Review Article

Quasi-Contractive Mappings in Modular Metric Spaces

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We prove the existence of fixed point and uniqueness of quasi-contractive mappings in modular metric spaces which was introduced by Ćirić

1. Introduction and Preliminaries

In this paper, we prove the existence and uniqueness of fixed points of quasi-contractive mappings in modular metric spaces which develop the theory of metric spaces generated by modulars. Throughout the paper \mathfrak{X} is a nonempty set and $\lambda > 0$. The notion of a metric modular was introduced by Chistyakov [1] as follows.

Definition 1.1. A function $\omega : (0, \infty) \times \mathfrak{X} \times \mathfrak{X} \to [0, \infty]$ is said to be a metric modular on \mathfrak{X} (or, simply, a modular if no ambiguity arises) if it satisfies three axioms:

(i) for any $x, y \in \mathfrak{X}$, $\omega_{\lambda}(x, y) = 0$ for all $\lambda > 0$ if and only if x = y;

(ii) $\omega_{\lambda}(x, y) = \omega_{\lambda}(y, x)$ for all $\lambda > 0$, and $x, y \in \mathfrak{X}$;

(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_{\lambda}(x, z) + \omega_{\mu}(z, y)$ for all $\lambda, \mu > 0$ and $x, y \in \mathfrak{X}$.

Definition 1.2. Let (\mathfrak{X}, ω) be a metric modular space.

(1) A sequence $\{x_n\}$ in \mathfrak{X}_{ω} is said to be ω -convergent to a point $x \in \mathfrak{X}$ if, for all $\lambda > 0$,

$$\omega_{\lambda}(x_n, x) \longrightarrow 0 \tag{1.1}$$

as $n \to \infty$.

- (2) A subset C of X_ω is said to be ω-closed if the ω-limit of a ω-convergent sequence of C always belongs to C.
- (3) A subset \mathfrak{C} of \mathfrak{X}_{ω} is said to be ω -complete if every ω -Cauchy sequence in \mathfrak{C} is ω -convergent and its ω -limit is in \mathfrak{C} .

Definition 1.3. The metric modular ω is said to have the Fatou property if

$$\omega_{\lambda}(x,y) \le \liminf_{n \to \infty} \omega(x_n,y) \tag{1.2}$$

for all $y \in \mathfrak{X}_{\omega}$ and $\lambda \in (0, \infty)$, where $\{x_n\}$ ω -converges to x.

2. Main Results

Definition 2.1. Let (\mathfrak{X}, ω) be a metric modular space, and let \mathfrak{C} be a nonempty subset of \mathfrak{X}_{ω} . The self-mapping $T : \mathfrak{C} \to \mathfrak{C}$ is said to be quasi-contraction if there exists 0 < k < 1 such that

$$\omega_{\lambda}(T(x), T(y)) \le k \max\{\omega_{\lambda}(x, y), \omega_{\lambda}(x, T(x)), \omega_{\lambda}(y, T(y)), \omega_{\lambda}(x, T(y)), \omega_{\lambda}(T(x), y)\}$$
(2.1)

for any $x, y \in \mathfrak{X}$ and $\lambda \in (0, \infty)$.

Let $T : \mathfrak{C} \to \mathfrak{C}$ be a mapping, and let \mathfrak{C} be a nonempty subset of \mathfrak{X}_{ω} . For any $x \in \mathfrak{C}$, define the orbit

$$\mathcal{O}(x) = \left\{ x, T(x), T^2(x), \ldots \right\}$$
(2.2)

and its ω -diameter by

$$\delta_{\omega}(x) = \operatorname{diam}(\mathcal{O}(x)) = \sup\{\omega_{\lambda}(T^{n}(x), T^{m}(x)) : n, m \in \mathbb{N}\}\}.$$
(2.3)

Lemma 2.2. Let (\mathfrak{X}, ω) be a metric modular space, and let \mathfrak{C} be a nonempty subset of \mathfrak{X}_{ω} . Let $T : \mathfrak{C} \to \mathfrak{C}$ be a quasi-contractive mapping, and let $x \in \mathfrak{C}$ be such that $\delta_{\omega}(x) < \infty$. Then, for any $n \ge 1$, one has

$$\delta_{\omega}(T(x)) \le k^n \delta_{\omega}(x), \tag{2.4}$$

where *k* is the constant associated with the mapping of *T*. Moreover, one has

$$\omega_{\lambda}(T^{n}(x), T^{n+m}(x)) \le k^{n} \delta_{\omega}(x)$$
(2.5)

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for any $n, m \ge 1$ and $\lambda \in (0, \infty)$.

Proof. For each $n, m \ge 1$, we have

$$\omega_{\lambda}(T^{n}(x), T^{m}(y)) \leq k \max\left\{\omega_{\lambda}\left(T^{n-1}(x), T^{m-1}(y)\right), \omega_{\lambda}\left(T^{n-1}(x), T^{n}(x)\right), \\ \omega_{\lambda}\left(T^{m-1}(y), T^{m}(y)\right), \omega_{\lambda}\left(T^{n-1}(x), T^{m}(y)\right), \omega_{\lambda}\left(T^{n}(x), T^{m-1}(y)\right)\right\}$$

$$(2.6)$$

for any $x, y \in \mathfrak{C}$ and $\lambda \in (0, \infty)$. This obviously implies that

$$\delta_{\omega}(T^{n}(x)) \le k \delta_{\omega} \Big(T^{n-1}(x) \Big)$$
(2.7)

for any $n \ge 1$. Hence, for any $n \ge 1$, we have

$$\delta_{\omega}(T^n(x)) \le k^n \delta_{\omega}(x). \tag{2.8}$$

Moreover, for any $n, m \ge 1$, we have

$$\omega_{\lambda}(T^{n}(x), T^{n+m}(x) \le \delta_{\omega}(T^{n}(x)) \le k^{n}\delta_{\omega}(x).$$
(2.9)

This completes the proof.

The next lemma is helpful to prove the main result in this paper.

Lemma 2.3. Let (\mathfrak{X}, ω) be a modular metric space, and let \mathfrak{C} be a ω -complete nonempty subset of \mathfrak{X}_{ω} . Let $T : \mathfrak{C} \to \mathfrak{C}$ be quasi-contractive mapping, and let $x \in \mathfrak{C}$ be such that $\delta_{\omega}(x) < \infty$. Then $\{T^n(x)\} \omega$ -converges to a point $v \in \mathfrak{C}$. Moreover, one has

$$\omega_{\lambda}(T^{n}(X) - \nu) \le k^{n} \delta_{\omega}(x) \tag{2.10}$$

for all $n \ge 1$ and $\lambda \in (0, \infty)$.

Proof. From Lemma 2.2, we know that $\{T^n(x)\}$ is a ω -Cauchy sequence in \mathfrak{C} . Since \mathfrak{C} is ω -complete, then there exists $\nu \in \mathfrak{C}$ such that $\{T^n(x)\}$ ω -converges to ν . Since

$$\omega_{\lambda}(T^{n}(x), T^{n+m}(x)) \le k^{n} \delta_{\omega}(x)$$
(2.11)

for any $n, m \ge 1$ and ω satisfies the Fatou property, and letting $m \to \infty$, we have

$$\omega_{\lambda}(T^{n}(x),\nu) \leq \liminf_{m \to \infty} \omega_{\lambda}(T^{n}(x),T^{n+m}(x)) \leq k^{n} \delta_{\omega}(x).$$
(2.12)

This completes the proof.

Next, we prove that v is, in fact, a fixed point of T and it is unique provided some extra assumptions.

Theorem 2.4. Let T, \mathfrak{C} , and x be as in Lemma 2.3. Suppose that $\omega_{\lambda}(v, T(v)) < \infty$ and $\omega_{\lambda}(x, T(x)) < \infty$ for all $\lambda \in (0, \infty)$. Then the ω -limit of $\{T^n(x)\}$ is a fixed point of T, that is, T(v) = v. Moreover, if v^* is any fixed point of T in \mathfrak{C} such that $\omega_{\lambda}(v, v^*) < \infty$ for all $\lambda \in (0, \infty)$, then one has $v = v^*$.

Proof. We have

$$\omega_{\lambda}(T(x), T(\nu)) \le k \max\{\omega_{\lambda}(x, \nu), \omega_{\lambda}(x, T(x)), \omega_{\lambda}(\nu, T(\nu)), \omega_{\lambda}(x, T(\nu)), \omega_{\lambda}(T(x), \nu)\}.$$
(2.13)

From Lemma 2.3, it follows that

$$\omega_{\lambda}(T(x), T(\nu)) \le k \max\{\delta_{\omega}(x), \omega_{\lambda}(\nu, T(\nu)), \omega_{\lambda}(x, T(\nu))\}.$$
(2.14)

Suppose that, for each $n \ge 1$,

$$\omega_{\lambda}(\mathbf{T}^{n}(x), T(\nu)) \leq \max\{k^{n}\delta_{\omega}(x), k\omega_{\lambda}(\nu, T(\nu)), k^{n}\omega_{\lambda}(x, T(\nu))\}.$$
(2.15)

Then we have

$$\omega_{\lambda}\left(T^{n+1}(x),T(\nu)\right) \leq k \max\left\{\omega_{\lambda}(T^{n}(x),\nu),\omega_{\lambda}\left(T^{n}(x),T^{n+1}(x)\right),\omega_{\lambda}(\nu,T(\nu)),\omega_{\lambda}(T^{n}(x),T(\nu)),\omega_{\lambda}\left(T^{n+1}(x),\nu\right)\right\}.$$
(2.16)

Hence we have

$$\omega_{\lambda}\Big(T^{n+1}(x), T(\nu)\Big) \le k \max\{k^n \delta_{\omega}(x), k\omega_{\lambda}(\nu, T(\nu)), \omega_{\lambda}(T^n(x), T(\nu))\}.$$
(2.17)

Using our previous assumption, we get

$$\omega_{\lambda}\left(T^{n+1}(x), T(\nu)\right) \le \max\left\{k^{n+1}\delta_{\omega}(x), k\omega_{\lambda}(\nu, T(\nu)), k^{n+1}\omega_{\lambda}(x, T(\nu))\right\}.$$
(2.18)

Thus, by induction, we have

$$\omega_{\lambda}(T^{n}(x), T(\nu)) \leq \max\{k^{n}\delta_{\omega}(x), k\omega_{\lambda}(\nu, T(\nu)), k^{n}\omega_{\lambda}(x, T(\nu))\}$$
(2.19)

for any $n \ge 1$ and $\lambda \in (0, \infty)$. Therefore, we have

$$\limsup_{n \to \infty} \omega_{\lambda}(T^{n}(x), T(x)) \le \omega(\nu, T(\nu))$$
(2.20)

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for all $\lambda \in (0, \infty)$. Using the Fatou property for the metric modular ω , we get

$$\omega_{\lambda}(\nu, T(\nu)) \liminf_{n \to \infty} \omega_{\lambda}(T^{n}(x), T(\nu)) \le k\omega(\nu, T(\nu))$$
(2.21)

for all $\lambda \in (0, \infty)$. Since k < 1, we get $\omega_{\lambda}(\nu, T(\nu)) = 0$ for all $\lambda \in (0, \infty)$, and so $T(\nu) = \nu$.

Let v^* be another fixed point of T such that $\omega_{\lambda}(v, v^*) < \infty$ for all $\lambda \in (0, \infty)$. Then we have

$$\omega_{\lambda}(\nu,\nu^{*}) = \omega_{\lambda}(T(\nu),T(\nu^{*})) \le k\omega_{\lambda}(\nu,\nu^{*}), \tag{2.22}$$

which implies that

$$\omega_{\lambda}(\nu,\nu^{*}) = 0 \tag{2.23}$$

for all $\lambda \in (0, \infty)$. Hence $\nu = \nu^*$. This complete the proof.

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