sum_{K\in\Gamma^h}\int_{\partial K}s\left(u-u^h\right)\cdot n\,ds. \tag{5.36}$$

Subtraction of (5.9) yields

$$\begin{aligned} &a_{0h}\left((u,B)-\left(u^h,B^h\right),\left(v^h,\Psi^h\right)\right) + a_{1h}\left((u,B)-\left(u^h,B^h\right),(u,B),\left(v^h,\Psi^h\right)\right)\\ &\quad + a_{1h}\left(\left(u^h,B^h\right),(u,B)-\left(u^h,B^h\right),\left(v^h,\Psi^h\right)\right) + b_h\left(\left(v^h,\Psi^h\right),p-p^h\right)\\ &\quad = E\left(\left(v^h,\Psi^h\right)\right). \end{aligned} \tag{5.37}$$

Note that

$$b_h\left((u,B)-\left(u^h,B^h\right),\phi^h\right) = 0, \quad \forall\phi^h\in M_h. \tag{5.38}$$

Now, setting $\psi = p - p^h$ in Problem $(\mathbf{I}_3)$, we have

$$b\left((w,\Phi),p-p^h\right) = 0. \tag{5.39}$$

From (5.35)–(5.39), we get

$$\begin{aligned} &\left\|(u,B)-\left(u^h,B^h\right)\right\|_0^2\\ &\quad = a_{0h}\left((u,B)-\left(u^h,B^h\right),(w,\Phi)-\left(v^h,\Psi^h\right)\right) + b_h\left((u,B)-\left(u^h,B^h\right),s-\phi^h\right)\\ &\qquad + b_h\left((w,\Phi)-\left(v^h,\Psi^h\right),p-p^h\right) + A_1 + A_2 + A_3, \end{aligned} \tag{5.40}$$

where

$$\begin{aligned} A_1 &= a_{1h}\left(\left(u^h,B^h\right),(u,B)-\left(u^h,\mathrm{B}^h\right),(w,\Phi)-\left(v^h,\Psi^h\right)\right)\\ &\quad + a_{1h}\left((u,B)-\left(u^h,B^h\right),(u,B),(w,\Phi)-\left(v^h,\Psi^h\right)\right)\\ &\quad + a_{1h}\left((u,B)-\left(u^h,B^h\right),(u,B)-\left(u^h,B^h\right),(w,\Phi)\right),\\ A_2 &= F\left(\left(u-u^h\right)\right) + E\left(\left(v^h,\Psi^h\right)\right),\\ A_3 &= -\frac{1}{2N}\sum_{K\in\Gamma^h}\int_K \operatorname{div}\left(u-u^h\right)(u\cdot w)d\mathbf{x} \end{aligned} \tag{5.41}$$

By (2.8), Lemma 4.4, and Theorem 4.6, we find

$$A_1 \le C\left(\left\|(u,B)-\left(u^h,B^h\right)\right\|_h\left\|(w,\Phi)-\left(v^h,\Psi^h\right)\right\|_h+\left\|(u,B)-\left(u^h,B^h\right)\right\|_h^2\|(w,\Phi)\|_W\right). \tag{5.42}$$

From [45], we know

$$F\left(\left(u-u^h\right)\right) \le Ch(|w|_2+\|s\|_1)\left\|u-u^h\right\|_{1h}. \tag{5.43}$$

By virtue of $u,w\in H^2(\Omega)^3 \hookrightarrow C^0(\Omega)^2$, we obtain

$$\begin{aligned} &E\left(\left(v^h,\Psi^h\right)\right) \\ &\quad= \sum_{K\in\Gamma^h}\int_{\partial K}\left[M^{-2}\frac{\partial u}{\partial n}\left(v^h-w\right)-p\left(v^h-w\right)\cdot n-(2N)^{-1}(u\cdot n)\left(u\cdot\left(v^h-w\right)\right)\right]ds \\ &\quad\le Ch\left(|u|_2+\|p\|_1\right)\left\|v^h-w\right\|_{1h}. \end{aligned} \tag{5.44}$$

Let a_K be a constant such that

$$\|u\cdot w-a_K\|_{0,K} \le Ch\|u\cdot w\|_{1,K} \le Ch\|u\|_{1,K}\|w\|_{2,K}. \tag{5.45}$$

Since div $u=0, b_h((u^h,B^h),q)=0, \forall q\in M_h$ and (5.45), we obtain

$$\begin{aligned} |A_3| &= \left|-\frac{1}{2N}\sum_{K\in\Gamma^h}\int_K \operatorname{div}\left(u-u^h\right)(u\cdot w-a_K)d\mathbf{x}\right| \\ &\le Ch\left\|u-u^h\right\|_{1h}\|u\|_1\|w\|_2. \end{aligned} \tag{5.46}$$

Thus, by (5.31) and the approximation theory, there hold

$$\begin{gathered} \inf_{(v^h,\Psi^h)\in W_h}\left\|(w,\Phi)-(v^h,\Psi^h)\right\|_h \le Ch\|(w,\Phi)\|_2 \le Ch\left\|(u,B)-(u^h,B^h)\right\|_0, \\ \inf_{v^h\in X_{2h}}\left\|w-v^h\right\|_{1h} \le Ch\|w\|_2 \le Ch\left\|(u,B)-(u^h,B^h)\right\|_0, \\ \inf_{\phi^h\in M_h}\left\|s-\phi^h\right\|_0 \le Ch\|s\|_1 \le Ch\left\|(u,B)-(u^h,B^h)\right\|_0, \\ \|(w,\Phi)\|_W \le \|(w,\Phi)\|_2 \le C\left\|(u,B)-(u^h,B^h)\right\|_0, \\ \left|b_h\left((u,B)-\left(u^h,B^h\right),s-\phi^h\right)\right| \le C\left\|(u,B)-\left(u^h,B^h\right)\right\|_h\left\|s-\phi^h\right\|_0, \\ \left|b_h\left((w,\Phi)-\left(v^h,\Psi^h\right),p-p^h\right)\right| \le C\left\|(u,B)-\left(u^h,B^h\right)\right\|_h\left\|s-\phi^h\right\|_0. \end{gathered} \tag{5.47}$$

Combining these inequalities and using Lemma 4.4 and the results from (5.39) to (5.46) yields the desired result. □

Remark 5.4. The results obtained in this paper are also valid to the MHD equations with the following boundary conditions $u = 0$, $n \times B = 0$, $(\nabla \times B) \cdot n = 0$ on $\partial\Omega$ when $u \in H_0^1(\Omega)^3$, $B \in H = \{B \in H^1(\Omega)^3; (B \times n)|_{\partial\Omega} = 0\}$.

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