Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 156139, 14 pages
http://dx.doi.org/10.1155/2013/156139

Research Article

Some Further Results on Traveling Wave Solutions for the ZK-BBM(m, n) Equations

Shaoyong Li[1,2] and Zhengrong Liu[2]

[1] *College of Mathematics and Information Sciences, Shaoguan University, Shaoguan, Guangdong 512005, China*
[2] *Department of Mathematics, South China University of Technology, Guangzhou, Guangdong 510640, China*

Correspondence should be addressed to Shaoyong Li; lishaoyongok@sohu.com

Received 30 June 2013; Accepted 4 September 2013

Academic Editor: K. S. Govinder

We investigate the traveling wave solutions for the ZK-BBM(m,n) equations $u_t + u_x - a(u^m)_x + (b(u^n)_{xt} + k(u^n)_{yt})_x = 0$ by using bifurcation method of dynamical systems. Firstly, for ZK-BBM(2, 2) equation, we obtain peakon wave, periodic peakon wave, and smooth periodic wave solutions and point out that the peakon wave is the limit form of the periodic peakon wave. Secondly, for ZK-BBM(3, 2) equation, we obtain some elliptic function solutions which include periodic blow-up and periodic wave. Furthermore, from the limit forms of the elliptic function solutions, we obtain some trigonometric and hyperbolic function solutions which include periodic blow-up, blow-up, and smooth solitary wave. We also show that our work extends some previous results.

1. Introduction

In recent years, many nonlinear wave equations have been derived from solid state physics, plasma physics, chemical physics, fluid mechanics, biology, and other fields. Thus, there has been considerable attention to find exact solutions of these problems. For this purpose, there have been many methods, such as inverse scattering transform method [1], Bäcklund and Darboux transforms [2, 3], Jacobi elliptic function method [4, 5], F-expansion and extended F-expansion method [6, 7], (G'/G)-expansion method [8, 9], and the bifurcation method of dynamical systems [10–14].

Zakharov-Kuznetsov (ZK) equation [15]

$$u_t + auu_x + \left(u_{xx} + u_{xy}\right)_x = 0 \tag{1}$$

is a two-dimensional space generalization of the KdV equation. The nonintegrable ZK equation governs the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [16, 17].

Benjamin-Bona-Mahony (BBM) equation [18]

$$u_t + u_x - a\left(u^2\right)_x - \tilde{b}u_{xxt} = 0 \tag{2}$$

is an alternative model to KdV equation for small-amplitude, surface waves of long wavelength in liquids, acoustic-gravity waves in compressible fluids, hydromagnetic waves in cold plasma, and acoustic waves in anharmonic crystals.

Combining the BBM equation with the sense of the ZK equation, Wazwaz [19] considered the following ZK-BBM equation:

$$u_t + u_x - a\left(u^2\right)_x - \left(\tilde{b}(u)_{xt} + \tilde{k}(u)_{yt}\right)_x = 0, \tag{3}$$

and its generalized form

$$u_t + u_x - a\left(u^n\right)_x - \left(\tilde{b}\left(u^n\right)_{xt} + \tilde{k}\left(u^n\right)_{yt}\right)_x = 0. \tag{4}$$

He presented a method called the extended tanh method to seek exact explicit compactons, solitons, solitary patterns, and plane periodic solutions of (3) and (4).

Wang and Tang [20] studied the following generalized ZK-BBM equations:

$$u_t + u_x - a\left(u^m\right)_x + \left(b\left(u^n\right)_{xt} + k\left(u^n\right)_{yt}\right)_x = 0. \tag{5}$$

By using the bifurcation theory of planar dynamical systems, they gave some exact explicit traveling wave solutions and the

sufficient conditions to guarantee the existence of smooth and nonsmooth traveling wave solutions.

In the present paper, we continue to study the traveling wave solutions for (5), which we denote by ZK-BBM(m,n) equations for convenience. Our results are as follows: (i) for ZK-BBM(2, 2) equation, we obtain peakon wave, periodic peakon wave, and smooth periodic wave solutions and point out that the peakon wave is the limit form of the periodic peakon wave; (ii) for ZK-BBM(3, 2) equation, we obtain some elliptic function solutions which include periodic blow-up and periodic wave. From the limit forms of the elliptic function solutions, we obtain some trigonometric and hyperbolic function solutions which include periodic blow-up, blow-up, and smooth solitary wave. We also check the correctness of these solutions by putting them back into the original equation.

This paper is organized as follows. In Section 2, we state our main results which are included in two propositions. In Sections 3 and 4, we give the derivations for the two propositions, respectively. A brief conclusion is given in Section 5.

2. Main Results and Remarks

In this section we list our main results and give some remarks. To begin with, let us recall some symbols. The symbols sn u and cn u denote the Jacobian elliptic functions sine amplitude u and cosine amplitude u. cosh u, sinh u, sech u, and csch u are the hyperbolic functions. For the sake of simplification, we only consider the case $a > 0$ (the other case $a < 0$ can be considered similarly). To relate conveniently, for given constant wave speed c, let

$$\begin{aligned} \xi &= x + y - ct, \\ g_1 &= -\frac{2(1-c)^2}{9a}, \\ g_2 &= \sqrt{\frac{80\,|1-c|^3}{829a}}. \end{aligned} \tag{6}$$

Via the following two propositions we state our main results.

Proposition 1. *Consider ZK-BBM(2, 2) equation*

$$u_t + u_x - a\left(u^2\right)_x + \left(b\left(u^2\right)_{xt} + k\left(u^2\right)_{yt}\right)_x = 0 \tag{7}$$

and its traveling wave equation

$$g + (1-c)\,\varphi - a\varphi^2 - 2c\,(b+k)\left(\varphi'\right)^2 - 2c\,(b+k)\,\varphi\varphi'' = 0. \tag{8}$$

There are the following results.

(1) When $c(b+k) < 0$, $c \neq 1$, and $g = g_1$, (7) has a peakon wave solution

$$u_1(\xi) = \frac{2\,(1-c)}{3a}\left(1 - e^{-\eta_1^-|\xi|}\right), \tag{9}$$

where

$$\eta_1^- = \sqrt{\frac{-a}{4c\,(b+k)}}. \tag{10}$$

(2) When $c(b+k) < 0$, $c \neq 1$, and $g_1 < g < 0$, (7) has a periodic peakon wave solution

$$u_2(\xi) = \begin{cases} \alpha e^{\eta_1^-|\xi|} + \beta e^{-\eta_1^-|\xi|} + \dfrac{2\,(1-c)}{3a}, & \text{for } c < 1, \\ \alpha e^{-\eta_1^-|\xi|} + \beta e^{\eta_1^-|\xi|} + \dfrac{2\,(1-c)}{3a}, & \text{for } c > 1, \end{cases} \tag{11}$$

where

$$\begin{aligned} &\xi \in [(2l-1)\,T, (2l+1)\,T), \\ &\alpha = \frac{c-1}{3a} + \sqrt{\frac{-g}{2a}}, \\ &\beta = \frac{c-1}{3a} - \sqrt{\frac{-g}{2a}}, \\ &l = 0, \pm 1, \pm 2, \ldots, \\ &T = \sqrt{\frac{4c\,(b+k)}{-a}} \ln\left|\frac{\sqrt{4(1-c)^2 + 18ag}}{3\sqrt{-2ag} - 2\,(1-c)}\right|. \end{aligned} \tag{12}$$

(3) When $c(b+k) > 0$, $c \neq 1$, and $g_1 < g$, (7) has two smooth periodic wave solutions

$$\begin{aligned} u_3(\xi) &= \frac{2\,(1-c)}{3a} \pm \gamma \cos\left(\eta_1^+ \xi\right), \\ u_4(\xi) &= \frac{2\,(1-c)}{3a} \pm \gamma \sin\left(\eta_1^+ \xi\right), \end{aligned} \tag{13}$$

where

$$\eta_1^+ = \sqrt{\frac{a}{4c\,(b+k)}}, \tag{14}$$

$$\gamma = \frac{\sqrt{4(1-c)^2 + 18ag}}{3a}. \tag{15}$$

Remark 2. When $c(b+k) < 0$, $c \neq 1$, and $g \to g_1 + 0$, the periodic peakon wave $u_2(\xi)$ becomes the peakon wave $u_1(\xi)$; the varying process is displayed in Figure 1.

Remark 3. When $c(b+k) > 0$, $c \neq 1$, and $g \to 0$, the smooth periodic wave $u_4(\xi)$ becomes

$$u_4^{\circ}(\xi) = \frac{2\,(1-c)}{3a}\left(1 \pm \sin\left(\frac{1}{2}\sqrt{\frac{a}{c\,(b+k)}}\,\xi\right)\right), \tag{16}$$

which can be found in [20]; this implies that we extend the previous result.

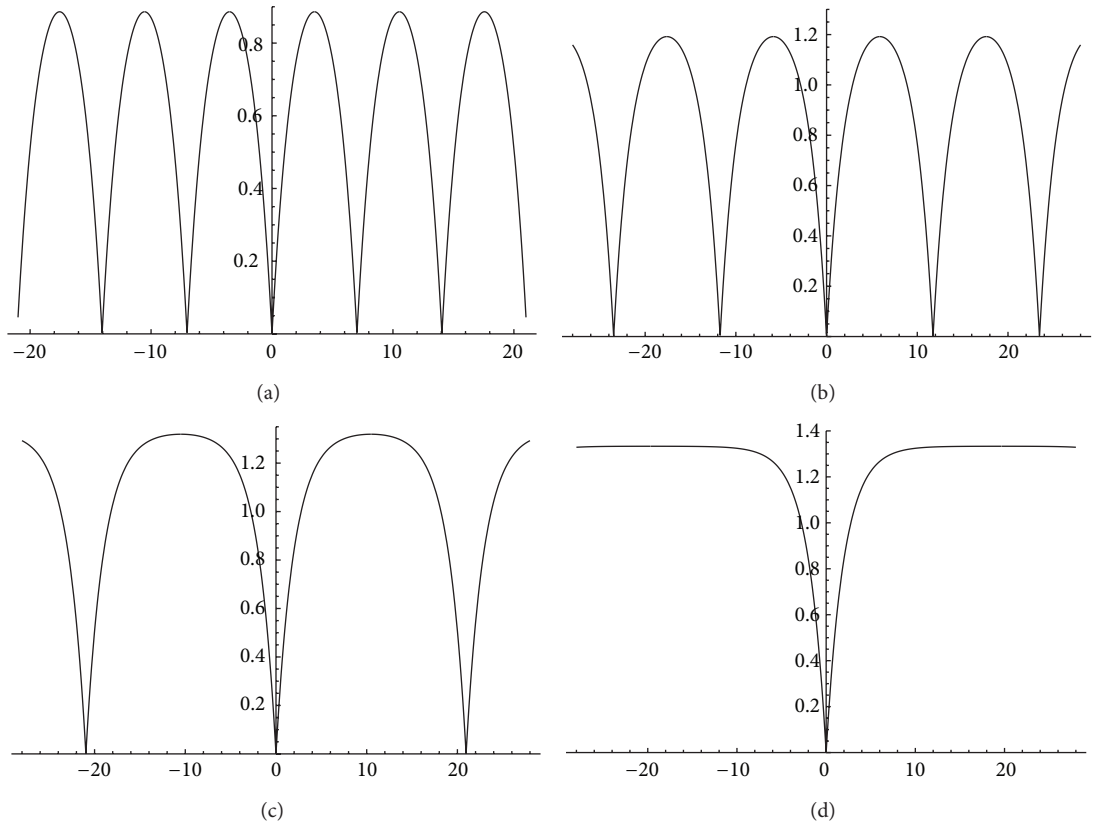


Figure 1: The varying process for the periodic peakon wave $u_2(\xi)$ tends to the peakon wave $u_1(\xi)$ when $g \to g_1 + 0$, where $a = 1$, $c = -1$, and $b = k = 1/2$, and (a) $g = g_1 + 10^{-1}$; (b) $g = g_1 + 10^{-2}$; (c) $g = g_1 + 10^{-4}$; (d) $g = g_1 + 10^{-8}$.

Proposition 4. *Consider ZK-BBM(3, 2) equation*

$$u_t + u_x - a(u^3)_x + (b(u^2)_{xt} + k(u^2)_{yt})_x = 0 \tag{17}$$

and its traveling wave equation

$$g + (1-c)\varphi - a\varphi^3 - 2c(b+k)(\varphi')^2 - 2c(b+k)\varphi\varphi'' = 0. \tag{18}$$

There are the following results.

(1°) When $c(b+k) > 0$, $c < 1$, and $g < -g_2$, (17) has two elliptic periodic blow-up solutions

$$\begin{aligned} u_5(\xi) &= \varphi_1 + A_1 - \frac{2A_1}{1 - \operatorname{cn}(\eta_3^+\xi, k_1^+)}, \\ u_6(\xi) &= \varphi_1 + A_1 - \frac{2A_1}{1 + \operatorname{cn}(\eta_3^+\xi, k_1^+)}, \end{aligned} \tag{19}$$

where

$$A_1 = \sqrt{(\varphi_1 - \varphi_2)(\varphi_1 - \varphi_3)}, \tag{20}$$

$$\eta_3^+ = \sqrt{\frac{aA_1}{5c(b+k)}}, \tag{21}$$

$$k_1^+ = \sqrt{\frac{2A_1 + 2\varphi_1 - \varphi_2 - \varphi_3}{4A_1}}, \tag{22}$$

$$\varphi_1 = \frac{2\sqrt[3]{100}a(c-1) - \sqrt[3]{10}\Omega^{2/3}}{6a\Omega^{1/3}}, \tag{23}$$

$$\varphi_2 = \frac{\sqrt[3]{5}\left(2a\sqrt[3]{10}\left(1-\sqrt{3}i\right)(1-c) + \left(1+\sqrt{3}i\right)\Omega^{2/3}\right)}{6\sqrt[3]{4}a\Omega^{1/3}}, \tag{24}$$

$$\varphi_3 = \frac{\sqrt[3]{5}\left(2a\sqrt[3]{10}\left(1+\sqrt{3}i\right)(1-c) + \left(1-\sqrt{3}i\right)\Omega^{2/3}\right)}{6\sqrt[3]{4}a\Omega^{1/3}}, \tag{25}$$

$$\Omega = \sqrt{729g^2a^4 - 80(1-c)^3a^3} - 27ga^2. \tag{26}$$

(2°) When $c(b+k) > 0$, $c < 1$, and $-g_2 < g < g_2$, (17) has two elliptic periodic blow-up solutions $u_7(\xi)$, $u_8(\xi)$ and two symmetric elliptic periodic wave solutions $u_9(\xi)$, $u_{10}(\xi)$

$$\begin{aligned} u_7(\xi) &= \varphi_3 - (\varphi_3 - \varphi_1)\operatorname{sn}^{-2}(\eta_4^+\xi, k_2), \\ u_8(\xi) &= \frac{\varphi_1 - \varphi_2\operatorname{sn}^2(\eta_4^+\xi, k_2)}{1 - \operatorname{sn}^2(\eta_4^+\xi, k_2)}, \\ u_9(\xi) &= \varphi_3 - (\varphi_3 - \varphi_2)\operatorname{sn}^2(\eta_4^+\xi, k_2), \\ u_{10}(\xi) &= \frac{\varphi_2 - \varphi_1 k_2^2\operatorname{sn}^2(\eta_4^+\xi, k_2)}{1 - k_2^2\operatorname{sn}^2(\eta_4^+\xi, k_2)}, \end{aligned} \tag{27}$$

where

$$\eta_4^+ = \sqrt{\frac{a(\varphi_3-\varphi_1)}{20c(b+k)}},$$
$$k_2 = \sqrt{\frac{\varphi_3-\varphi_2}{\varphi_3-\varphi_1}}. \tag{28}$$

(3°) When $c(b+k) > 0$, $c < 1$, and $g > g_2$, (17) has two elliptic periodic blow-up solutions

$$u_{11}(\xi) = \varphi_3 + A_2 - \frac{2A_2}{1-\mathrm{cn}(\eta_5^+\xi, k_3^+)},$$
$$u_{12}(\xi) = \varphi_3 + A_2 - \frac{2A_2}{1+\mathrm{cn}(\eta_5^+\xi, k_3^+)}, \tag{29}$$

where

$$A_2 = \sqrt{(\varphi_3-\varphi_2)(\varphi_3-\varphi_1)},$$
$$\eta_5^+ = \sqrt{\frac{aA_2}{5c(b+k)}},$$
$$k_3^+ = \sqrt{\frac{2A_2+2\varphi_3-\varphi_2-\varphi_1}{4A_2}}. \tag{30}$$

(4°) When $c(b+k) < 0$, $c < 1$, and $g > g_2$, (17) has two elliptic periodic blow-up solutions

$$u_{13}(\xi) = \varphi_3 - A_2 + \frac{2A_2}{1-\mathrm{cn}(\eta_5^-\xi, k_3^-)},$$
$$u_{14}(\xi) = \varphi_3 - A_2 + \frac{2A_2}{1+\mathrm{cn}(\eta_5^-\xi, k_3^-)}, \tag{31}$$

where

$$\eta_5^- = \sqrt{\frac{-aA_2}{5c(b+k)}},$$
$$k_3^- = \sqrt{\frac{2A_2-2\varphi_3+\varphi_2+\varphi_1}{4A_2}}. \tag{32}$$

(5°) When $c(b+k) < 0$, $c < 1$, and $-g_2 < g < g_2$, (17) has two elliptic periodic blow-up solutions $u_{15}(\xi)$, $u_{16}(\xi)$ and two symmetric elliptic periodic wave solutions $u_{17}(\xi)$, $u_{18}(\xi)$

$$u_{15}(\xi) = \varphi_1 + (\varphi_3-\varphi_1)\,\mathrm{sn}^{-2}(\eta_4^-\xi, k_4),$$
$$u_{16}(\xi) = \frac{\varphi_3-\varphi_2\mathrm{sn}^2(\eta_4^-\xi, k_4)}{1-\mathrm{sn}^2(\eta_4^-\xi, k_4)},$$
$$u_{17}(\xi) = \varphi_1 + (\varphi_2-\varphi_1)\,\mathrm{sn}^2(\eta_4^-\xi, k_4),$$
$$u_{18}(\xi) = \frac{\varphi_2-\varphi_3k_4^2\mathrm{sn}^2(\eta_4^-\xi, k_4)}{1-k_4^2\mathrm{sn}^2(\eta_4^-\xi, k_4)}, \tag{33}$$

where

$$\eta_4^- = \sqrt{\frac{-a(\varphi_3-\varphi_1)}{20c(b+k)}},$$
$$k_4 = \sqrt{\frac{\varphi_2-\varphi_1}{\varphi_3-\varphi_1}}. \tag{34}$$

(6°) When $c(b+k) < 0$, $c < 1$, and $g < -g_2$, (17) has two elliptic periodic blow-up solutions

$$u_{19}(\xi) = \varphi_1 - A_1 + \frac{2A_1}{1+\mathrm{cn}(\eta_3^-\xi, k_1^-)},$$
$$u_{20}(\xi) = \varphi_1 - A_1 + \frac{2A_1}{1-\mathrm{cn}(\eta_3^-\xi, k_1^-)}, \tag{35}$$

where

$$\eta_3^- = \sqrt{\frac{-aA_1}{5c(b+k)}},$$
$$k_1^- = \sqrt{\frac{2A_1-2\varphi_1+\varphi_2+\varphi_3}{4A_1}}. \tag{36}$$

Remark 5. When $c(b+k) > 0$, $c < 1$, and $g \to -g_2$, the periodic blow-up solutions $u_5(\xi)$ (or $u_7(\xi)$) and $u_6(\xi)$ (or $u_8(\xi)$) tend to two trigonometric periodic blow-up solutions, respectively,

$$u_{21}(\xi) = \sqrt{\frac{5(1-c)}{9a}}\left(1-3\csc^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)\right), \tag{37}$$

$$u_{22}(\xi) = \sqrt{\frac{5(1-c)}{9a}}\left(1-3\sec^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)\right). \tag{38}$$

The symmetric elliptic periodic wave solutions $u_9(\xi)$ and $u_{10}(\xi)$ become a trivial solution $u_{23}(\xi) = \sqrt{5(1-c)/9a}$.

Remark 6. When $c(b+k) > 0$, $c < 1$, and $g \to g_2$, the periodic blow-up solution $u_7(\xi)$ (or $u_{11}(\xi)$) tends to a hyperbolic blow-up solution

$$u_{24}(\xi) = -\sqrt{\frac{5(1-c)}{9a}}\left(1+3\,\mathrm{csch}^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)\right). \tag{39}$$

The elliptic periodic wave solution $u_9(\xi)$ (or the elliptic periodic blow-up solution $u_{12}(\xi)$) tends to a hyperbolic smooth solitary wave solution

$$u_{25}(\xi) = -\sqrt{\frac{5(1-c)}{9a}}\left(1-3\,\mathrm{sech}^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)\right). \tag{40}$$

For the varying process, see Figures 2, 3, and 4. The elliptic solutions $u_8(\xi)$ and $u_{10}(\xi)$ tend to a trivial solution $u_{26}(\xi) = -\sqrt{(5(1-c))/9a}$.

Remark 7. When $c(b+k) < 0$, $c < 1$, and $g \to g_2$, the periodic blow-up solutions $u_{13}(\xi)$ (or $u_{15}(\xi)$) and $u_{14}(\xi)$

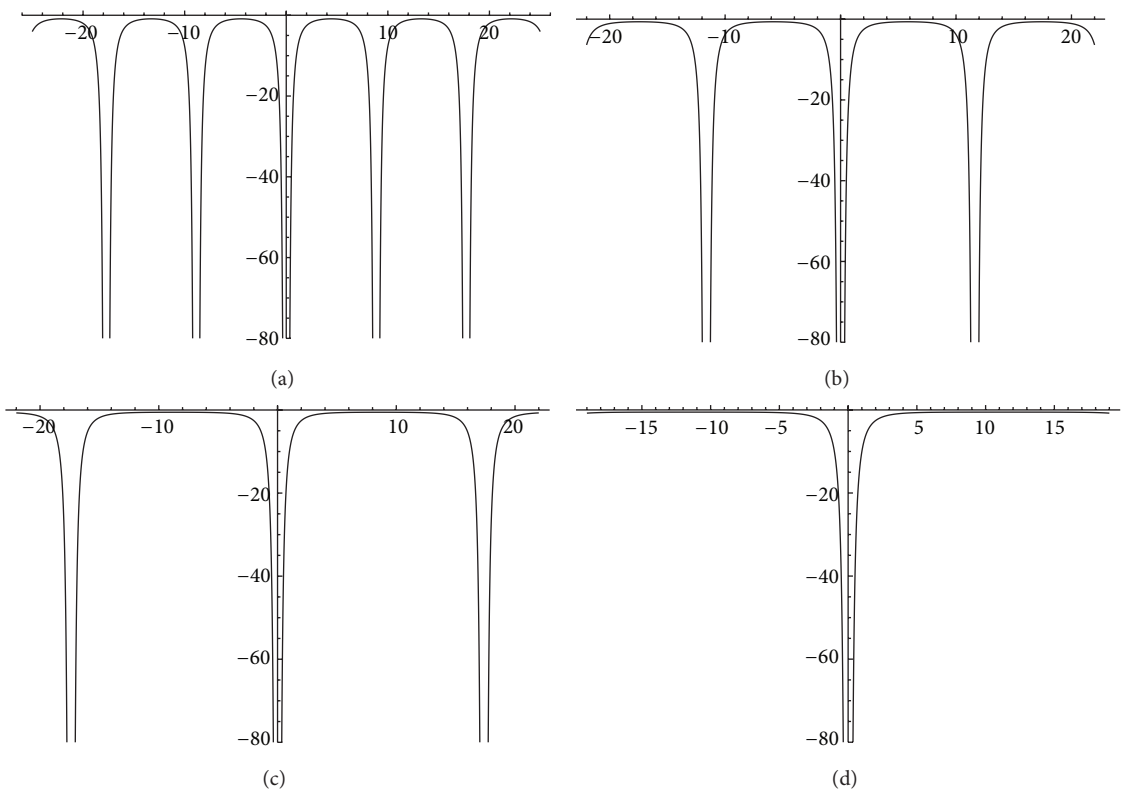


FIGURE 2: The varying process for graphs of $u_7(\xi)$ when $g \to g_2 - 0$, where $a = 1$, $c = b = k = 1/2$, and (a) $g = g_2 - 10^{-1}$; (b) $g = g_2 - 10^{-2}$; (c) $g = g_2 - 10^{-4}$; (d) $g = g_2 - 10^{-6}$.

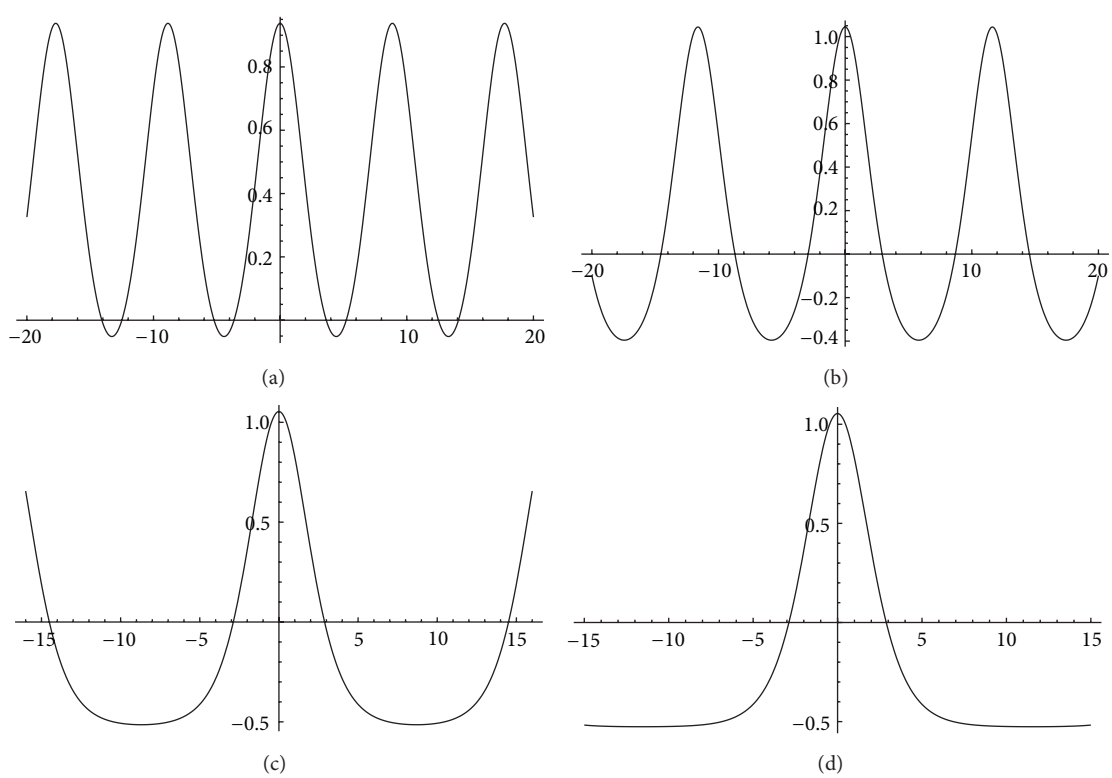


FIGURE 3: The varying process for graphs of $u_9(\xi)$ when $g \to g_2 - 0$, where $a = 1$, $c = b = k = 1/2$, and (a) $g = g_2 - 10^{-1}$; (b) $g = g_2 - 10^{-2}$; (c) $g = g_2 - 10^{-4}$; (d) $g = g_2 - 10^{-6}$.

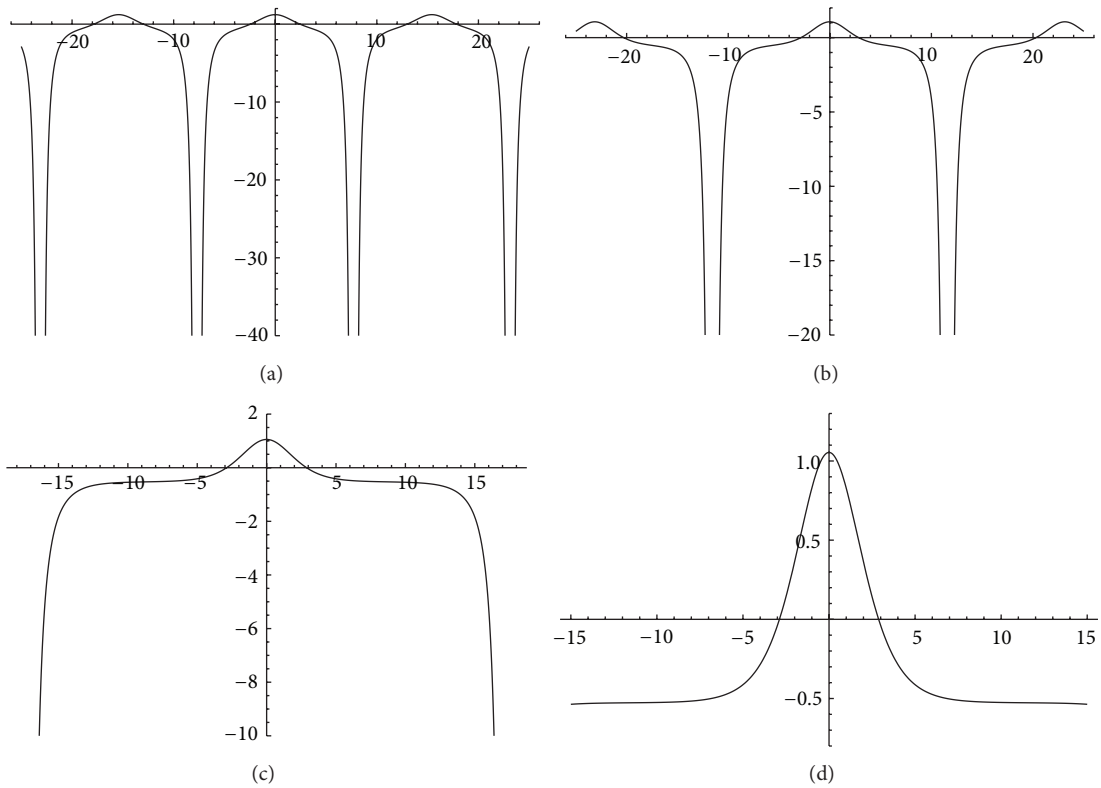


Figure 4: The varying process for graphs of $u_{12}(\xi)$ when $g \to g_2 + 0$, where $a = 1$, $c = b = k = 1/2$, and (a) $g = g_2 + 1/5$; (b) $g = g_2 + 10^{-2}$; (c) $g = g_2 + 10^{-4}$; (d) $g = g_2 + 10^{-6}$.

(or $u_{16}(\xi)$) tend to two trigonometric periodic blow-up solutions, respectively,

$$u_{27}(\xi) = -\sqrt{\frac{5(1-c)}{9a}}\left(1 - 3\sec^2\left(\sqrt{\frac{-a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)\right),$$
$$u_{28}(\xi) = -\sqrt{\frac{5(1-c)}{9a}}\left(1 - 3\csc^2\left(\sqrt{\frac{-a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)\right). \tag{41}$$

The symmetric elliptic periodic wave solutions $u_{17}(\xi)$ and $u_{18}(\xi)$ become the trivial solution $u_{26}(\xi)$.

Remark 8. When $c(b+k) < 0$, $c < 1$, and $g \to -g_2$, the periodic blow-up solution $u_{15}(\xi)$ (or $u_{20}(\xi)$) tends to a hyperbolic blow-up solution

$$u_{29}(\xi) = \sqrt{\frac{5(1-c)}{9a}}\left(1 + 3\operatorname{csch}^2\left(\sqrt{\frac{-a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)\right). \tag{42}$$

The elliptic periodic wave solution $u_{17}(\xi)$ (or the elliptic periodic blow-up solution $u_{19}(\xi)$) tends to a hyperbolic smooth solitary wave solution

$$u_{30}(\xi) = \sqrt{\frac{5(1-c)}{9a}}\left(1 - 3\operatorname{sech}^2\left(\sqrt{\frac{-a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)\right). \tag{43}$$

The varying process is similar to those in Figures 2–4. The elliptic solutions $u_{16}(\xi)$ and $u_{18}(\xi)$ tend to the trivial solution $u_{23}(\xi)$.

Remark 9. When $g \to 0$, the solutions $u_{10}(\xi)$ and $u_{18}(\xi)$, respectively, become

$$\begin{aligned} u_{10}^{\circ}(\xi) = {} & \left(\sqrt{\frac{5(1-c)}{3a}}\operatorname{sn}^2\left(\sqrt{\frac{a}{10c(b+k)}}\right.\right. \\ & \left.\left.\times\sqrt[4]{\frac{5(1-c)}{3a}}\xi, \frac{\sqrt{2}}{2}\right)\right) \\ & \times\left(2 - \operatorname{sn}^2\left(\sqrt{\frac{a}{10c(b+k)}}\right.\right. \\ & \left.\left.\times\sqrt[4]{\frac{5(1-c)}{3a}}\xi, \frac{\sqrt{2}}{2}\right)\right)^{-1}, \end{aligned}$$

for $c(b+k) > 0$, $c < 1$,

$$\begin{aligned} u_{18}^{\circ}(\xi) = {} & \left(\sqrt{\frac{5(1-c)}{3a}}\operatorname{sn}^2\left(\sqrt{\frac{-a}{10c(b+k)}}\right.\right. \\ & \left.\left.\times\sqrt[4]{\frac{5(1-c)}{3a}}\xi, \frac{\sqrt{2}}{2}\right)\right) \end{aligned}$$

$$\times\left(\mathrm{sn}^2\left(\sqrt{\frac{-a}{10c(b+k)}}\times\sqrt[4]{\frac{5(1-c)}{3a}}\xi,\frac{\sqrt{2}}{2}\right)-2\right)^{-1},\quad \text{for } c(b+k)<0,\ c<1, \tag{44}$$

which can be found in [20]; this implies that we extend the previous results.

3. The Derivations for Proposition 1

In this section, we derive the precise expressions of the traveling wave solutions for ZK-BBM(2, 2) equation. Substituting $u=\varphi(\xi)$ with $\xi=x+y-ct$ into (7), it follows that

$$(1-c)\varphi'-2a\varphi\varphi'-2c(b+k)\left(3\varphi'\varphi''+\varphi\varphi''\right)=0. \tag{45}$$

Integrating (45) once, we have

$$g+(1-c)\varphi-a\varphi^2-2c(b+k)\left(\varphi'\right)^2+2c(b+k)\varphi\varphi''=0, \tag{46}$$

where g is an integral constant.

Letting $\psi=\varphi'$, we obtain the following planar system:

$$\begin{aligned}\frac{d\varphi}{d\xi}&=\psi,\\ \frac{d\psi}{d\xi}&=\frac{g+(1-c)\varphi-a\varphi^2-2c(b+k)\psi^2}{2c(b+k)\varphi}.\end{aligned} \tag{47}$$

Under the transformation $d\xi=2c(b+k)\varphi d\tau$, system (47) becomes

$$\begin{aligned}\frac{d\varphi}{d\tau}&=2c(b+k)\varphi\psi,\\ \frac{d\psi}{d\tau}&=g+(1-c)\varphi-a\varphi^2-2c(b+k)\psi^2.\end{aligned} \tag{48}$$

Clearly, system (47) and system (48) have the same first integral

$$2c(b+k)\varphi^2\psi^2+\frac{a}{2}\varphi^4-\frac{2(1-c)}{3}\varphi^3-g\varphi^2=h, \tag{49}$$

where h is an integral constant. Consequently, these two systems have the same topological phase portraits except for the straight line $\varphi=0$. Thus, we can understand the phase portraits of system (47) from those of system (48).

When the integral constant $h=0$, (49) becomes

$$2c(b+k)\psi^2+\frac{a}{2}\varphi^2-\frac{2(1-c)}{3}\varphi-g=0. \tag{50}$$

Solving equation $(a/2)\varphi^2-(2(1-c)/3)\varphi-g=0$, we get two roots

$$\varphi_\pm^*=\frac{2(1-c)\pm\sqrt{4(1-c)^2+18ag}}{3a},\quad \text{where } g\geq-\frac{2(1-c)^2}{9a}. \tag{51}$$

On the other hand, solving equation $g+(1-c)\varphi-a\varphi^2$, we obtain

$$\varphi_\pm^\circ=\frac{1-c\pm\sqrt{(1-c)^2+4ag}}{2a},\quad \text{where } g\geq-\frac{(1-c)^2}{4a}. \tag{52}$$

According to the qualitative theory, we obtain the phase portraits of system (48) as shown in Figure 5.

When $g=g_1$, on $\varphi-\psi$ plane the orbit Γ_1 has expression

$$\Gamma_1:\psi=\begin{cases}\pm\eta_1^-\left(\dfrac{2(1-c)}{3a}-\varphi\right), & \text{for } c(b+k)<0,\ c<1,\\ 0\leq\varphi<\dfrac{2(1-c)}{3a}\\ \pm\eta_1^-\left(\varphi-\dfrac{2(1-c)}{3a}\right), & \text{for } c>1,\ b+k<0.\\ \dfrac{2(1-c)}{3a}<\varphi\leq0\end{cases} \tag{53}$$

Substituting (53) into $d\varphi/d\xi$ and integrating it along the orbit Γ_1, we obtain the peakon wave solution $u_1(\xi)$ as (9).

When $g_1<g<0$, on $\varphi-\psi$ plane the orbit Γ_2 has expression

$$\Gamma_2:\psi=\begin{cases}\pm\eta_1^-\sqrt{(\varphi_+^*-\varphi)(\varphi_-^*-\varphi)}, & \text{for } c(b+k)<0,\ c<1,\\ 0\leq\varphi\leq\varphi_-^*\\ \pm\eta_1^-\sqrt{(\varphi-\varphi_+^*)(\varphi-\varphi_-^*)}, & \text{for } c>1,\ b+k<0.\\ \varphi_+^*\leq\varphi\leq0\end{cases} \tag{54}$$

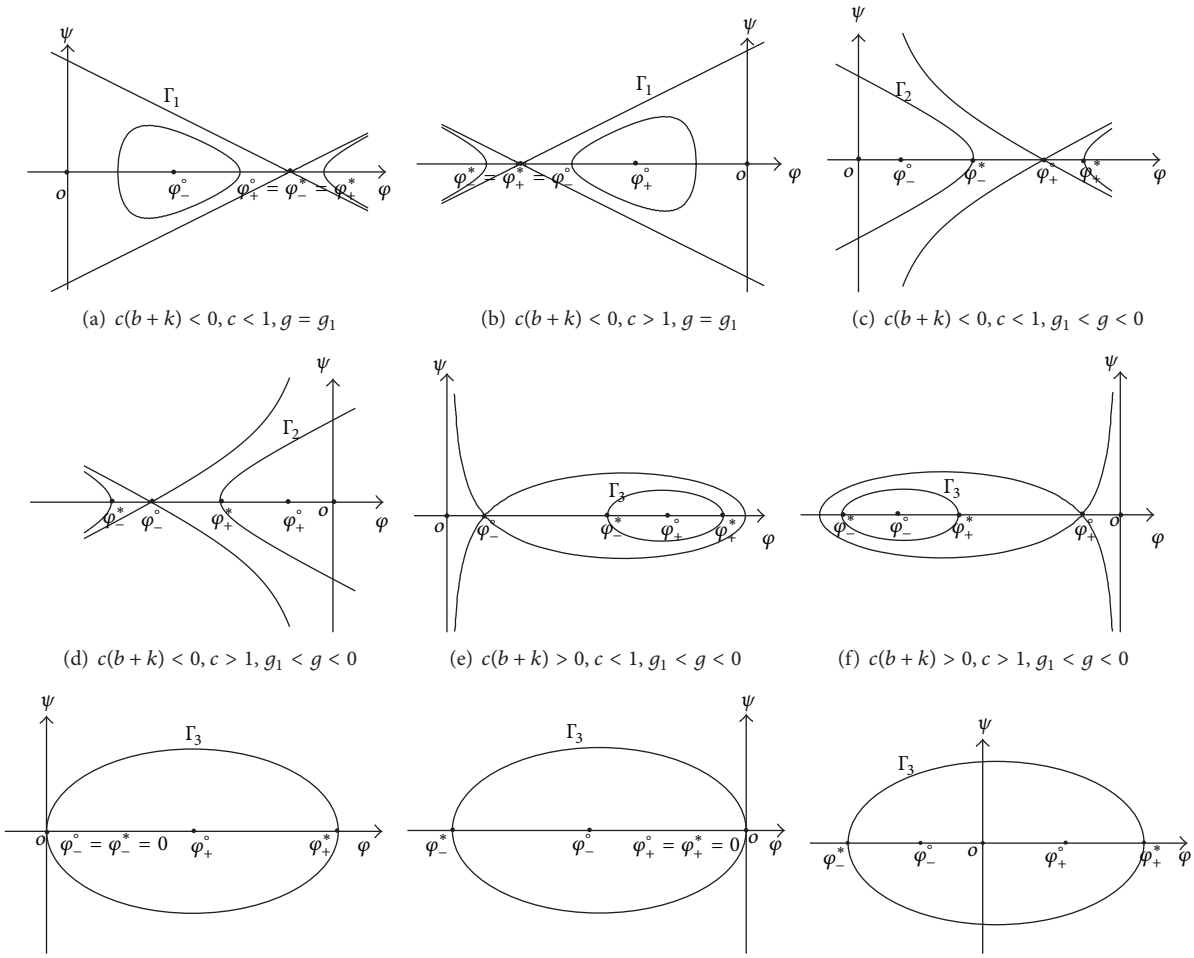
Substituting (54) into $d\varphi/d\xi$ and integrating it along the orbit Γ_2, we obtain the periodic peakon wave solution $u_2(\xi)$ as (11), where

$$\begin{aligned}T&=\frac{1}{\eta_1^-}\left|\int_0^{\varphi_-^*}\frac{ds}{\sqrt{(\varphi_+^*-s)(\varphi_-^*-s)}}\right|\\ &=\sqrt{\frac{4c(b+k)}{-a}}\ln\left|\frac{\sqrt{4(1-c)^2+18ag}}{3\sqrt{-2ag}-2(1-c)}\right|.\end{aligned} \tag{55}$$

If $g\to g_1+0$, it follows that $\alpha\to(c-1)/3a$, $\beta\to0$ (or $\alpha\to0$, $\beta\to(c-1)/3a$), and $T\to+\infty$. This implies that the periodic peakon wave solution $u_2(\xi)$ tends to the peakon wave solution $u_1(\xi)$.

When $g_1<g$ and $c(b+k)>0$, on $\varphi-\psi$ plane the orbit Γ_3 has expression

$$\Gamma_3:\psi=\pm\eta_1^+\sqrt{(\varphi-\varphi_-^*)(\varphi_+^*-\varphi)},\quad \varphi_-^*\leq\varphi\leq\varphi_+^*. \tag{56}$$



(a) $c(b+k)<0, c<1, g=g_1$ (b) $c(b+k)<0, c>1, g=g_1$ (c) $c(b+k)<0, c<1, g_1<g<0$

(d) $c(b+k)<0, c>1, g_1<g<0$ (e) $c(b+k)>0, c<1, g_1<g<0$ (f) $c(b+k)>0, c>1, g_1<g<0$

(g) $c(b+k)>0, c<1, g=0$ (h) $c(b+k)>0, c>1, g=0$ (i) $c(b+k)>0\ g>0$

Figure 5: The phase portraits of system (48).

Substituting (56) into $d\varphi/d\xi$ and integrating it along the orbit Γ_3, we obtain the smooth periodic wave solutions $u_3(\xi)$ and $u_4(\xi)$ as (13).

Hereto, we have completed the derivations for Proposition 1.

4. The Derivations for Proposition 4

In this section, we derive the explicit elliptic function solutions and their limit forms for ZK-BBM(3, 2) equation. Similar to the derivations in Section 3, substituting $u=\varphi(\xi)$ with $\xi=x+y-ct$ into (17) and integrating it, we have the following planar system:

$$\begin{aligned} \frac{d\varphi}{d\xi} &= \psi, \\ \frac{d\psi}{d\xi} &= \frac{g+(1-c)\varphi-a\varphi^3-2c(b+k)\psi^2}{2c(b+k)\varphi}. \end{aligned} \tag{57}$$

Similarly, under the transformation $d\xi=2c(b+k)\varphi d\tau$, system (57) becomes

$$\begin{aligned} \frac{d\varphi}{d\tau} &= 2c(b+k)\varphi\psi, \\ \frac{d\psi}{d\tau} &= g+(1-c)\varphi-a\varphi^3-2c(b+k)\psi^2, \end{aligned} \tag{58}$$

which has the first integral

$$2c(b+k)\varphi^2\psi^2+\frac{2a}{5}\varphi^5-\frac{2(1-c)}{3}\varphi^3-g\varphi^2=h. \tag{59}$$

When the integral constant $h=0$, (59) becomes

$$c(b+k)\varphi^2\psi^2-\varphi^2\left(\frac{g}{2}+\frac{1-c}{3}\varphi-\frac{a}{5}\varphi^3\right)=0. \tag{60}$$

Solving equation $g/2+((1-c)/3)\varphi-(a/5)\varphi^3=0$, we get three roots φ_1, φ_2, and φ_3 as (23), (24), and (25). On the other hand, solving equations

$$\begin{aligned} \psi &= 0, \\ g+(1-c)\varphi-a\varphi^3-2c(b+k)\psi^2 &= 0, \end{aligned} \tag{61}$$

we get three equilibrium points $(\varphi_i^*, 0)$ $(i = 1, 2, 3)$ of system (58), where

$$\begin{aligned}
\varphi_1^* &= \frac{2\sqrt[3]{18}a(c-1) - \sqrt[3]{12}\Delta^{2/3}}{6a\Delta^{1/3}},\\
\varphi_2^* &= \frac{2a\sqrt[3]{2}\sqrt[6]{3}\left(\sqrt{3}-3i\right)(1-c) + \sqrt[3]{4}\sqrt[6]{9}\left(1+\sqrt{3}i\right)\Delta^{2/3}}{12a\Delta^{1/3}},\\
\varphi_3^* &= \frac{2a\sqrt[3]{2}\sqrt[6]{3}\left(\sqrt{3}+3i\right)(1-c) + \sqrt[3]{4}\sqrt[6]{9}\left(1-\sqrt{3}i\right)\Delta^{2/3}}{12a\Delta^{1/3}},\\
\Delta &= \sqrt{81g^2a^4 + 12(c-1)^3a^3} - 9ga^2.
\end{aligned} \tag{62}$$

According to the qualitative theory, we obtain the phase portraits of system (58) as shown in Figure 6.

Now using planar system (57) and the phase portraits in Figure 6, we derive the explicit expressions of solutions for the ZK-BBM(3, 2) equation respectively.

When $c(b+k) > 0$, $c < 1$, and $g < -g_2$, Γ_4 has the expression

$$\Gamma_4 : \psi = \pm\sqrt{\frac{a}{5c(b+k)}}\sqrt{(\varphi_1-\varphi)(\varphi-\varphi_2)(\varphi-\varphi_3)}, \quad \varphi \le \varphi_1, \tag{63}$$

where φ_2 and φ_3 are complex numbers.

Substituting (63) into $d\varphi/d\xi = \psi$ and integrating it, we have

$$\begin{aligned}
\int_{-\infty}^{\varphi} \frac{ds}{\sqrt{(\varphi_1-s)(s-\varphi_2)(s-\varphi_3)}} &= \sqrt{\frac{a}{5c(b+k)}}\,|\xi|,\\
\int_{\varphi}^{\varphi_1} \frac{ds}{\sqrt{(\varphi_1-s)(s-\varphi_2)(s-\varphi_3)}} &= \sqrt{\frac{a}{5c(b+k)}}\,|\xi|.
\end{aligned} \tag{64}$$

Completing the integrals in the above two equations and noting that $u = \varphi(\xi)$, we obtain $u_5(\xi)$ and $u_6(\xi)$ as (19).

When $c(b+k) > 0$, $c < 1$, and $-g_2 < g < g_2$, Γ_4 and Γ_5 have the expressions

$$\begin{aligned}
\Gamma_4 : y &= \pm\sqrt{\frac{a}{5c(b+k)}}\sqrt{(\varphi_1-\varphi)(\varphi_2-\varphi)(\varphi_3-\varphi)}, \quad \varphi \le \varphi_1,\\
\Gamma_5 : y &= \pm\sqrt{\frac{a}{5c(b+k)}}\sqrt{(\varphi-\varphi_1)(\varphi-\varphi_2)(\varphi_3-\varphi)}, \quad \varphi_2 \le \varphi \le \varphi_3.
\end{aligned} \tag{65}$$

Substituting (65) into $d\varphi/d\xi = \psi$ and integrating them, we have

$$\begin{aligned}
\int_{-\infty}^{\varphi} \frac{ds}{\sqrt{(\varphi_1-s)(\varphi_2-s)(\varphi_3-s)}} &= \sqrt{\frac{a}{5c(b+k)}}\,|\xi|,\\
\int_{\varphi}^{\varphi_1} \frac{ds}{\sqrt{(\varphi_1-s)(\varphi_2-s)(\varphi_3-s)}} &= \sqrt{\frac{a}{5c(b+k)}}\,|\xi|,\\
\int_{\varphi_2}^{\varphi} \frac{ds}{\sqrt{(s-\varphi_1)(s-\varphi_2)(\varphi_3-s)}} &= \sqrt{\frac{a}{5c(b+k)}}\,|\xi|,\\
\int_{\varphi}^{\varphi_3} \frac{ds}{\sqrt{(s-\varphi_1)(s-\varphi_2)(\varphi_3-s)}} &= \sqrt{\frac{a}{5c(b+k)}}\,|\xi|.
\end{aligned} \tag{66}$$

Completing the integrals in the above four equations and noting that $u = \varphi(\xi)$, we obtain $u_i(\xi)$ $(i = 7\text{–}10)$ as (27).

When $c(b+k) > 0$, $c < 1$, and $g_2 < g$, Γ_6 has the expression

$$\Gamma_6 : y = \pm\sqrt{\frac{a}{5c(b+k)}}\sqrt{(\varphi_3-\varphi)(\varphi-\varphi_2)(\varphi-\varphi_1)}, \quad \varphi \le \varphi_3, \tag{67}$$

where φ_1 and φ_2 are complex numbers.

Substituting (67) into $d\varphi/d\xi = \psi$ and integrating it, we have

$$\begin{aligned}
\int_{-\infty}^{\varphi} \frac{ds}{\sqrt{(\varphi_3-s)(s-\varphi_2)(s-\varphi_1)}} &= \sqrt{\frac{a}{5c(b+k)}}\,|\xi|,\\
\int_{\varphi}^{\varphi_3} \frac{ds}{\sqrt{(\varphi_3-s)(s-\varphi_2)(s-\varphi_1)}} &= \sqrt{\frac{a}{5c(b+k)}}\,|\xi|.
\end{aligned} \tag{68}$$

Completing the integrals in the above two equations and noting that $u = \varphi(\xi)$, we obtain $u_{11}(\xi)$ and $u_{12}(\xi)$ as (29).

When $c(b+k) < 0$, $c < 1$, and $g_2 < g$, Γ_7 has the expression

$$\Gamma_7 : y = \pm\sqrt{\frac{-a}{5c(b+k)}}\sqrt{(\varphi-\varphi_3)(\varphi-\varphi_2)(\varphi-\varphi_1)}, \quad \varphi \ge \varphi_3, \tag{69}$$

where φ_1 and φ_2 are complex numbers.

Substituting (69) into $d\varphi/d\xi = \psi$ and integrating it, we have

$$\begin{aligned}
\int_{\varphi_3}^{\varphi} \frac{ds}{\sqrt{(s-\varphi_3)(s-\varphi_2)(s-\varphi_1)}} &= \sqrt{\frac{-a}{5c(b+k)}}\,|\xi|,\\
\int_{\varphi}^{+\infty} \frac{ds}{\sqrt{(s-\varphi_3)(s-\varphi_2)(s-\varphi_1)}} &= \sqrt{\frac{-a}{5c(b+k)}}\,|\xi|.
\end{aligned} \tag{70}$$

Completing the integrals in the above two equations and noting that $u = \varphi(\xi)$, we obtain $u_{13}(\xi)$ and $u_{14}(\xi)$ as (31).

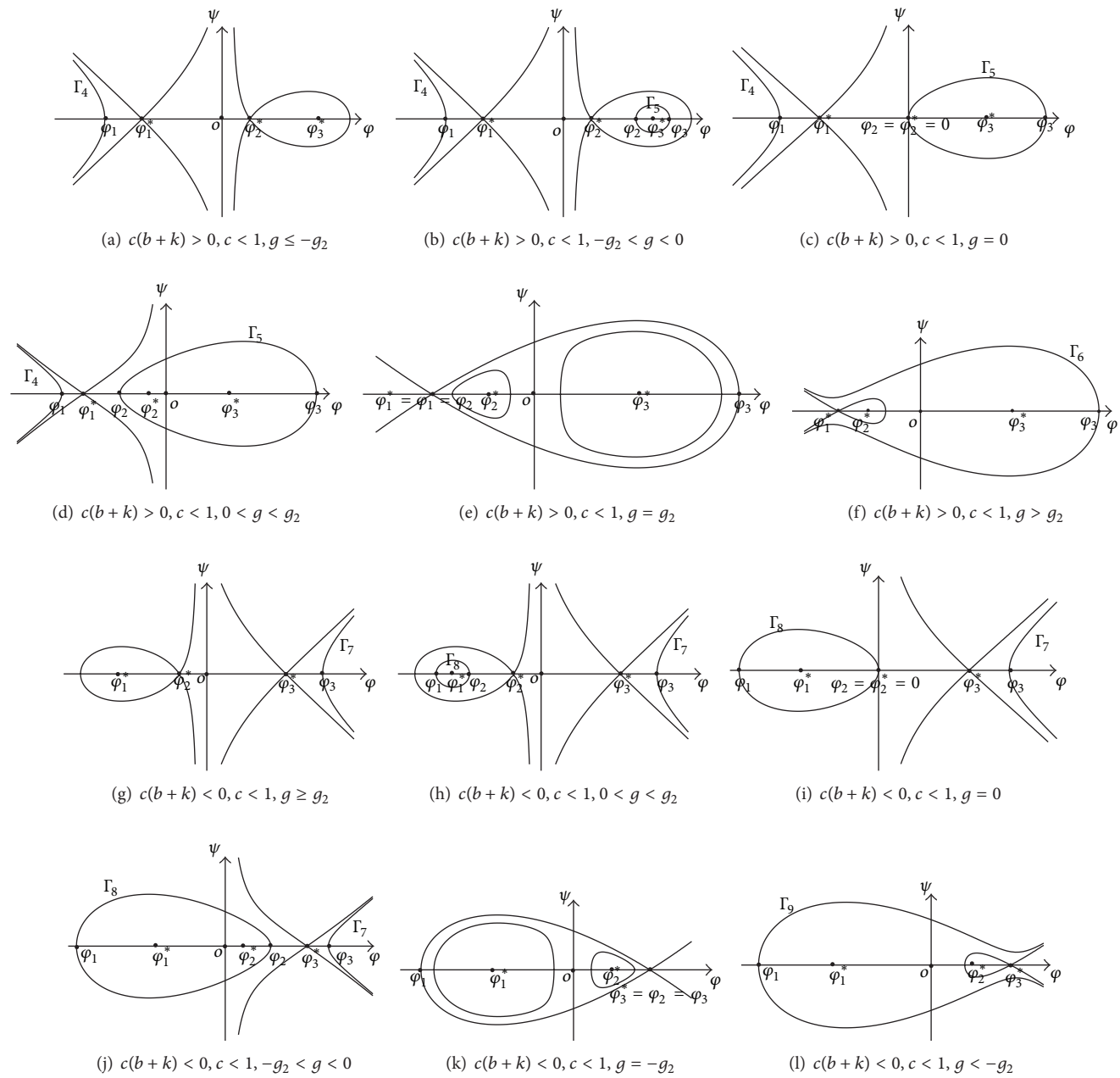


Figure 6: The phase portraits of system (58).

When $c(b+k)<0$, $c<1$, and $-g_2<g<g_2$, Γ_7 and Γ_8 have the expressions

$$\begin{aligned}\Gamma_7 : y &= \pm\sqrt{\frac{-a}{5c(b+k)}}\sqrt{(\varphi-\varphi_1)(\varphi-\varphi_2)(\varphi-\varphi_3)}, \\ &\qquad \varphi\ge\varphi_3, \\ \Gamma_8 : y &= \pm\sqrt{\frac{-a}{5c(b+k)}}\sqrt{(\varphi_3-\varphi)(\varphi_2-\varphi)(\varphi-\varphi_1)}, \\ &\qquad \varphi_1\le\varphi\le\varphi_2.\end{aligned} \tag{71}$$

Substituting (71) into $d\varphi/d\xi=\psi$ and integrating them, we have

$$\int_{\varphi_3}^{\varphi}\frac{ds}{\sqrt{(s-\varphi_1)(s-\varphi_2)(s-\varphi_3)}}=\sqrt{\frac{-a}{5c(b+k)}}\,|\xi|,$$

$$\int_{\varphi}^{+\infty}\frac{ds}{\sqrt{(s-\varphi_1)(s-\varphi_2)(s-\varphi_3)}}=\sqrt{\frac{-a}{5c(b+k)}}\,|\xi|,$$

$$\int_{\varphi_1}^{\varphi} \frac{ds}{\sqrt{(\varphi_3 - s)(\varphi_2 - s)(s - \varphi_1)}} = \sqrt{\frac{-a}{5c(b+k)}}\,|\xi|,$$
$$\int_{\varphi}^{\varphi_2} \frac{ds}{\sqrt{(\varphi_3 - s)(\varphi_2 - s)(s - \varphi_1)}} = \sqrt{\frac{-a}{5c(b+k)}}\,|\xi|. \tag{72}$$

Completing the integrals in the above four equations and noting that $u = \varphi(\xi)$, we obtain $u_i(\xi)$ $(i = 15\text{–}18)$ as (33).

When $c(b+k) < 0$, $c < 1$, and $g < -g_2$, Γ_9 has the expression

$$\Gamma_9 : y = \pm\sqrt{\frac{-a}{5c(b+k)}}\sqrt{(\varphi - \varphi_1)(\varphi - \varphi_2)(\varphi - \varphi_3)}, \quad \varphi \geq \varphi_1, \tag{73}$$

where φ_2 and φ_3 are complex numbers.

Substituting (73) into $d\varphi/d\xi = \psi$ and integrating it, we have

$$\int_{\varphi_1}^{\varphi} \frac{ds}{\sqrt{(s - \varphi_1)(s - \varphi_2)(s - \varphi_3)}} = \sqrt{\frac{-a}{5c(b+k)}}\,|\xi|,$$
$$\int_{\varphi}^{+\infty} \frac{ds}{\sqrt{(s - \varphi_1)(s - \varphi_2)(s - \varphi_3)}} = \sqrt{\frac{-a}{5c(b+k)}}\,|\xi|. \tag{74}$$

Completing the integrals in the above two equations and noting that $u = \varphi(\xi)$, we obtain $u_{19}(\xi)$ and $u_{20}(\xi)$ as (35).

Hereto, we have finished the derivations for the solutions $u_i(\xi)$ $(i = 5\text{–}20)$. In what follows, we will derive the limit forms of these solutions.

When $c(b+k) > 0$, $c < 1$, and $g \to -g_2$, it follows that

$$\varphi_1 \longrightarrow -\sqrt{\frac{20(1-c)}{9a}},$$
$$\varphi_2 \longrightarrow \sqrt{\frac{5(1-c)}{9a}},$$
$$\varphi_3 \longrightarrow \sqrt{\frac{5(1-c)}{9a}},$$
$$A_1 \longrightarrow \sqrt{\frac{5(1-c)}{a}},$$
$$\eta_3^+ \longrightarrow \sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{5a}}, \qquad \eta_4^+ \longrightarrow \sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}},$$
$$k_1^+ \longrightarrow 0, \qquad k_2 \longrightarrow 0,$$
$$\operatorname{cn}(\eta_3^+\xi, k_1^+) \longrightarrow \operatorname{cn}\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{5a}}\xi, 0\right) = \cos\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{5a}}\xi\right),$$
$$\operatorname{sn}(\eta_4^+\xi, k_2) \longrightarrow \operatorname{sn}\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi, 0\right) = \sin\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right). \tag{75}$$

Thus we have

$$\begin{aligned}
u_5(\xi) &\longrightarrow -\sqrt{\frac{20(1-c)}{9a}} + \sqrt{\frac{5(1-c)}{a}} - \frac{2\sqrt{5(1-c)/a}}{1 - \operatorname{cn}\left(\sqrt{a/c(b+k)}\sqrt[4]{(1-c)/5a}\xi, 0\right)} \\
&= \sqrt{\frac{5(1-c)}{9a}} - \frac{2\sqrt{5(1-c)/a}}{1 - \cos\left(\sqrt{a/c(b+k)}\sqrt[4]{(1-c)/5a}\xi\right)} \\
&= \sqrt{\frac{5(1-c)}{9a}} - \sqrt{\frac{5(1-c)}{a}} \times \csc^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right) \\
&= u_{21}(\xi)\ (\text{see } (37)),
\end{aligned}$$

$$\begin{aligned}
u_6(\xi) &\longrightarrow -\sqrt{\frac{20(1-c)}{9a}} + \sqrt{\frac{5(1-c)}{a}} - \frac{2\sqrt{5(1-c)/a}}{1 + \operatorname{cn}\left(\sqrt{a/c(b+k)}\sqrt[4]{((1-c)/5a)}\xi, 0\right)} \\
&= \sqrt{\frac{5(1-c)}{9a}} - \frac{2\sqrt{5(1-c)/a}}{1 + \cos\left(\sqrt{a/c(b+k)}\sqrt[4]{((1-c)/5a)}\xi\right)} \\
&= \sqrt{\frac{5(1-c)}{9a}} - \sqrt{\frac{5(1-c)}{a}}\sec^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right) \\
&= u_{22}(\xi)\ (\text{see } (38)),
\end{aligned}$$

$$\begin{aligned}
u_7(\xi) &\longrightarrow \sqrt{\frac{5(1-c)}{9a}} - \sqrt{\frac{5(1-c)}{a}} \times \operatorname{sn}^{-2}\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi, 0\right) \\
&= \sqrt{\frac{5(1-c)}{9a}} - \sqrt{\frac{5(1-c)}{a}} \times \sin^{-2}\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right) \\
&= \sqrt{\frac{5(1-c)}{9a}} - \sqrt{\frac{5(1-c)}{a}} \times \csc^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right) \\
&= u_{21}(\xi)\ (\text{see } (37)),
\end{aligned}$$

$$
\begin{aligned}
u_8(\xi) \longrightarrow{}& \left(-\sqrt{\frac{20(1-c)}{9a}} - \sqrt{\frac{5(1-c)}{9a}}\right.\\
&\left.\times \operatorname{sn}^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi, 0\right)\right)\\
&\times\left(1-\operatorname{sn}^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi, 0\right)\right)^{-1}\\
={}& \left(-\sqrt{\frac{20(1-c)}{9a}} - \sqrt{\frac{5(1-c)}{9a}}\right.\\
&\left.\times \sin^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)\right)\\
&\times\left(1-\sin^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)\right)^{-1}\\
={}& \left(-\sqrt{\frac{5(1-c)}{a}} + \sqrt{\frac{5(1-c)}{9a}}\right.\\
&\left.\times \cos^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)\right)\\
&\times\left(\cos^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)\right)^{-1}\\
={}& \sqrt{\frac{5(1-c)}{9a}} - \sqrt{\frac{5(1-c)}{a}}\\
&\times \sec^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)\\
={}& u_{22}(\xi)\ (\text{see } (38)).
\end{aligned}
\tag{76}
$$

When $c(b+k) > 0$, $c < 1$, and $g \to g_2$, it follows that

$$
\begin{gathered}
\varphi_1 \longrightarrow -\sqrt{\frac{5(1-c)}{9a}},\\
\varphi_2 \longrightarrow -\sqrt{\frac{5(1-c)}{9a}},\\
\varphi_3 \longrightarrow \sqrt{\frac{20(1-c)}{9a}},\\
A_2 \longrightarrow \sqrt{\frac{5(1-c)}{a}},\\
\eta_4^+ \longrightarrow \sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}, \qquad \eta_5^+ \longrightarrow \sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{5a}},\\
k_2 \longrightarrow 1, \qquad k_3^+ \longrightarrow 1,\\
\operatorname{sn}(\eta_4^+\xi, k_2) \longrightarrow \operatorname{sn}\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi, 1\right)\\
= \tanh\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right),\\
\operatorname{cn}(\eta_5^+\xi, k_3^+) \longrightarrow \operatorname{cn}\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{5a}}\xi, 1\right)\\
= \operatorname{sech}\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{5a}}\xi\right).
\end{gathered}
\tag{77}
$$

Thus we have

$$
\begin{aligned}
u_7(\xi) \longrightarrow{}& \sqrt{\frac{20(1-c)}{9a}} - \sqrt{\frac{5(1-c)}{a}}\\
&\times \operatorname{sn}^{-2}\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi, 1\right)\\
={}& \sqrt{\frac{20(1-c)}{9a}} - \sqrt{\frac{5(1-c)}{a}}\\
&\times \tanh^{-2}\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)\\
={}& -\sqrt{\frac{5(1-c)}{9a}} - \sqrt{\frac{5(1-c)}{a}}\\
&\times \operatorname{csch}^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)\\
={}& u_{24}(\xi)\ (\text{see } (39)),\\
u_9(\xi) \longrightarrow{}& \sqrt{\frac{20(1-c)}{9a}} - \sqrt{\frac{5(1-c)}{a}}\\
&\times \operatorname{sn}^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi, 1\right)\\
={}& \sqrt{\frac{20(1-c)}{9a}} - \sqrt{\frac{5(1-c)}{a}}\\
&\times \tanh^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)\\
={}& -\sqrt{\frac{5(1-c)}{9a}} + \sqrt{\frac{5(1-c)}{a}}\\
&\times \operatorname{sech}^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)\\
={}& u_{25}(\xi)\ (\text{see } (40)),\\
u_{11}(\xi) \longrightarrow{}& \sqrt{\frac{20(1-c)}{9a}} + \sqrt{\frac{5(1-c)}{a}}\\
&- \frac{2\sqrt{5(1-c)/a}}{1-\operatorname{cn}\left(\sqrt{a/c(b+k)}\sqrt[4]{((1-c)/5a)}\xi, 1\right)}\\
={}& 5\sqrt{\frac{5(1-c)}{9a}} - \left(2\sqrt{\frac{5(1-c)}{a}}\right)\\
&\times\left(1-\operatorname{sech}\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{5a}}\xi\right)\right)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&= 5\sqrt{\frac{5(1-c)}{9a}} \\
&\quad - \left(2\sqrt{\frac{5(1-c)}{a}} + 4\sqrt{\frac{5(1-c)}{a}}\right. \\
&\qquad \left.\times \sinh^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)\right) \\
&\quad \times 2\sinh^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)^{-1} \\
&= -\sqrt{\frac{5(1-c)}{9a}} - \sqrt{\frac{5(1-c)}{a}} \\
&\quad \times \operatorname{csch}^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right) \\
&= u_{24}(\xi)\ (\text{see } (39)),
\end{aligned}
$$

$$
\begin{aligned}
u_{12}(\xi) &\longrightarrow \sqrt{\frac{20(1-c)}{9a}} + \sqrt{\frac{5(1-c)}{a}} \\
&\quad - \frac{2\sqrt{5(1-c)/a}}{1+\operatorname{cn}\left(\sqrt{a/(c(b+k))}\sqrt[4]{((1-c)/5a)}\xi, 1\right)} \\
&= 5\sqrt{\frac{5(1-c)}{9a}} - \left(2\sqrt{\frac{5(1-c)}{a}}\right) \\
&\quad \times \left(1+\operatorname{sech}\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{(1-c)}{5a}}\xi\right)\right)^{-1} \\
&= 5\sqrt{\frac{5(1-c)}{9a}} \\
&\quad - \left(-2\sqrt{\frac{5(1-c)}{a}} + 4\sqrt{\frac{5(1-c)}{a}}\right. \\
&\qquad \left.\times \cosh^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)\right) \\
&\quad \times \left(2\cosh^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right)\right)^{-1} \\
&= -\sqrt{\frac{5(1-c)}{9a}} + \sqrt{\frac{5(1-c)}{a}} \\
&\quad \times \operatorname{sech}^2\left(\sqrt{\frac{a}{c(b+k)}}\sqrt[4]{\frac{1-c}{80a}}\xi\right) \\
&= u_{25}(\xi)\ (\text{see } (40)).
\end{aligned}
\tag{78}
$$

The limit forms of the other solutions can be derived similarly, so here we omit them. Hereto, we have completed the derivations for Proposition 4.

5. Conclusion

In this paper, we have investigated ZK-BBM$(m, 2)$ $(m = 2, 3)$ equations. For ZK-BBM(2, 2) equation, we obtain peakon wave, periodic peakon wave, and smooth periodic wave solutions (see $u_i(\xi)$ $(i = 1\text{–}4)$). For ZK-BBM(3, 2) equation, we obtain some elliptic function solutions (see $u_i(\xi)$ $(i = 5\text{–}20)$). Furthermore, from the limit forms of these solutions, we obtain some trigonometric and hyperbolic function solutions (see Remarks 5–8 and the corresponding derivations). We also showed that some previous results are our special cases (see Remarks 3 and 9). We would like to study the ZK-BBM(m, n) equations further.

Acknowledgment

This study is supported by the National Natural Science Foundation (no. 11171115).

References

[1] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, vol. 149, Cambridge University Press, Cambridge, UK, 1991.

[2] C. Rogers and W. K. Schief, *Bäcklund and Darboux Transformations*, Geometry and Modern Applications in Soliton Theory, Cambridge University Press, Cambridge, UK, 2002.

[3] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons*, Springer, Berlin, Germany, 1991.

[4] S. K. Liu, Z. T. Fu, S. D. Liu, and Q. Zhao, "Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations," *Physics Letters A*, vol. 289, no. 1-2, pp. 69–74, 2001.

[5] G. T. Liu and T. Y. Fan, "New applications of developed Jacobi elliptic function expansion methods," *Physics Letters A*, vol. 345, no. 1–3, pp. 161–166, 2005.

[6] M. L. Wang and Y. B. Zhou, "The periodic wave solutions for the Klein-Gordon-Schrödinger equations," *Physics Letters A*, vol. 318, no. 1-2, pp. 84–92, 2003.

[7] M. A. Abdou, "The extended F-expansion method and its application for a class of nonlinear evolution equations," *Chaos, Solitons and Fractals*, vol. 31, no. 1, pp. 95–104, 2007.

[8] M. L. Wang, X. Z. Li, and J. L. Zhang, "The G'/G-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics," *Physics Letters A*, vol. 372, no. 4, pp. 417–423, 2008.

[9] M. Song and Y. L. Ge, "Application of the G'/G-expansion method to (3+1)-dimensional nonlinear evolution equations," *Computers & Mathematics with Applications*, vol. 60, no. 5, pp. 1220–1227, 2010.

[10] J. B. Li and Z. R. Liu, "Smooth and non-smooth traveling waves in a nonlinearly dispersive equation," *Applied Mathematical Modelling*, vol. 25, pp. 41–56, 2000.

[11] Z. R. Liu, T. P. Jiang, P. Qin, and Q. F. Xu, "Trigonometric function periodic wave solutions and their limit forms for the KdV-like and the PC-like equations," *Mathematical Problems in Engineering*, vol. 2011, Article ID 810217, 23 pages, 2011.

[12] M. Song, S. Y. Li, and J. Cao, "New exact solutions for the (2+1)-dimensional Broer-Kaup-Kupershmidt equations,"

Abstract and Applied Analysis, vol. 2010, Article ID 652649, 10 pages, 2010.

[13] S. Y. Li and R. Liu, "Some explicit expressions and interesting bifurcation phenomena for nonlinear waves in generalized Zakharov equations," *Abstract and Applied Analysis*, vol. 2013, Article ID 869438, 19 pages, 2013.

[14] Z. S. Wen, "Bifurcation of traveling wave solutions for a two-component generalized-equation," *Mathematical Problems in Engineering*, vol. 2012, Article ID 597431, 17 pages, 2012.

[15] V. E. Zakharov and E. A. Kunznetsov, "Three-dimensional solitons," *Soviet Physics*, vol. 39, pp. 285–286, 1974.

[16] S. Monro and E. J. Parkes, "The derivation of a modified Zakharov-Kuznetsov equation and the stability of its solutions," *Journal of Plasma Physics*, vol. 62, pp. 305–317, 1999.

[17] S. Monro and E. J. Parkes, "Stability of solitary-wave solutions to a modified Zakharov-Kuznetsov equation," *Journal of Plasma Physics*, vol. 64, pp. 411–426, 2000.

[18] S. Micu, "On the controllability of the linearized Benjamin-Bona-Mahony equation," *SIAM Journal on Control and Optimization*, vol. 39, no. 6, pp. 1677–1696, 2001.

[19] A. M. Wazwaz, "The extended tanh method for new compact and noncompact solutions for the KP-BBM and the ZK-BBM equations," *Chaos, Solitons and Fractals*, vol. 38, no. 5, pp. 1505–1516, 2008.

[20] Z. J. Wang and S. Q. Tang, "Bifurcation of travelling wave solutions for the generalized ZK-BBM equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 7, pp. 2948–2955, 2009.