Research Article
Some Fixed Point Results in *GP*-Metric Spaces

Hassen Aydi,[1] Erdal Karapınar,[2] and Peyman Salimi[3]

[1] *Institut Supérieur d'Informatique et des Technologies de Communication de Hammam Sousse,*
Université de Sousse, Route GP1-4011, Hammam Sousse, Tunisia
[2] *Department of Mathematics, Atilim University, Incek, 06836 Ankara, Turkey*
[3] *Department of Mathematics, Sahand University of Technology, Tabriz, Iran*

Correspondence should be addressed to Erdal Karapınar, erdalkarapinar@yahoo.com

Following a recent paper of Zand and Nezhad (2011), we establish some fixed point results in *GP*-metric spaces. The presented theorems generalize and improve several existing results in the literature. Also, some examples are presented.

1. Introduction

Partial metric space is a generalized metric space introduced by Matthews [1] in which each object does not necessarily have to have a zero distance from itself. A motivation is to introduce this space to give a modified version of the Banach contraction principle [2]. Subsequently, several authors studied the problem of existence and uniqueness of a fixed point for mappings satisfying different contractive conditions, see [3–23].

On the other hand, in 2006 Mustafa and Sims [24] introduced a new notion of generalized metric spaces called *G*-metric spaces. Based on the notion of a *G*-metric space, many fixed point results for different contractive conditions have been presented, for more details see [25–42].

Recently, based on the two above notions, Zand and Nezhad [43] introduced a new generalized metric space as both a generalization of a partial metric space and a *G*-metric space. It is given as follows.

Definition 1.1 (see [43]). Let X be a nonempty set. A function $G_p : X \times X \times X \to [0, +\infty)$ is called a *GP*-metric if the following conditions are satisfied:

(GP1) $x = y = z$ if $G_p(x, y, z) = G_p(z, z, z) = G_p(y, y, y) = G_p(x, x, x)$;

(GP2) $0 \leq G_p(x,x,x) \leq G_p(x,x,y) \leq G_p(x,y,z)$ for all $x,y,z \in X$;

(GP3) $G_p(x,y,z) = G_p(x,z,y) = G_p(y,z,x) = \cdots$, symmetry in all three variables;

(GP4) $G_p(x,y,z) \leq G_p(x,a,a) + G_p(a,y,z) - G_p(a,a,a)$ for any $x,y,z,a \in X$.

Then the pair (X,G) is called a *GP*-metric space.

Example 1.2 (see [43]). Let $X = [0,\infty)$ and define $G_p(x,y,z) = \max\{x,y,z\}$, for all $x,y,z \in X$. Then (X,G_p) is a *GP*-metric space.

Proposition 1.3 (see [43]). *Let (X,G_p) be a GP-metric space, then for any x,y,z and $a \in X$ it follows that*

(i) $G_p(x,y,z) \leq G_p(x,x,y) + G_p(x,x,z) - G_p(x,x,x)$;

(ii) $G_p(x,y,y) \leq 2G_p(x,x,y) - G_p(x,x,x)$;

(iii) $G_p(x,y,z) \leq G_p(x,a,a) + G_p(y,a,a) + G_p(z,a,a) - 2G_p(a,a,a)$;

(iv) $G_p(x,y,z) \leq G_p(x,a,z) + G_p(a,y,z) - G_p(a,a,a)$.

Proposition 1.4 (see [43]). *Every GP-metric space (X,G_p) defines a metric space (X,D_{G_p}), where*

$$D_{G_p}(x,y) = G_p(x,y,y) + G_p(y,x,x) - G_p(x,x,x) - G_p(y,y,y) \quad \text{for all } x,y \in X. \tag{1.1}$$

Definition 1.5 (see [43]). Let (X,G_p) be a *GP*-metric space and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ or $x_n \to x$ if

$$\lim_{m,m \to \infty} G_p(x,x_m,x_n) = G_p(x,x,x). \tag{1.2}$$

Proposition 1.6 (see [43]). *Let (X,G_p) be a GP-metric space. Then, for any sequence $\{x_n\}$ in X, and a point $x \in X$ the following are equivalent:*

(A) $\{x_n\}$ *is GP-convergent to x;*

(B) $G_p(x_n,x_n,x) \to G_p(x,x,x)$ *as $n \to \infty$;*

(C) $G_p(x_n,x,x) \to G_p(x,x,x)$ *as $n \to \infty$.*

Definition 1.7 (see [43]). Let (X,G_p) be a *GP*-metric space.

(S1) A sequence $\{x_n\}$ is called a *GP*-Cauchy if and only if $\lim_{m,n \to \infty} G_p(x_n,x_m,x_m)$ exists (and is finite).

(S2) A *GP*-partial metric space (X,G_p) is said to be *GP*-complete if and only if every *GP*-Cauchy sequence in X is *GP*-convergent to $x \in X$ such that $G_p(x,x,x) = \lim_{m,n \to \infty} G_p(x_n,x_m,x_m)$.

Now, we introduce the following.

Definition 1.8. Let (X,G_p) be a *GP*-metric space.

(M1) A sequence $\{x_n\}$ is called a 0-*GP*-Cauchy if and only if $\lim_{m,n\to\infty}G_p(x_n, x_m, x_m) = 0$;

(M2) A *GP*-metric space (X, G_p) is said to be 0-*GP*-complete if and only if every 0-*GP*-Cauchy sequence is GP-convergent to a point $x \in X$ such that $G_p(x, x, x) = 0$.

Example 1.9. Let $X = [0, +\infty)$ and define $G_p(x, y, z) = \max\{x, y, z\}$, for all $x, y, z \in X$. Then (X, G_p) is a *GP*-complete *GP*-metric space. Moreover, if $X = \mathbb{Q} \cap [0, +\infty)$ (where $\mathbb{Q}$ denotes the set of rational numbers), then (X, G_p) is a 0-*GP*-complete *GP*-metric space.

Lemma 1.10. *Let (X, G_p) be a GP-metric space. Then*

(A) *if $G_p(x, y, z) = 0$, then $x = y = z$;*

(B) *if $x \neq y$, then $G_p(x, y, y) > 0$.*

Proof. By (GP2) we have

$$G_p(x, x, x),\ G_p(y, y, y),\ G_p(x, y, y), G_p(x, x, y) \leq G_p(x, y, z) = 0. \tag{1.3}$$

Then, by Proposition 1.4, we have $D_{G_p}(x, y) = 0$, that is, $x = y$. Similarly, we can obtain that $y = z$. The assertion (A) is proved.

On the other hand, if $x \neq y$ and $G_p(x, y, y) = 0$, then by (A), $x = y$ which is a contradiction and so (B) holds. $\qquad\square$

In this paper, we establish some fixed point results in *GP*-metric spaces analogous to results of Ilić et al. [44] which were proved in partial metric spaces. Also, some examples are provided to illustrate our results. To our knowledge, we are the first to give some fixed point results in *GP*-metric spaces, and so is the novelty and original contributions of this paper. This opens the door to other possible fixed (common fixed) point results.

2. Main Results

We start by stating a fixed point result of Ilić et al. [44].

Theorem 2.1 (see [44]). *Let (X, p) be a complete partial metric space. Let f be a self-mapping on X. Suppose that for all $x, y, z \in X$ the following condition holds:*

$$p(fx, fy) \leq \max\{\alpha p(x, y), p(x, x), p(y, y)\}, \tag{2.1}$$

where $0 \leq \alpha < 1$. Then

(1) *the set $X_P = \{y \in X : p(x, x) = \inf_{y \in X} p(y, y)\}$ is nonempty;*

(2) *there is a unique $u \in X_P$ such that $fu = u$;*

(3) *for all $x \in X_P$ the sequence $\{f^n x\}$ converges to u with respect to the metric d_p (where $d_p(x, y) = p(x, y) - p(x, x) - p(y, y)$ for $x, y \in X$).*

The analog of Theorem 2.1 in *GP*-metric spaces is given as follows.

Theorem 2.2. *Let* (X, G_p) *be a GP-complete GP-metric space. Let* f *be a self-mapping on X. Suppose that for all* $x, y, z \in X$ *the following condition holds:*

$$G_p(fx, fy, fz) \leq \max\{rG_p(x, y, z), G_p(x, x, x), G_p(y, y, y), G_p(z, z, z)\}, \qquad (2.2)$$

where $0 \leq r < 1$. *Then*

(T1) *the set* $X_{G_p} = \{y \in X : G_p(x, x, x) = \inf_{y \in X} G_p(y, y, y)\}$ *is nonempty;*

(T2) *there is a unique* $x^* \in X_{G_p}$ *such that* $fx^* = x^*$;

(T3) *for all* $x \in X_{G_p}$ *the sequence* $\{f^n x\}$ *converges to* x^* *with respect to the metric* D_{G_p}.

Proof. Let $x \in X$. By (2.2), we have

$$G_p(fx, fx, fx) \leq \max\{rG_p(x, x, x), G_p(x, x, x)\} = G_p(x, x, x). \qquad (2.3)$$

Hence, $\{G_p(f^n x, f^n x, f^n x)\}_{n \geq 0}$ is a nonincreasing sequence. Put

$$S_x := \lim_n G_p(f^n x, f^n x, f^n x) = \inf_n G_p(f^n x, f^n x, f^n x), \qquad (2.4)$$

$$\Gamma_x := \frac{1}{1-r} G_p(x, fx, fx) + G_p(x, x, x). \qquad (2.5)$$

We shall show that

$$G_p\left(f^i x, f^j x, f^j x\right) \leq 3\Gamma_x \quad \text{for all } i, j \geq 0. \qquad (2.6)$$

Again, by (2.2), we have for all $m > n \geq 0$

$$G_p(f^n x, f^m x, f^m x) \leq \max\left\{rG_p\left(f^{n-1}x, f^{m-1}x, f^{m-1}x\right), G_p(f^n x, f^n x, f^n x)\right\}. \qquad (2.7)$$

At first

$$\begin{aligned}
G_p\left(x, f^j x, f^j x\right) &\leq G_p(x, fx, fx) + G_p\left(fx, f^j x, f^j x\right) \\
&\leq G_p(x, fx, fx) + \max\left\{rG_p\left(x, f^{j-1}x, f^{j-1}x\right), G_p(x, x, x)\right\}.
\end{aligned} \qquad (2.8)$$

Similarly

$$G_p\left(x, f^{j-1}x, f^{j-1}x\right) \leq G_p(x, fx, fx) + \max\left\{rG_p\left(x, f^{j-2}x, f^{j-2}x\right), G_p(x, x, x)\right\}. \qquad (2.9)$$

Then we have

$$G_p\left(x, f^j x, f^j x\right) \leq G_p(x, fx, fx) + \max\left\{rG_p(x, fx, fx) + r^2 G_p\left(x, f^{j-2}x, f^{j-2}x\right), G_p(x, x, x)\right\}. \qquad (2.10)$$

By continuing this process, we get

$$
\begin{aligned}
G_p\left(x, f^j x, f^j x\right) & \\
&\leq G_p(x, fx, fx) + \max\left\{ rG_p(x, fx, fx) + \cdots + r^{j-1}G_p(x, fx, fx), G_p(x, x, x) \right\} \\
&\leq G_p(x, fx, fx) + \max\left\{ \frac{r}{1-r}G_p(x, fx, fx), G_p(x, x, x) \right\} \\
&\leq G_p(x, fx, fx) + \frac{r}{1-r}G_p(x, fx, fx) + G_p(x, x, x) = \Gamma_x.
\end{aligned}
\tag{2.11}
$$

Now, by Proposition 1.3 (ii), we have $G_P(f^i x, x, x) \leq 2G_P(x, f^i x, f^i x) \leq 2\Gamma_x$. Hence

$$
G_p\left(f^i x, f^i x, f^j x\right) \leq G_p\left(f^i x, x, x\right) + G_p\left(x, f^j x, f^j x\right) \leq 3\Gamma_x,
\tag{2.12}
$$

that is, (2.6) holds. On the other hand, by (GP2), we have

$$
S_x \leq G_p(f^n x, f^n x, f^n x) \leq G_p(f^n x, f^m x, f^m x).
\tag{2.13}
$$

Given any $\epsilon > 0$, by (2.4), there exists $n_0 \in \mathbb{N}$ such that $G_p(f^{n_0} x, f^{n_0} x, f^{n_0} x) < S_x + \epsilon$. Since $0 \leq r < 1$, so without loss of generality, we have $3\Gamma_x r^{n_0} < S_x + \epsilon$. Therefore, for all $m, n \geq 2n_0$

$$
\begin{aligned}
G_p(f^n x, f^m x, f^m x) &\leq \max\left\{
\begin{array}{l}
rG_p(f^{n-1}x, f^{m-1}x, f^{m-1}x), \\
G_p(f^{n-1}x, f^{n-1}x, f^{n-1}x), \\
G_p(f^{m-1}x, f^{m-1}x, f^{m-1}x)
\end{array}
\right\} \\
&\leq \max\left\{
\begin{array}{l}
r^2 G_p(f^{n-2}x, f^{m-2}x, f^{m-2}x), \\
G_p(f^{n-2}x, f^{n-2}x, f^{n-2}x), \\
G_p(f^{m-2}x, f^{m-2}x, f^{m-2}x)
\end{array}
\right\} \leq \cdots \\
&\leq \max\left\{
\begin{array}{l}
r^{n_0} G_p(f^{n-n_0}x, f^{m-n_0}x, f^{m-n_0}x), \\
G_p(f^{n-n_0}x, f^{n-n_0}x, f^{n-n_0}x), \\
G_p(f^{m-n_0}x, f^{m-n_0}x, f^{m-n_0}x)
\end{array}
\right\} < S_x + \epsilon.
\end{aligned}
\tag{2.14}
$$

Then, $S_x = \lim_{m,n \to \infty} G_p(f^n x, f^m x, f^m x)$ and so $\{f^n x\}$ is a GP-Cauchy sequence. Since (X, G_p) is GP-complete, then there exists $x^* \in X$ such that $\{f^n x\}$ GP-converges to x^*, that is,

$$
G_p(x^*, x^*, x^*) = \lim_{m,n \to \infty} G_p(f^n x, f^m x, f^m x).
\tag{2.15}
$$

Since $\{f^n x\}$ GP-converges to x^*, then Proposition 1.6 yields that

$$G_p(x^*, x^*, x^*) = \lim_{n \to \infty} G_p(x^*, f^n x, f^n x) = \lim_{n \to \infty} G_p(f^n x, x^*, x^*). \tag{2.16}$$

We obtain that

$$S_x = G_p(x^*, x^*, x^*) = \lim_{n \to \infty} G_p(x^*, f^n x, f^n x) \tag{2.17}$$

$$= \lim_{n \to \infty} G_p(f^n x, x^*, x^*) = \lim_{m,n \to \infty} G_p(f^n x, f^m x, f^m x). \tag{2.18}$$

For all $n \in \mathbb{N}$

$$G_p(x^*, fx^*, fx^*) \le G_p(x^*, f^n x, f^n x) + G_p(f^n x, fx^*, fx^*) - G_p(f^n x, f^n x, f^n x). \tag{2.19}$$

By taking the limit as $n \to \infty$ in the above inequality, we get

$$G_p(x^*, fx^*, fx^*) \le \lim_{n \to \infty} G_p(f^n x, fx^*, fx^*). \tag{2.20}$$

On the other hand, from (2.2), we have

$$\lim_{n \to \infty} G_p(f^n x, fx^*, fx^*) \le \max \left\{ \begin{array}{c} r \lim_{n \to \infty} G_p(f^{n-1} x, x^*, x^*), \\ \lim_{n \to \infty} G_p(f^{n-1} x, f^{n-1} x, f^{n-1} x), \\ G_p(x^*, x^*, x^*) \end{array} \right\} = G_p(x^*, x^*, x^*). \tag{2.21}$$

Thus, $G_p(x^*, fx^*, fx^*) \le G_p(x^*, x^*, x^*)$. By (GP2), we deduce that

$$G_p(x^*, fx^*, fx^*) = G_p(x^*, x^*, x^*). \tag{2.22}$$

Now we show that X_{G_p} is nonempty. Let $\Omega = \inf_{y \in X} G_p(y, y, y)$. For all $k \in \mathbb{N}$, pike $x_k \in X$ with $G_p(x_k, x_k, x_k) < \Omega + 1/k$. Define $x_n^* = f^n x^*$ for all $n \ge 1$. Let us show that

$$\lim_{m,n \to \infty} G_p(x_n^*, x_m^*, x_m^*) = \Omega. \tag{2.23}$$

Given $\epsilon > 0$, put $n_0 := [3/\epsilon(1-r)] + 1$. If $k \ge n_0$, then we have

$$\Omega \le G_p(fx_k^*, fx_k^*, fx_k^*) \le G_p(x_k^*, x_k^*, x_k^*) = S_{x_k} \le G_p(x_k, x_k, x_k) < \Omega + \frac{1}{k}$$

$$< \Omega + \frac{1}{n_0} < \Omega + \frac{3}{\epsilon(1-r)} \tag{2.24}$$

$$\le G_p(fx_k^*, fx_k^*, fx_k^*) + \frac{3}{\epsilon(1-r)}.$$

Therefore, we have

$$T_k := G_p(x_k^*, x_k^*, x_k^*) - G_p(fx_k^*, fx_k^*, fx_k^*) < \frac{3}{\epsilon(1-r)}. \tag{2.25}$$

On the other hand, if $k \geq n_0$, then $G_p(x_k^*, x_k^*, x_k^*) = S_{x_k} < \Omega + 1/n_0$. It follows that

$$G_p(x_k^*, x_k^*, x_k^*) < \Omega + \frac{3}{\epsilon(1-r)}. \tag{2.26}$$

By (GP4), (2.22), and (2.25), we can obtain

$$\begin{aligned}
G_p(x_n^*, x_m^*, x_m^*) &\leq G_p(x_n^*, fx_n^*, fx_n^*) + G_p(fx_n^*, x_m^*, x_m^*) - G_p(fx_n^*, fx_n^*, fx_n^*) \\
&= T_n + G_p(fx_n^*, x_m^*, x_m^*), \\
G_p(fx_n^*, x_m^*, x_m^*) &\leq G_p(fx_n^*, fx_m^*, fx_m^*) + G_p(fx_m^*, x_m^*, x_m^*) - G_p(fx_m^*, fx_m^*, fx_m^*) \\
&= G_p(fx_n^*, fx_m^*, fx_m^*) + T_m.
\end{aligned} \tag{2.27}$$

Thus

$$\begin{aligned}
G_p(x_n^*, x_m^*, x_m^*) &\leq T_m + T_n + G_p(fx_n^*, fx_m^*, fx_m^*) \\
&\leq T_m + T_n + \max\{rG_p(x_n^*, x_m^*, x_m^*), G_p(x_n^*, x_n^*, x_n^*), G_p(x_m^*, x_m^*, x_m^*)\}.
\end{aligned} \tag{2.28}$$

Now, by (2.25) and (2.26), we have

$$\begin{aligned}
\Omega \leq G_p(x_n^*, x_m^*, x_m^*) &\leq \max\left\{\frac{2}{3}\epsilon, \frac{2}{3}\epsilon(1-r) + G_p(x_n^*, x_n^*, x_n^*), \frac{2}{3}\epsilon(1-r) + G_p(x_m^*, x_m^*, x_m^*)\right\} \\
&\leq \max\left\{\frac{2}{3}\epsilon, \Omega + \epsilon(1-r)\right\} < \Omega + \epsilon,
\end{aligned} \tag{2.29}$$

that is, (2.23) holds. Again, Since (X, G_p) is GP-complete, then there exists $y \in X$ such that

$$G_p(y, y, y) = \lim_{n \to \infty} G_p(y, x_n^*, x_n^*) = \lim_{n \to \infty} G_p(x_n^*, y, y) = \lim_{m,n \to \infty} G_p(x_n^*, x_m^*, x_m^*) = \Omega. \tag{2.30}$$

This leads that $y \in X_{G_p}$, so X_{G_p} is nonempty.

Let $x \in X$. By (2.22), we get

$$\Omega \leq G_p(fx^*, fx^*, fx^*) \leq G_p(x^*, fx^*, fx^*) = G_p(x^*, x^*, x^*) = S_x = \Omega. \tag{2.31}$$

From (GP1), it follows that $x^* = fx^*$. By (2.17), we have

$$
\begin{aligned}
\lim_{n \to \infty} D_{G_p}(f^n x, x^*) &= \lim_{n \to \infty} G_p(f^n x, x^*, x^*) + \lim_{n \to \infty} G_p(x^*, f^n x, f^n x) \\
&\quad - \lim_{n \to \infty} G_p(f^n x, f^n x, f^n x) - G_p(x^*, x^*, x^*) \\
&= 0.
\end{aligned}
\tag{2.32}
$$

Therefore, for all $x \in X_P$ the sequence $\{f^n x\}$ converges with respect to the metric D_{G_p} to x^*. The uniqueness of the fixed point follows easily from (2.2). $\qquad\square$

We illustrate Theorem 2.2 by the following examples.

Example 2.3. Let $X = [0, \infty)$ and define $G_p(x, y, z) = \max\{x, y, z\}$, for all $x, y, z \in X$. Then (X, G_p) is a complete GP-metric space. Clearly, (X, G) is not a G-metric space. Consider $f : X \to X$ defined by $fx = x^2/(1 + x)$. Without loss of generality, take $x \le y \le z$. We have

$$
\begin{aligned}
G_p(fx, fy, fz) &= \frac{z^2}{1 + z} \\
&\le z = \max\{G_p(x, x, x), G_p(y, y, y), G_p(z, z, z)\} \\
&= \max\{rG_p(x, y, z), G_p(x, x, x), G_p(y, y, y), G_p(z, z, z)\},
\end{aligned}
\tag{2.33}
$$

for all $r \in [0, 1)$. So, (2.2) holds. Here, $u = 0$ is the unique fixed point of f.

Example 2.4. Let $X = [0, \infty)$. Define $G_p : X^3 \to [0, \infty)$ by $G_p(x, y, z) = \max\{x, y, z\}$. Clearly, (X, G_p) is a GP-metric space. Define $f : X \to X$ by

$$
fx = \begin{cases}
x^2, & 0 \le x \le \dfrac{1}{3} \\[2ex]
\dfrac{(1 - x)}{2}, & \dfrac{1}{3} < x \le \dfrac{1}{2} \\[2ex]
\dfrac{1}{(2x + 1)} & x > \dfrac{1}{2}.
\end{cases}
\tag{2.34}
$$

Then, the inequality (2.2) of Theorem 2.2 holds. Here, $u = 0$ is the unique fixed point of f.

Proof. Clearly, $M(x, y, z) := \max\{rG_p(x, y, z), G_p(x, x, x), G_p(y, y, y), G_p(z, z, z)\} = \max\{x, y, z\}$, for all $r \in (0, 1)$. We have the following cases.
Case 1 $(0 \le x, y, z \le 1/3)$.
Consider the following:

$$
G_p(fx, fy, fz) = \max\{x^2, y^2, z^2\} \le \max\{x, y, z\} = M(x, y, z).
\tag{2.35}
$$

Case 2 $(1/3 < x, y, z \leq 1/2)$.
  Consider the following:

$$G_p(fx, fy, fz) = \max\left\{\frac{(1-x)}{2}, \frac{(1-y)}{2}, \frac{(1-z)}{2}\right\} \leq \max\{x, y, z\} = M(x, y, z). \quad (2.36)$$

Case 3 $(x, y, z > 1/2)$.
  Consider the following:

$$G_p(fx, fy, fz) = \max\left\{\frac{1}{(2x+1)}, \frac{1}{(2y+1)}, \frac{1}{(2z+1)}\right\} \leq M(x, y, z). \quad (2.37)$$

Case 4 $(0 \leq x \leq 1/3, 1/3 < y \leq 1/2 \text{ and } z > 1/2)$.
  Consider the following:

$$G_p(fx, fy, fz) = \max\left\{x^2, \frac{(1-y)}{2}, \frac{1}{(2z+1)}\right\} \leq M(x, y, z). \quad (2.38)$$

Case 5 $(0 \leq x, y \leq 1/3, 1/3 < z \leq 1/2)$.
  Consider the following:

$$G_p(fx, fy, fz) = \max\left\{x^2, y^2, \frac{(1-x)}{2}\right\} \leq M(x, y, z). \quad (2.39)$$

Case 6 $(0 \leq x, y \leq 1/3, z > 1/2)$.
  Consider the following:

$$G_p(fx, fy, fz) = \max\left\{x^2, y^2, \frac{1}{(2x+1)}\right\} \leq M(x, y, z). \quad (2.40)$$

Case 7 $(1/3 < x, y \leq 1/2, 0 \leq z \leq 1/3)$.
  Consider the following:

$$G_p(fx, fy, fz) = \max\left\{\frac{(1-x)}{2}, \frac{(1-y)}{2}, z^2\right\} \leq M(x, y, z). \quad (2.41)$$

Case 8 $(1/3 \leq x, y \leq 1/2, z > 1/2)$.
  Consider the following:

$$G_p(fx, fy, fz) = \max\left\{\frac{(1-x)}{2}, \frac{(1-y)}{2}, \frac{1}{(2z+1)}\right\} \leq M(x, y, z). \quad (2.42)$$

Case 9 $(x, y > 1/2, 0 \leq z \leq 1/3)$.
    Consider the following:

$$G_p(fx, fy, fz) = \max\left\{\frac{1}{(2x + 1)}, \frac{1}{(2y + 1)}, z^2\right\} \leq M(x, y, z). \tag{2.43}$$

Case 10 $(x, y > 1/2, 1/3 < z \leq 1/2)$.
    Consider the following:

$$G_p(fx, fy, fz) = \max\left\{\frac{1}{(2x + 1)}, \frac{1}{(2y + 1)}, \frac{(1 - z)}{2}\right\} \leq M(x, y, z). \tag{2.44}$$

Thus, the inequality (2.2) holds. Applying Theorem 2.2, we get $u = 0$ is the unique point fixed point of f. $\qquad\square$

Also, Ilić et al. [44] proved the following result.

Theorem 2.5 (see [44])**.** *Let* (X, P) *be a complete partial metric space. Let* f *be a self-mapping on* X. *Suppose that for all* $x, y, z \in X$ *the following condition holds:*

$$p(fx, fy) \leq \max\left\{\alpha p(x, y), \frac{p(x, x) + p(y, y)}{2}\right\}, \tag{2.45}$$

where $0 \leq \alpha < 1$. *Then*

(i) *the set* $X_P = \{y \in X : p(x, x) = \inf_{y \in X} p(y, y)\}$ *is nonempty;*

(ii) *there is a unique* $x^* \in X_P$ *such that* $fx^* = x^*$;

(iii) *for all* $x \in X_P$, *the sequence* $\{f^n x\}$ *converges to* x^* *with respect to the metric* d_p.

The analog of Theorem 2.5 in GP-metric spaces is stated as follows.

Theorem 2.6. *Let* (X, G_p) *be a GP-complete GP-metric space. Let* f *be a self-mapping on* X. *Suppose that for all* $x, y, z \in X$ *the following condition holds:*

$$G_p(fx, fy, fz) \leq \max\left\{rG_p(x, y, z), \frac{G_p(x, x, x) + G_p(y, y, y) + G_p(z, z, z)}{3}\right\}, \tag{2.46}$$

where $0 \leq r < 1$. *Then*

(R1) *the set* $X_P = \{y \in X : G_p(x, x, x) = \inf_{y \in X} G_p(y, y, y)\}$ *is nonempty;*

(R2) *there is a unique* $x^* \in X_P$ *such that* $fx^* = x^*$;

(R3) *for all* $x \in X_{G_p}$, *the sequence* $\{f^n x\}$ *converges to* x^* *with respect to the metric* D_{G_p}.

Proof. Since

$$G_p(x,x,x), G_p(y,y,y), G_p(z,z,z)$$
$$\leq \max\{rG_p(x,y,z), G_p(x,x,x), G_p(y,y,y), G_p(z,z,z)\}, \tag{2.47}$$

then

$$\frac{G_p(x,x,x) + G_p(y,y,y) + G_p(z,z,z)}{3}$$
$$\leq \max\{rG_p(x,y,z), G_p(x,x,x), G_p(y,y,y), G_p(z,z,z)\}. \tag{2.48}$$

Thus

$$G_p(fx, fy, fz) \leq \max\{rG_p(x,y,z), G_p(x,x,x), G_p(y,y,y), G_p(z,z,z)\}. \tag{2.49}$$

Then, the conditions of **Theorem 2.1** hold. Hence, it follows that (R1), (R2), and (R3) hold. □

Example 2.7. Let $X = [0,1]$ and define $G_p(x,y,z) = \max\{x,y,z\}$, for all $x,y,z \in X$. We have (X, G_p) is a complete GP-metric space. Take $fx = x^2/2$ and $r = 1/2$. For all $x \leq y \leq z$, we have

$$G_p(fx, fy, fz) = \frac{z^2}{2}$$
$$\leq rG_p(x,y,z) \tag{2.50}$$
$$\leq \max\{rG_p(x,y,z), G_p(x,x,x), G_p(y,y,y), G_p(z,z,z)\},$$

that is, (2.2) holds. Here, $u = 0$ is the unique fixed point of f.

Similarly, we have the following.

Theorem 2.8. *Let (X, G_p) be a GP-complete GP-metric space. Let f be a self-mapping on X. Suppose that for all $x, y, z \in X$ the following condition holds:*

$$G_p(fx, fy, fz)^3 \leq rG_p(x,x,x)G_p(y,y,y)G_p(z,z,z), \tag{2.51}$$

where $0 \leq r < 1$. Then

   (N1) *the set $X_{G_p} = \{y \in X : G_p(x,x,x) = \inf_{y \in X} G_p(y,y,y)\}$ is nonempty;*

   (N2) *there is a unique $x^* \in X_{G_p}$ such that $fx^* = x^*$;*

   (N3) *for all $x \in X_{G_p}$, the sequence $\{f^n x\}$ converges with respect to the metric D_{G_p} to x^*.*

The following lemma is useful.

Lemma 2.9. *Let* (X, G_p) *be a GP-metric space and* $\{x_n\}$ *be a sequence in* X. *Assume that* $\{x_n\} GP$ *converges to a point* $x \in X$ *with* $G_p(x, x, x) = 0$. *Then* $\lim_{n \to +\infty} G_p(x_n, y, y) = G_p(x, y, y)$ *for all* $y \in X$. *Moreover,* $\lim_{m,n \to +\infty} G_p(x_n, x_m, x) = 0$.

Proof. By (GP4), we have

$$G_p(x, y, y) - G_p(x, x_n, x_n) \le G_p(x_n, y, y) \le G_p(x_n, x, x) + G_p(x, y, y), \qquad (2.52)$$

and so $\lim_{n \to +\infty} G_p(x_n, y, y) = P(x, y, y)$. Again by (GP4), we get

$$G_p(x_n, x_m, x) \le G_p(x_n, x, x) + G_p(x, x, x_m), \qquad (2.53)$$

and hence $\lim_{m,n \to +\infty} G_p(x_n, x_m, x) = 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 2.10. *Let* (X, G_p) *be a 0-GP-complete GP-metric space and* f *be a self-mapping on* X. *Assume that* $(1/3) G_p(x, fx, fx) < G_p(x, y, y)$ *implies*

$$G_p(fx, fy, fy) \le \alpha G_p(x, y, y) + \beta G_p(x, fx, fx) + \gamma G_p(y, fy, fy) \qquad (2.54)$$

for all $x, y \in X$, *where* $\alpha, \beta, \gamma \ge 0$ *with* $\alpha + \beta + \gamma < 1$. *Then* f *has a unique fixed point.*

Proof. If $x = fx$, then x is a fixed point for f. Assume that $x \ne fx$. So by Lemma 1.10, it follows that $G_p(x, fx, fx) > 0$. Therefore, $(1/3) G_p(x, fx, fx) < G_p(x, fx, fx)$ and so from (2.54), we have

$$G_p\left(fx, f^2x, f^2x\right) \le r G_p(x, fx, fx) \quad \text{for all } x \in X, r = \frac{\alpha + \beta}{1 - \gamma} < 1. \qquad (2.55)$$

Let $x_0 \in X$ and define a sequence $\{x_n\}$ by $x_n = f^n x_0$ for all $n \in \mathbb{N}$. Now by (2.55), we can obtain that

$$G_p\left(f^n x_0, f^{n+1} x_0, f^{n+1} x_0\right) \le r G_p\left(f^{n-1} x_0, f^n x_0, f^n x_0\right) \le \cdots \le r^n G_p(x_0, fx_0, fx_0). \qquad (2.56)$$

Then, for any $m > n$, by (2.56), we get

$$
\begin{aligned}
G_p\left(f^n x_0, f^m x_0, f^m x_0\right) &\leq G_p\left(f^n x_0, f^{n+1} x_0, f^{n+1} x_0\right) + G_p\left(f^{n+1} x_0, f^m x_0, f^m x_0\right) \\
&\leq G_p\left(f^n x_0, f^{n+1} x_0, f^{n+1} x_0\right) + G_p\left(f^{n+1} x_0, f^{n+2} x_0, f^{n+2} x_0\right) \\
&\quad + G_p\left(f^{n+2} x_0, f^m x_0, f^m x_0\right) \\
&\leq G_p\left(f^n x_0, f^{n+1} x_0, f^{n+1} x_0\right) + G_p\left(f^{n+1} x_0, f^{n+2} x_0, f^{n+2} x_0\right) \\
&\quad + G_p\left(f^{n+2} x_0, f^{n+3} x_0, f^{n+3} x_0\right) + \cdots + G_p\left(f^{m-1} x_0, f^m x_0, f^m x_0\right) \\
&\leq \frac{r^n}{1-r} G_p\left(x_0, f x_0, f x_0\right).
\end{aligned}
$$

(2.57)

It implies that $\lim_{m,n \to \infty} G_p(f^n x_0, f^m x_0, f^m x_0) = 0$; that is, $\{x_n\}$ is a $0\text{-}GP$-Cauchy sequence. Since X is $0\text{-}GP$-complete, so $\{x_n\}$ GP converges to some point $z \in X$ with $G_p(z, z, z) = 0$, that is,

$$
\lim_{n \to +\infty} G_p(x_n, z, z) = \lim_{n \to \infty} G_p(x_n, x_n, z) = G_P(z, z, z) = 0.
$$

(2.58)

Now, we suppose that the following inequality holds:

$$
\frac{1}{3} G_p(x, fx, fx) \geq G_p(x, y, y), \qquad \frac{1}{3} G_p\left(fx, f^2 x, f^2 x\right) \geq G_p(fx, y, y),
$$

(2.59)

for some $x, y \in X$. Then, by Proposition 1.3 (iii) and (2.55), we have

$$
\begin{aligned}
G_p(x, fx, fx) &\leq G_p(x, y, y) + 2G_p(fx, y, y) \\
&\leq \frac{1}{3} G_p(x, fx, fx) + \frac{2}{3} G_p\left(fx, f^2 x, f^2 x\right) \\
&\leq \frac{1}{3} G_p(x, fx, fx) + \frac{2r}{3} G_p(x, fx, fx) < G_p(x, fx, fx)
\end{aligned}
$$

(2.60)

which is a contradiction. Thus, for all $x, y \in X$, either

$$
\frac{1}{3} G_p(x, fx, fx) < G_p(x, y, y) \quad \text{or} \quad \frac{1}{3} G_p\left(fx, f^2 x, f^2 x\right) < G_p(fx, y, y)
$$

(2.61)

holds. Therefore, either

$$
\frac{1}{3} G_p(x_{2n}, x_{2n+1}, x_{2n+1}) \leq G_p(x_{2n}, z, z) \quad \text{or} \quad \frac{1}{3} G_p(x_{2n+1}, x_{2n+2}, x_{2n+2}) \leq G_p(x_{2n+1}, z, z)
$$

(2.62)

holds for every $n \in \mathbb{N}$.

On the other hand, by (2.54), it follows that

$$G_p(x_{2n+1}, fz, fz) \leq \alpha G_p(x_{2n}, z, z) + \beta G_p(x_{2n}, x_{2n+1}, x_{2n+1}) + \gamma G_p(z, fz, fz),$$

$$G_p(x_{2n+2}, fz, fz) \leq \alpha G_p(x_{2n+1}, z, z) + \beta G_p(x_{2n+1}, x_{2n+2}, x_{2n+2}) + \gamma G_p(z, fz, fz).$$

(2.63)

If we take the limit as $n \to \infty$ in each of these inequalities, having in mind (2.58), (2.62), and applying Lemma 2.9, then we get $(1 - \gamma)G_p(z, fz, fz) \leq 0$, that is, $z = fz$. The uniqueness of the fixed point follows easily from (2.54).

As a consequence of Theorem 2.10, we may state the following corollaries. □

Corollary 2.11. *Let* (X, G_p) *be a 0-GP-complete GP-metric space and* f *be a self-mapping on* X. *Assume that*

$$\frac{1}{3}G_p(x, fx, fx) < G_p(x, y, y) \quad \text{implies} \quad G_p(fx, fy, fy) \leq rG_p(x, fx, fx)$$

(2.64)

for all $x, y \in X$, *where* $0 \leq r < 1$. *Then* f *has a unique fixed point.*

Corollary 2.12. *Let* (X, G_p) *be a 0-GP-complete GP-metric space and* f *be a self-mapping on* X. *Assume that*

$$\frac{1}{3}G_p(x, fx, fx) < G_p(x, y, y) \quad \text{implies} \quad G_p(fx, fy, fy) \leq rG_p(y, fy, fy)$$

(2.65)

for all $x, y \in X$, *where* $0 \leq r < 1$. *Then* f *has a unique fixed point.*

3. Conclusion

In [43], Zand and Nezhad initiated the notion of a *GP*-metric space. Also, they studied fully its topology. Based on this new space, in this paper we present some fixed point results for self mappings involving different contractive conditions. They are illustrated by some examples. The presented theorems are the first results in fixed point theory on *GP*-metric spaces.

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