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Research Article

Commutators with Lipschitz Functions and Nonintegral Operators

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Let T be a singular nonintegral operator; that is, it does not have an integral representation by a kernel with size estimates, even rough. In this paper, we consider the boundedness of commutators with T and Lipschitz functions. Applications include spectral multipliers of self-adjoint, positive operators, Riesz transforms of second-order divergence form operators, and fractional power of elliptic operators.

1. Introduction

Let T be a bounded operator on $L^p(\mathbb{R}^n)$ for some p, $1 < p < \infty$. A measurable function $K(x, y)$ is called an associated kernel of T if

$$Tf(x) = \int_X K(x, y) f(y)\,dy \tag{1}$$

holds for each continuous function f with compact support and for almost all x not in the support of f.

The kernel $K(x, y)$ is said to satisfy the following.

(i) The pointwise Hörmander condition on x variable if there exist $0 < \alpha \leq 1$ and $c, c_1 \geq 1$ such that

$$|K(x, y) - K(z, y)| \leq c\frac{|x-z|^{\alpha}}{|x-y|^{n+\alpha}}, \tag{2}$$

when $|x - y| \geq c_1|x - z|$, and $B(x, r)$ denotes the ball with center x, radius r.

(ii) The integral Hörmander condition on y variable if there exist constants C and $c_2 \geq 1$ such that

$$\int_{|x-y|\geq c_2|z-y|} |K(x, y) - K(x, z)|\,dx \leq C, \tag{3}$$

for all $y, z \in \mathbb{R}^n$.

It is well known that if T is bounded on $L^q(\mathbb{R}^n)$ for some q, $1 < q < \infty$, and $b \in \mathrm{BMO}$, the two Hörmander conditions (i) and (ii) above are sufficient to imply that the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for all p, $1 < p < \infty$, with norm

$$\|[b, T](f)\|_p \leq C\|b\|_*\|f\|_p, \tag{4}$$

where the commutator $[b, T]$ is defined by $[b, T](f) = T(bf) - bT(f)$ and $\|b\|_*$ is the BMO seminorm of b. See [1, 2] for BMO functions on Euclidean spaces $\mathbb{R}^n$ and [3] for spaces of homogeneous type.

A particular case of the result of Janson [2] states that $[b, T] : L^p \to L^q$ is bounded, $1 < p < q < \infty$, if $b \in \dot{\Lambda}_{\beta}$, $\beta = n(1/p - 1/q)$. Here, $\dot{\Lambda}_{\beta}$ is the homogeneous Lipschitz space determined by the first difference operator.

In [4], Duong and Yan have replaced the two Hörmander conditions (2) and (3) by the following weaker conditions (5) and (6) below which previously appeared in [5] and still concluded that the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for all p, $1 < p < \infty$. And in [6], Hu and Yang obtained the weighted boundedness of maximal commutator when T satisfy (5) and (6). Roughly speaking, we assume the following.

(iii) There exists a class of operators A_t with kernels $a_t(x, y)$, which satisfy the condition (23) in Section 2, so that the kernels $k_t(x, y)$ of the operators $(T - A_t T)$ satisfy the condition

$$\left|k_t(x, y)\right| \le c \frac{t^{\gamma/m}}{|x-y|^{n+\gamma}}, \tag{5}$$

when $|x - y| \ge c_2 t^{1/m}$ for some $\gamma, m > 0$, where c is a positive constant.

(iv) There exists a class of operators B_t with kernels $b_t(x, y)$, which satisfy the condition (23), such that $(T - TB_t)$ have associated kernels $K_t(x, y)$ and there exist positive constants c_3, c_4 such that

$$\int_{|x-y|\ge c_3 t^{1/m}} \left|K_t(x, y)\right| dx \le c_4, \quad \forall y \in \mathbb{R}^n. \tag{6}$$

Under conditions (5) and (6), if T is bounded on $L^q(\mathbb{R}^n)$ for some q, $1 < q < \infty$, then the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for all p, $1 < p < \infty$.

In [7], Auscher and Martell have considered the commutators of singular nonintegral operators, where the implicit terminology has been introduced in [8]. By this we mean that they are still of order 0, but they do not have an integral representation by a kernel with size and/or smoothness estimates. Let $1 \le p_0 < q_0 \le \infty$. Suppose that the singular nonintegral operator T is a sublinear operator bounded on $L^{p_0}(\mathbb{R}^n)$ and that $\{A_r\}_{r>0}$ is a family of operators acting from $L^\infty_c(\mathbb{R}^n)$ into $L^{p_0}(\mathbb{R}^n)$. Auscher and Martell assume the following.

(v) For all $f \in L^\infty_c(\mathbb{R}^n)$ and all balls B where $r(B)$ denotes its radius,

$$\begin{aligned}&\left(\frac{1}{|B|}\int_B \left|T\left(I - A_{r(B)}\right) f\right|^{p_0} dx\right)^{1/p_0}\\ &\quad \le C \sum_{j=1}^{\infty} \alpha_j \left(\frac{1}{\left|2^{j+1}B\right|}\int_{2^{j+1}B} |f|^{p_0} dx\right)^{1/p_0}.\end{aligned} \tag{7}$$

(vi) For all $f \in L^\infty_c(\mathbb{R}^n)$ and all balls B where $r(B)$ denotes its radius,

$$\begin{aligned}&\left(\frac{1}{|B|}\int_B \left|TA_{r(B)} f\right|^{q_0} dx\right)^{1/q_0}\\ &\quad \le C \sum_{j=1}^{\infty} \alpha_j \left(\frac{1}{\left|2^{j+1}B\right|}\int_{2^{j+1}B} |Tf|^{p_0} dx\right)^{1/p_0}.\end{aligned} \tag{8}$$

Let $p_0 < p < q_0$ and $w \in A_{p/p_0} \cap RH_{(q_0/p)'}$ (for the definitions of A_{p/p_0} and $RH_{(q_0/p)'}$ see Section 2). Under conditions (7) and (8), if $\sum_{j=1}^{\infty} \alpha_j\, j < \infty$, then the commutator $[b, T]$ is bounded on $L^p(w)$; that is, $\|[b, T] f\|_{L^p(w)} \le C\|b\|_* \|f\|_{L^p(w)}$ for all $f \in L^\infty_c(\mathbb{R}^n)$.

The main object of this paper is the commutators of nonintegral operators $[b, T]$. Compared to the result in [7], we can obtain a more general result for b belongs to the Lipschitz spaces $\dot{\Lambda}_{\beta_i}(X)$. To be more specific, we can obtain the following.

Theorem 1. *Let $0 \le \alpha < 1$, $1 \le p_0 \le s_0 < q_0 \le \infty$ such that $1/s_0 = 1/p_0 - \alpha/n$. Suppose that T is a sublinear operator bounded from $L^{p_0}(\mathbb{R}^n)$ to $L^{s_0}(\mathbb{R}^n)$ and that $\{A_r\}_{r>0}$ is a family of operators acting from $L^\infty_c(\mathbb{R}^n)$ into $L^{p_0}(\mathbb{R}^n)$. Assume that*

$$\begin{aligned}&\left(\frac{1}{|B|}\int_B \left|T\left(I - A_{r(B)}\right) f\right|^{s_0} dx\right)^{1/s_0}\\ &\quad \le C \sum_{j=1}^{\infty} \alpha_j \left|2^{j+1}B\right|^{\alpha/n} \left(\frac{1}{\left|2^{j+1}B\right|}\int_{2^{j+1}B} |f|^{p_0} dx\right)^{1/p_0},\end{aligned} \tag{9}$$

$$\begin{aligned}&\left(\frac{1}{|B|}\int_B \left|TA_{r(B)} f\right|^{q_0} dx\right)^{1/q_0}\\ &\quad \le C \sum_{j=1}^{\infty} \alpha_j \left(\frac{1}{\left|2^{j+1}B\right|}\int_{2^{j+1}B} |Tf|^{s_0} dx\right)^{1/s_0},\end{aligned} \tag{10}$$

for all $f \in L^\infty_c(\mathbb{R}^n)$ and all balls B, where $r(B)$ denotes its radius. Let $0 < \beta < 1$ such that $\alpha + \beta < 1$. Let $p_0 < p < q < q_0$ and $1/q = 1/p - (\alpha + \beta)/n$. If $\sum_{j=1}^{\infty} \alpha_j < \infty$, then there is a constant C such that

$$\|[b, T] f\|_{L^q} \le C\|b\|_{\dot{\Lambda}_\beta} \|f\|_{L^p}, \tag{11}$$

for all $f \in L^\infty_c(\mathbb{R}^n)$ and for all $b \in \dot{\Lambda}_\beta$.

The case $q_0 = \infty$ is understood in the sense that the L^{q_0}-average in (10) is indeed an essential supremum.

Remark 2. Let $1 \le p_0 < p < q < q_0$ be such that $1/q = 1/p - \alpha/n$. Under the assumptions above, we know that if $\sum_{j=1}^{\infty} \alpha_j < \infty$, then T is bounded from L^p to L^q. See Theorem 2.2 in [9].

In the limiting case $\alpha = 0$, from the assumptions (9) and (10), we deduce

$$\begin{aligned}\left(\frac{1}{|B|}\int_B \left|T\left(I - A_{r(B)}\right) f\right|^{p_0}\right)^{1/p_0} &\le CM\left(|f|^{p_0}\right)^{1/p_0}(x),\\ \left(\frac{1}{|B|}\int_B \left|TA_{r(B)} f\right|^{q_0}\right)^{1/q_0} &\le CM\left(|Tf|^{p_0}\right)^{1/p_0}(x).\end{aligned} \tag{12}$$

Consequently, from the Theorem 3.7 in [7], we know that if $\sum_{j=1}^{\infty} \alpha_j < \infty$, then $\|Tf\|_{L^p(w)} \le C\|f\|_{L^p(w)}$ for $p_0 < p < q_0$ and for all $w \in A_{p/p_0} \cap RH_{(q_0/p)'}$.

Theorem 3. *Let $1 \le p_0 < q_0 \le \infty$. Suppose that T is a sublinear operator bounded on $L^{P_0}(\mathbb{R}^n)$ and that $\{A_r\}_{r>0}$ is a family of operators acting from $L^\infty_c(\mathbb{R}^n)$ to $L^{P_0}(\mathbb{R}^n)$. Assume that T satisfy (9) and (10) with $\alpha = 0$. Let $0 < \beta < \min\{1, n/p_0\}$, $p_0 < p < q < q_0$, $b \in \dot{\Lambda}_\beta$ and $w, v \in A_{p/p_0} \cap RH_{(q_0/p)'}$. Assume that there exists a constant $1 < s < \min\{n/\beta p_0, p/p_0\}$ such that $(w, v) \in A(p/p_0 s, q/p_0 s, \beta p_0 s/n)$. If $\sum_{j=1}^{\infty} \alpha_j < \infty$, then there is a constant C such that*

$$\|[b, T] f\|_{L^q(v)} \le C\|b\|_{\dot{\Lambda}_\beta} \|f\|_{L^p(w)}, \tag{13}$$

for all $f \in L^\infty_c$.

The class $A(p, q, s)$ is defined in Section 2.

2. Definitions and Preliminary Results

We use the notation

$$\fint_E f = \frac{1}{|E|}\int_E f(x)\,dx, \tag{14}$$

and we often ignore the Lebesgue measure and the variable of the integrand in writing integrals, unless this is needed to avoid confusions.

A weight w is a nonnegative locally integrable function. We say that $w \in A_p$, $1 < p < \infty$, if there exists a constant C such that for every ball $B \subset X$

$$\left(\fint_B w\right)\left(\fint_B w^{1-p'}\right)^{p-1} \le C. \tag{15}$$

For $p = 1$, we say that $w \in A_1$ if there is a constant C such that for every ball $B \subset \mathbb{R}^n$, $\fint_B w \le Cw(x)$, for a.e. $x \in B$, or, equivalently, $M(w) \le Cw$ a.e., where $M(w)$ denotes the classical Hardy-Littlewood maximal function of w. The reverse Hölder classes are defined in the following way: $w \in RH_q$, $1 < q < \infty$, if there is a constant C such that for every ball $B \subset \mathbb{R}^n$

$$\left(\fint_B w^q\right)^{1/q} \le \fint_B w. \tag{16}$$

The endpoint $q = \infty$ is given by the condition: $w \in RH_\infty$ whenever, for any ball B,

$$w(x) \le \fint_B w, \quad \text{for a.e. } x \in B. \tag{17}$$

The homogenous Lipschitz function space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ is the space of functions f such that

$$\|f\|_{\dot{\Lambda}_\beta} = \sup_{x,h\in\mathbb{R}^n, h\neq 0} \frac{\left|\Delta_h^{[\beta]+1} f(x)\right|}{|h|^\beta} < \infty, \tag{18}$$

where Δ_h^k denotes the kth difference operator (see [10]). That is, $\Delta_h^1 f(x) = \Delta_h f(x) = f(x+h) - f(x)$, $\Delta_h^{k+1} f(x) = \Delta_h^k f(x+h) - \Delta_h^k f(x)$, $k \ge 1$.

We have the following lemmas.

Lemma 4 (see [10]). *For $0 < \beta < 1$, $1 \le q < \infty$, one has*

$$\begin{aligned}\|f\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} &\approx \sup_B \frac{1}{|B|^{1+\beta/n}}\int_B |f - f_B|\,dx \\ &\approx \sup_B \frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B |f - f_B|^q\right)^{1/q} dx.\end{aligned} \tag{19}$$

For $q = \infty$, the last formula should be modified appropriately.

Lemma 5 (see [10]). *Let $B^* \subset B \subset \mathbb{R}^n$, and then $|f_{B^*} - f_B| \le C\|f\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}|B|^{\beta/n}$.*

Lemma 6 (see [11]). *For $1 \le \gamma < \infty$ and $\beta > 0$, let*

$$M_{\beta,\gamma}(f)(x) = \sup_{B\ni x}\left(\frac{1}{|B|^{1-\beta\gamma/n}}\int_B |f(y)|^\gamma dy\right)^{1/\gamma}. \tag{20}$$

Suppose that $\gamma < p < n/\beta$ and $1/q = 1/p - \beta/n$, and then

$$\left\|M_{\beta,\gamma}(f)\right\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}. \tag{21}$$

Theorem A (see [7]). *Fix $1 < q \le \infty$, $a \ge 1$, and $\omega \in RH_{s'}$, $1 \le s < \infty$. Then, there exist $C = C(q, n, a, \omega, s)$ and $K_0 = K_0(n, a) \ge 1$ with the following property: assume that F, G, H_1, and H_2 are nonnegative measurable functions on $\mathbb{R}^n$ such that for any cube Q there exist nonnegative functions G_Q and H_Q with $F(x) \le G_Q(x) + H_Q(x)$ for a.e. $x \in Q$ and*

$$\left(\fint_Q H_Q^q\right)^{1/q} \le a\left(MF(x) + MH_1(x) + H_2(\overline{x})\right), \quad \forall x, \overline{x} \in Q, \tag{22}$$

$$\fint_Q G_Q \le G(x), \quad \forall x \in Q.$$

Then for all $\lambda > 0$, $K \ge K_0$ and $0 < \gamma < 1$

$$\begin{aligned}&\omega\{MF > K\lambda,\ G + H_2 \le \gamma\lambda\} \\ &\quad \le C\left(\frac{a^q}{K^q} + \frac{\gamma}{K}\right)^{1/s}\omega\{MF + MH_1 > \lambda\}.\end{aligned} \tag{23}$$

As a consequence, for all $0 < p < 1/s$, one has

$$\begin{aligned}&\|MF\|_{L^p(\omega)} \\ &\quad \le C\left(\|G\|_{L^p(\omega)} + \|MH_1\|_{L^p(\omega)} + \|H_2\|_{L^p(\omega)}\right),\end{aligned} \tag{24}$$

provided $\|MF\|_{L^p(\omega)} < \infty$, and

$$\begin{aligned}&\|MF\|_{L^{p,\infty}(\omega)} \\ &\quad \le C\left(\|G\|_{L^{p,\infty}(\omega)} + \|MH_1\|_{L^{p,\infty}(\omega)} + \|H_2\|_{L^{p,\infty}(\omega)}\right),\end{aligned} \tag{25}$$

provided $\|MF\|_{L^{p,\infty}(\omega)} < \infty$. Furthermore, if $p \ge 1$, then (24) and (25) hold, provided $F \in L^1$(whether or not $MF \in L^p(\omega)$).

For $0 < s < 1$ and $1 \le \gamma < \infty$, we denote

$$\mathscr{M}_{s,\gamma}(f)(x) = \sup_{B\ni x}\left(\frac{1}{|B|^{1-s}}\int_B |f(y)|^\gamma dy\right)^{1/\gamma}, \tag{26}$$

where the supremum is taken with respect to all balls B of positive measure containing the point x.

Theorem B. *Let $1 < p < q < \infty$, $0 < s < 1$, and let v and w be the weight functions. For a constant $C > 0$ to exist so that the inequality*

$$\begin{aligned}&\left(\int_{\mathbb{R}^n}\left(\mathscr{M}_{s,1}(f)(x)\right)^q v(x)\,dx\right)^{1/q} \\ &\quad \le C\left(\int_{\mathbb{R}^n}|f(x)|^p w(x)\,dx\right)^{1/p}\end{aligned} \tag{27}$$

would hold, it is necessary and sufficient that the condition

$$\sup_{x\in\mathbb{R}^n, r>0}\left(w^{1-p'}B(x,6r)\right)^{1/p'} \times\left(\int_{\mathbb{R}^n\setminus B(x,r)} v(y)|x-y|^{(s-1)qn}dy\right)^{1/q}<\infty, \tag{28}$$

where $1/p+1/p'=1$, *be fulfilled.*

For the proof of this theorem, see [12].

Definition 7. (w,v) is said to belong to $A(p,q,s)$ $(1<p<q<\infty, 0<s<1)$ if (28) holds.

Lemma 8. *Let* $1\le\gamma<p<q<\infty$, $0<s<1$. *If* $(w,v)\in A(p/\gamma,q/\gamma,s)$, *then*

$$\left\|\mathscr{M}_{s,\gamma}f\right\|_{L^q(v)}\le C\|f\|_{L^p(w)}. \tag{29}$$

Proof. Since $\mathscr{M}_{s,\gamma}(f)(x)=(\mathscr{M}_{s,1}(|f|^\gamma)(x))^{1/\gamma}$, we have

$$\begin{aligned}\left\|\mathscr{M}_{s,\gamma}f\right\|_{L^q(v)}&=\left\|\left(\mathscr{M}_{s,1}\left(|f|^\gamma\right)\right)^{1/\gamma}\right\|_{L^q(v)}\\&=\|\mathscr{M}_{s,1}(|f|^\gamma)\|^{1/\gamma}_{L^{q/\gamma}(v)}.\end{aligned} \tag{30}$$

By Theorem B, we have

$$\begin{aligned}\|\mathscr{M}_{s,1}(|f|^\gamma)\|_{L^{q/\gamma}(v)}&\le C\left\||f|^\gamma\right\|_{L^{p/\gamma}(w)}\\&=C\|f\|^\gamma_{L^p(w)}.\end{aligned} \tag{31}$$

Thus,

$$\left\|\mathscr{M}_{s,\gamma}f\right\|_{L^q(v)}\le C\|f\|_{L^p(w)}. \tag{32}$$

□

3. The Proof of the Main Theorems

In order to prove Theorem 1, we need the following lemma.

Lemma 9. *Let* $1\le p_0\le s_0$, $p_0<p<q<\infty$, *and* $w,v\in A_\infty$. *Let* T *be a sublinear operator bounded from* L^{p_0} *to* L^{s_0}.

(i) *If* $b\in\dot{\Lambda}_\beta\cap L^\infty$ *and* $f\in L^\infty_c$, *then* $[b,T]f\in L^{s_0}$.

(ii) *Assume that for any* $b\in\dot{\Lambda}_\beta\cap L^\infty$ *and for any* $f\in L^\infty_c$ *one has that*

$$\|[b,T]f\|_{L^q(v)}\le C\|b\|_{\dot{\Lambda}_\beta}\|f\|_{L^p(w)}, \tag{33}$$

where C *does not depend on* b *and* f. *Then for all* $b\in\dot{\Lambda}_\beta$, (33) *holds.*

Proof. The ideas of the following argument are taken from [7].

Fix $f\in L^\infty_c$. Note that (i) follows easily observing that

$$\begin{aligned}[b,T]f(x)&\le|b(x)|\left|Tf(x)\right|+\left|T(bf)(x)\right|\\&\le\|b\|_{L^\infty}\left|Tf(x)\right|+\left|T(bf)(x)\right|\in L^{s_0}\end{aligned} \tag{34}$$

since $b\in L^\infty$, $f\in L^\infty_c$ imply that $f,bf\in L^\infty_c\subset L^{p_0}$ and hence, by assumption, $T(f),T(bf)\in L^{s_0}$.

To obtain (ii), we fix $b\in\dot{\Lambda}_\beta$ and $f\in L^\infty_c$. Let Q_0 be a cube such that $\operatorname{supp}f\subset Q_0$. We may assume that $b_{Q_0}=0$ since otherwise we can work with $\bar{b}=b-b_{Q_0}$ and observe that

$$[b,T]=\left[\bar{b},T\right],\qquad\|b\|_{\dot{\Lambda}_\beta}=\left\|\bar{b}\right\|_{\dot{\Lambda}_\beta}. \tag{35}$$

Note that for $m=0,1$, we have that $|b^mf|$ and $|T(b^mf)|$ are finite almost everywhere since they belong to L^{p_0}.

Let $N>0$ and define b_N as follows:

$$b_N(x)=\begin{cases}-N, & b(x)<-N,\\ b(x), & -N\le b(x)\le N,\\ N, & b(x)>N.\end{cases} \tag{36}$$

Then, it is immediate to see that $|b_N(x)-b_N(y)|\le|b(x)-b(y)|$ for all x,y. Thus, $\|b_N\|_{\dot{\Lambda}_\beta}\le\|b\|_{\dot{\Lambda}_\beta}$. As $b_N\in L^\infty$, we can use (33) and

$$\begin{aligned}\left\|[b_N,T]f\right\|_{L^q(v)}&\le C\|b_N\|_{\dot{\Lambda}_\beta}\|f\|_{L^p(w)}\\&\le C\|b\|_{\dot{\Lambda}_\beta}\|f\|_{L^p(w)}<\infty.\end{aligned} \tag{37}$$

To conclude, by Fatou's lemma, it suffices to show that $|[b_{N_j},T]f(x)|\to|[b,T]f(x)|$ for a.e. $x\in\mathbb{R}^n$ and for some subsequence $\{N_j\}_j$ such that $N_j\to\infty$.

As $|b_N|\le|b|\in L^p(Q_0)$, for any $1\le p<\infty$, the dominated convergence theorem yields that $b_Nf\to bf$ in L^{p_0} as $N\to\infty$. Therefore, T is bounded from L^{p_0} to L^{s_0}. It follows that $T(b_Nf-bf)\to 0$ in L^{s_0}. Thus, there exists a subsequence $N_j\to\infty$ such that $T(b_{N_j}f-bf)\to 0$ for a.e. $x\in\mathbb{R}^n$. In this way we obtain

$$\begin{aligned}&\left|\left|\left[b_{N_j},T\right]f(x)\right|-\left|[b,T]f(x)\right|\right|\\&\quad\le\left|\left[b_{N_j},T\right]f(x)-[b,T]f(x)\right|\\&\quad\le\left|T\left(b_{N_j}f-bf\right)(x)\right|+\left|b_{N_j}(x)-b(x)\right|\left|Tf(x)\right|\end{aligned} \tag{38}$$

as desired, and we get that $|[b_{N_j},T]f(x)|\to|[b,T]f(x)|$ for a.e $x\in\mathbb{R}^n$. □

Proof of Theorem 1. We assume that $q_0<\infty$, for $q_0=\infty$, and the main ideas are the same and details are left to the interested reader. Lemma 9 ensures that it suffices to consider the case $b\in\dot{\Lambda}_\beta\cap L^\infty$. Let $f\in L^\infty_c$ and set $F=|[b,T]f|^{s_0}$. Note that $F\in L^1$ by (i) of Lemma 9. Given a ball B, we set $f_{B,b}=(b_{4B}-b)f$ and decompose $[b,T]f$ as follows:

$$\begin{aligned}&\left|[b,T]f(x)\right|\\&\quad=\left|T\left((b(x)-b)f\right)(x)\right|\\&\quad\le\left|b(x)-b_{4B}\right|\left|Tf(x)\right|+\left|T\left((b_{4B}-b)f\right)(x)\right|\\&\quad\le\left|b(x)-b_{4B}\right|\left|Tf(x)\right|+\left|T\left(I-A_{r(B)}\right)f_{B,b}(x)\right|\\&\quad\quad+\left|TA_{r(B)}f_{B,b}(x)\right|.\end{aligned} \tag{39}$$

We observe that $F \le G_B + H_B$, where

$$\begin{aligned} G_B &= 4^{s_0-1}\left(G_{B,1}+G_{B,2}\right) \\ &= 4^{s_0-1}\left(\left|b-b_{4B}\right|^{s_0}|Tf|^{s_0}+\left|T\left(I-A_{r(B)}\right)f_{B,b}\right|^{s_0}\right) \end{aligned} \quad (40)$$

and $H_B = 2^{s_0-1}|TA_{r(B)}f_{B,b}|^{s_0}$.

We first estimate the average of G_B on B. Fix any $x \in B$. Let $1 < s < \infty$. Using Lemma 4,

$$\begin{aligned} \left(⨍_B G_{B,1}\right)^{1/s_0} &= \left(\frac{1}{|B|}\int_B \left|b-b_{4B}\right|^{s_0}|Tf|^{s_0}\right)^{1/s_0} \\ &\le \left(\frac{1}{|B|}\int_B \left|b-b_{4B}\right|^{s_0 s'}\right)^{1/(s_0 s')} \\ &\quad\times\left(\frac{1}{|B|}\int_B |Tf|^{s_0 s}\right)^{1/(s_0 s)} \\ &= \frac{1}{|B|^{\beta/n}}\left(\frac{1}{|B|}\int_B \left|b-b_{4B}\right|^{s_0 s'}\right)^{1/(s_0 s')} \\ &\quad\times\left(\frac{1}{|B|^{1-s_0 s\beta/n}}\int_B |Tf|^{s_0 s}\right)^{1/(s_0 s)} \\ &\le C\|b\|_{\dot{\Lambda}_\beta} M_{\beta,s_0 s}(Tf)(x). \end{aligned} \quad (41)$$

Using (9) and Lemmas 4 and 5,

$$\begin{aligned} \left(⨍_B G_{B,2}\right)^{1/s_0} &= \left(⨍_B \left|T\left(I-A_{r(B)}\right)f_{B,b}\right|^{s_0}\right)^{1/s_0} \\ &\le C\sum_{j=1}^{\infty}\alpha_j\left|2^{j+1}B\right|^{\alpha/n}\left(⨍_{2^{j+1}B}\left|f_{B,b}\right|^{p_0}\right)^{1/p_0} \\ &\le C\sum_{j=1}^{\infty}\alpha_j\left|2^{j+1}B\right|^{\alpha/n} \\ &\quad\times\left(\frac{1}{\left|2^{j+1}B\right|}\int_{2^{j+1}B}\left|b-b_{2^{j+1}B}\right|^{p_0}|f|^{p_0}\right)^{1/p_0} \\ &\quad+C\sum_{j=1}^{\infty}\alpha_j\left|2^{j+1}B\right|^{\alpha/n} \\ &\quad\times\left(\frac{1}{\left|2^{j+1}B\right|}\left|b_{2^{j+1}B}-b_{4B}\right|^{p_0}\int_{2^{j+1}B}|f|^{p_0}\right)^{1/p_0} \\ &\le C\sum_{j=1}^{\infty}\alpha_j\|b\|_{\dot{\Lambda}_\beta} M_{\alpha+\beta,p_0 s}(f)(x) \\ &\quad+C\sum_{j=1}^{\infty}\alpha_j\|b\|_{\dot{\Lambda}_\beta}\left|2^{j+1}B\right|^{(\alpha+\beta)/n} \\ &\quad\times\left(\frac{1}{\left|2^{j+1}B\right|}\int_{2^{j+1}B}|f|^{p_0 s}\right)^{1/(p_0 s)} \\ &\le C\|b\|_{\dot{\Lambda}_\beta} M_{\alpha+\beta,p_0 s}(f)(x) \end{aligned} \quad (42)$$

since $\sum_{j=1}^{\infty}\alpha_j < \infty$. Hence, for any $x \in B$,

$$\begin{aligned} ⨍_B G_B &\le C\left(\|b\|_{\dot{\Lambda}_\beta}^{s_0} M_{\beta,s_0 s}(Tf)^{s_0}(x)\right. \\ &\quad\left.+\|b\|_{\dot{\Lambda}_\beta}^{s_0} M_{\alpha+\beta,p_0 s}(f)^{s_0}(x)\right) \equiv G(x). \end{aligned} \quad (43)$$

We next estimate the average of $H_B^{q'}$ on B with $q' = q_0/s_0$. Using (10) and proceeding as before, we see that

$$\begin{aligned} &\left(⨍_B H_B^{q'}\right)^{1/q_0} \\ &\quad= 2^{(s_0-1)/s_0}\left(⨍_B \left|TA_{r(B)}f_{B,b}\right|^{q_0}\right)^{1/q_0} \\ &\quad\le C\sum_{j=1}^{\infty}\alpha_j\left(⨍_{2^{j+1}B}\left|Tf_{B,b}\right|^{s_0}\right)^{1/s_0} \\ &\quad\le C\sum_{j=1}^{\infty}\alpha_j\left(⨍_{2^{j+1}B}\left|T_b f\right|^{s_0}\right)^{1/s_0} \\ &\quad\quad+C\sum_{j=1}^{\infty}\alpha_j\left(⨍_{2^{j+1}B}\left|b-b_{4B}\right|^{s_0}|Tf|^{s_0}\right)^{1/s_0} \\ &\quad\le C(MF)^{1/s_0}(x)+C\|b\|_{\dot{\Lambda}_\beta} M_{\beta,s_0 s}(Tf)(\overline{x}), \end{aligned} \quad (44)$$

for any $x, \overline{x} \in B$. Thus we have obtained

$$\begin{aligned} \left(⨍_B H_B^{q'}\right)^{1/q'} &\le C\left(MF(x)+\|b\|_{\dot{\Lambda}_\beta}^{s_0} M_{\beta,s_0 s}(Tf)^{s_0}(\overline{x})\right) \\ &\equiv C\left(MF(x)+H_2(\overline{x})\right). \end{aligned} \quad (45)$$

For $p_0 < p < q < q_0$ and $1/q = 1/p - (\alpha+\beta)/n$, we can find a $1 < s < \infty$ such that $s_0 s < 1/(1/p - \alpha/n)$ and $p_0 s < p$. As mentioned before $F \in L^1$. Applying Theorem A and Remark 2 with q/s_0 in place of p, we obtain

$$\begin{aligned} &\|[b,T]f\|_q^{s_0} \\ &\quad\le \|MF\|_{q/s_0} \le C\left(\|G\|_{q/s_0}+\left\|H_2\right\|_{q/s_0}\right) \\ &\quad\le C\|b\|_{\dot{\Lambda}_\beta}^{s_0}\left(\left\|M_{\beta,s_0 s}(Tf)\right\|_q^{s_0}+\left\|M_{\alpha+\beta,p_0 s}(f)\right\|_q^{s_0}\right) \\ &\quad\le C\|b\|_{\dot{\Lambda}_\beta}^{s_0}\left(\|Tf\|_{1/(1/p-\alpha/n)}^{s_0}+\|f\|_p^{s_0}\right) \\ &\quad\le C\|b\|_{\dot{\Lambda}_\beta}^{s_0}\|f\|_p^{s_0}, \end{aligned} \quad (46)$$

where we have used Lemma 6. This implies that

$$\|[b,T]f\|_q \le C\|b\|_{\dot{\Lambda}_\beta}\|f\|_p. \tag{47}$$

□

Proof of Theorem 3. Let F, G, and H_2 be the same as those used in the proof of Theorem 1. As mentioned before $F \in L^1$. Since $v \in A_{p/p_0} \cap RH_{(q_0/p)'}$, applying Theorem A with p/p_0 in place of p and $s = q_0/p$, we obtain

$$\begin{aligned}
&\|[b,T]f\|_{L^q(v)}^{p_0} \\
&\quad\le \|MF\|_{L^{q/p_0}(v)} \le C\left(\|G\|_{L^{q/p_0}(v)} + \|H_2\|_{L^{q/p_0}(v)}\right) \\
&\quad\le C\|b\|_{\dot{\Lambda}_\beta}^{p_0}\left(\left\|M_{\beta,p_0 s}(Tf)\right\|_{L^q(v)}^{p_0} + \left\|M_{\beta,p_0 s}(f)\right\|_{L^q(v)}^{p_0}\right) \\
&\quad= C\|b\|_{\dot{\Lambda}_\beta}^{p_0}\left(\left\|\mathscr{M}_{\beta p_0 s/n, p_0 s}(Tf)\right\|_{L^q(v)}^{p_0}\right. \\
&\qquad\qquad\left.+\left\|\mathscr{M}_{\beta p_0 s/n, p_0 s}(f)\right\|_{L^q(v)}^{p_0}\right).
\end{aligned} \tag{48}$$

Noting that $(w,v) \in A(p/p_0 s, q/p_0 s, \beta p_0 s/n)$, Lemma 8 and Remark 2 give us that

$$\begin{aligned}
\left\|\mathscr{M}_{\beta p_0 s/n, p_0 s}(Tf)\right\|_{L^q(v)} &\le C\|Tf\|_{L^p(w)} \\
&\le C\|f\|_{L^p(w)}.
\end{aligned} \tag{49}$$

This implies that

$$\|[b,T]f\|_{L^q(v)} \le C\|b\|_{\dot{\Lambda}_\beta}\|f\|_{L^p(w)}. \tag{50}$$

□

4. Applications

4.1. Spectral Multipliers: Off-Diagonal Estimates. Suppose that L is a self-adjoint nonnegative definite operator on $L^2(\mathbb{R}^n)$. Let $E(\lambda)$ be the spectral resolution of L. For any bounded Borel function $m : [0,\infty) \to \mathbb{C}$, by using the spectral theorem, we can define the operator

$$m(L) = \int_0^\infty m(\lambda)\, dE(\lambda). \tag{51}$$

This is of course bounded on $L^2(\mathbb{R}^n)$.

The following will be assumed throughout this subsection.

(H1) L is a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$.

(H2) The operator L generates an analytic semigroup $\{e^{-tL}\}_{t>0}$ which satisfies the Davies-Gaffney condition. That is, there exist constants $C, c > 0$ such that for any open subsets $U_1, U_2 \subset \mathbb{R}^n$,

$$\begin{aligned}
&\left|\left\langle e^{-tL} f_1, f_2\right\rangle\right| \\
&\quad\le C\exp\left(-\frac{\operatorname{dist}(U_1,U_2)^2}{ct}\right) \\
&\qquad\times\|f_1\|_{L^2(\mathbb{R}^n)}\|f_2\|_{L^2(\mathbb{R}^n)}, \quad \forall t > 0,
\end{aligned} \tag{52}$$

for every $f_i \in L^2(\mathbb{R}^n)$ with supp $f_i \subset U_i$, $i = 1,2$, where $\operatorname{dist}(U_1,U_2) := \inf_{x\in U_1, y\in U_2} d(x,y)$.

(H3) Suppose $2 < q_0 \le \infty$. Assume that the analytic semigroup e^{-tL} generated by L satisfies "$L^2 - L^{q_0}$ off-diagonal" estimates: there exist coefficients $\{a_j\}_{j\ge0}$ satisfying $\sum_{j=0}^\infty a_j < \infty$ such that for all balls B and for all functions $f \in L^2(\mathbb{R}^n)$

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B \left|e^{-r_B^2 L} f\right|^{q_0} dx\right)^{1/q_0} \\
&\quad\le \sum_{j=0}^\infty a_j\left(\frac{1}{|2^j B|}\int_{2^j B} |f|^2 dx\right)^{1/2}.
\end{aligned} \tag{53}$$

Let ϕ be a nonnegative C_0^∞ function such that

$$\operatorname{supp}\phi \subset \left(\frac{1}{4}, 1\right), \qquad \sum_{l\in\mathbb{Z}} \phi\left(2^{-l}\lambda\right) = 1, \quad \forall\lambda > 0. \tag{54}$$

For $s \ge 0$, let $[s]$ denote the integer part of s. Recall that C^s is the space of functions m on $\mathbb{R}$ for which

$$\begin{aligned}
&\|m\|_{C^s} \\
&\quad= \begin{cases}
\displaystyle\sum_{k=0}^{s}\sup_{\lambda\in\mathbb{R}}\left|m^{(k)}(\lambda)\right| & \text{if } s \in \mathbb{Z}, \\
\displaystyle\left\|m^{([s])}\right\|_{\operatorname{Lip}(s-[s])} + \sum_{k=0}^{[s]}\sup_{\lambda\in\mathbb{R}}\left|m^{(k)}(\lambda)\right| & \text{if } s \notin \mathbb{Z}
\end{cases}
\end{aligned} \tag{55}$$

is finite.

Then the following result holds.

Theorem 10. *Let L satisfy assumptions (H1)–(H3). Let ϕ be a nonnegative C_0^∞ function satisfying* (54)*, and suppose that the bounded measurable function $m : [0,\infty) \to \mathbb{C}$ satisfies*

$$C_{\phi,s} = \sup_{t>0}\|\phi(\cdot)m(t\cdot)\|_{C^s} + |m(0)| < \infty \tag{56}$$

for some $s > n/2$. Then

(i) *let $0 < \beta < 1$. If $2 < p < 1/(1/q_0 + \beta/n)$ and $1/q = 1/p - \beta/n$, then there is a constant C such that*

$$\|[b,m(L)]f\|_{L^q} \le C\|b\|_{\dot{\Lambda}_\beta}\|f\|_{L^p}, \tag{57}$$

for all $f \in L_c^\infty$ and for all $b \in \dot{\Lambda}_\beta$.

(ii) *Let $0 < \beta < \min\{1, n/2\}$, $2 < p < q < q_0$, and $w, v \in A_{p/p_0} \cap RH_{(q_0/p)'}$. If there exists a constant $1 < s < \min\{n/\beta 2, p/2\}$ such that $(w,v) \in A(p/2s, q/2s, \beta 2s/n)$, then there is a constant C such that*

$$\|[b,m(L)]f\|_{L^q(v)} \le C\|b\|_{\dot{\Lambda}_\beta}\|f\|_{L^p(w)}, \tag{58}$$

for all $f \in L_c^\infty$ and for all $b \in \dot{\Lambda}_\beta$.

Proof. Estimate (57) follows from Theorem 1 with $\alpha = 0$ and estimate (58) follows from Theorem 3, applied to $Tf = m(L)f$ and $A_r = I - (I - e^{-r^2L})^M$ with $M \in \mathbb{N}$ and $M > s/2$. It suffices to show that there exist coefficients $\{a_j\}_{j\geq 0}$ satisfying $\sum_{j=1}^{\infty} a_j < \infty$ such that (9) and (10) hold for all $f \in L_c^{\infty}(\mathbb{R}^n)$.

Fix $1 \leq k \leq M$. From (53), we deduce that

$$\left(\frac{1}{|B|}\int_B \left|e^{-kr_B^2L}f\right|^{q_0}dx\right)^{1/q_0} \leq \sum_{j=0}^{\infty} Ca_j\left(\frac{1}{|2^jB|}\int_{2^jB}|f|^2dx\right)^{1/2}. \tag{59}$$

This estimate with $m(L)f$ in place of f yields (10). Since, by functional calculus, $m(L)e^{-kr^2L}f = e^{-kr^2L}m(L)f$, (9) was proved in [13]. □

4.2. Riesz Transforms. Let A be an $n \times n$ matrix of complex and L^{∞}-valued coefficients on $\mathbb{R}^n$. We assume that this matrix satisfies the following ellipticity (or "accretivity") condition: there exist $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda|\xi|^2 \leq \operatorname{Re} A(x)\,\xi\cdot\overline{\xi}, \qquad \left|A(x)\,\xi\cdot\overline{\zeta}\right| \leq \Lambda|\xi||\zeta|, \tag{60}$$

for all $\xi, \zeta \in \mathbb{C}^n$ and almost every $x \in \mathbb{R}^n$. Associated with this matrix we define the second-order divergence form operator

$$L = -\operatorname{div}(A\nabla). \tag{61}$$

The Riesz transforms associated to L are $\partial_j L^{-1/2}$, $1 \leq j \leq n$. Set $\nabla L^{-1/2} = (\partial_1 L^{-1/2}, \ldots, \partial_n L^{-1/2})$. The solution of the Kato conjecture [14] implies that this operator extends boundedly to L^2. This allows the representation

$$\nabla L^{-1/2}f = \frac{1}{\sqrt{\pi}}\int_0^{\infty}\nabla e^{-tL}f\frac{dt}{\sqrt{t}} \tag{62}$$

in which the integral converges strongly in L^2 both at 0 and ∞ when $f \in L^2$.

Define $\vartheta \in [0, \pi/2)$ by

$$\vartheta = \sup\left\{\left|\arg\langle Lf, f\rangle\right| : f \in \mathcal{D}(L)\right\}. \tag{63}$$

We write for $0 < \theta < \infty$, $\Sigma_\theta = \{z \in \mathbb{C}\setminus\{0\} : |\arg z| < \theta\}$.

We extract from [15] some definitions and results on unweighted off-diagonal estimates.

Definition 11. Let $1 \leq p \leq q \leq \infty$. One says that a family $\{T_t\}_{t>0}$ of sublinear operators satisfies $L^p - L^q$ full off-diagonal estimates, in short $T_t \in \mathcal{F}(L^p - L^q)$, if for some $c > 0$, for all closed sets E and F, all f, and all $t > 0$, we have

$$\left(\int_F |T_t(\chi_E f)|^q dx\right)^{1/q} \leq Ct^{-(1/2)(n/p-n/q)}e^{-cd^2(E,F)/2}\left(\int_E |f|^p dx\right)^{1/p}. \tag{64}$$

If I is a subinterval of $[1, \infty]$, Int I denotes the interior in $\mathbb{R}$ of $I \cap \mathbb{R}$.

Proposition 12 (see [15]). *Fix $m \in \mathbb{N}$ and $0 < \mu < \pi/2 - \vartheta$.*

(a) *There exists a nonempty maximal interval in $[1, \infty]$, denoted by $\mathcal{J}(L)$, such that if $p, q \in \mathcal{J}(L)$ with $p \leq q$, then $\{(zL)^m e^{-zL}\}_{z\in\Sigma_\mu}$ satisfies $L^p - L^q$ full off-diagonal estimates and is a bounded set in $\mathcal{L}(L^p)$.*

(b) *There exists a nonempty maximal interval in $[1, \infty]$, denoted by $\mathcal{K}(L)$, such that if $p, q \in \mathcal{K}(L)$ with $p \leq q$, then $\{\sqrt{z}\nabla(zL)^m e^{-zL}\}_{z\in\Sigma_\mu}$ satisfies $L^p - L^q$ full off-diagonal estimates and is a bounded set in $\mathcal{L}(L^p)$.*

(c) *$\mathcal{K}(L) \subset \mathcal{J}(L)$ and, for $p < 2$, we have $p \in \mathcal{K}(L)$ if and only if $p \in \mathcal{J}(L)$.*

(d) *Denote by $p_-(L)$, $p_+(L)$ the lower and upper bounds of $\mathcal{J}(L)$ and by $q_-(L)$, $q_+(L)$ those of $\mathcal{K}(L)$. We have $p_-(L) = q_-(L)$ and $(q_-(L))^* \leq p_+(L)$. (We have set $q^* = (qn/(n-q))$, the Sobolev exponent of q when $q < n$ and $q^* = \infty$, otherwise.)*

(e) *If $n = 1$, $\mathcal{J}(L) = \mathcal{K}(L) = [1, \infty]$. If $n = 2$, $\mathcal{J}(L) = [1, \infty]$ and $\mathcal{K}(L) \supset [1, q_+(L))$ with $q_+(L) > 2$.*

(f) *If $n \geq 3$, $p_-(L) < 2n/(n+2)$, $p_+(L) > 2n/(n-2)$, and $q_+(L) > 2$.*

Then for $q_- < p_0 < q_0 < q_+$, $T = \nabla L^{-1/2}$ satisfy (9) and (10) with $\alpha = 0$ and $A_r = I - (I - e^{-r^2L})^M$, where M is a large enough integer. For the proof of this argument, see [15]. So Theorem 1 with $\alpha = 0$ and Theorem 3 can be applied to $T = \nabla L^{-1/2}$.

4.3. Fractional Operators. Let $L = -\operatorname{div}(A\nabla)$. The fractional power of an elliptic operator L on R^n is given formally by

$$L^{-\alpha/2} = \frac{1}{\Gamma(\alpha/2)}\int_0^{\infty} t^{\alpha/2}e^{-tL}\frac{dt}{t}, \tag{65}$$

with $\alpha > 0$. There exist $p_- = p_-(L)$ and $p_+ = p_+(L)$, $1 \leq p_- < 2 < p_+ \leq \infty$ such that the semigroup $\{e^{-tL}\}_{t>0}$ is uniformly bounded on L^p for every $p_- < p < p_+$ (see Proposition 12). We have the following results.

Lemma 13 (see [9]). *Let $p_- < p_0 < s_0 < q_0 < p_+$ so that $1/p_0 - 1/s_0 = \alpha/n$. Fix a ball B with radius r. For $f \in L_c^{\infty}$ and M large enough, one has*

$$\left(⨍_B\left|L^{-\alpha/2}\left(I - e^{-r^2L}\right)^M f\right|^{s_0}\right)^{1/s_0} \leq C\sum_{j=1}^{\infty} g_1(j)\left|2^{j+1}B\right|^{\alpha/n}\left(⨍_{2^{j+1}B}|f|^{p_0}\right)^{1/p_0}, \tag{66}$$

and for $1 \le l \le M$

$$\left(⨍_B \left|L^{-\alpha/2}e^{-lr^2L}f\right|^{q_0}\right)^{1/q_0} \le C\sum_{j=1}^{\infty} g_2(j)\left(⨍_{2^{j+1}B}\left|L^{-\alpha/2}f\right|^{s_0}\right)^{1/s_0}, \quad (67)$$

where $g_j = C2^{-j(2M-n/s_0)}$ *and* $g_2(j) = Ce^{-c4^j}$.

Theorem 14. *Let* $p_- < p < q < p_+$, $0 < \alpha, \beta, \alpha + \beta < 1$, *and* $1/q = 1/p - (\alpha + \beta)/n$. *Given* $b \in \dot{\Lambda}_\beta$, *one has*

$$\left\|[b, L^{-\alpha/2}]f\right\|_q \le C\|b\|_{\dot{\Lambda}_\beta}\|f\|_p. \quad (68)$$

Proof. We are going to apply Theorem 1 to the linear operator $T = L^{-\alpha/2}$. We fix $p_- < p < q < p_+$, α, and β so that $1/q = 1/p - (\alpha + \beta)/n$. Then we can find p_0, q_0, s_0 such that $1/p_0 - 1/s_0 = \alpha/n$, $p_- < p_0 < s_0 < q_0 < p_+$, and $p_0 < p < q < q_0$. Notice that as $1 \le p_- < p_+ \le \infty$, we have that $1 < p_0 < s_0 < q_0 < \infty$. By Theorem 1.2 in [9], we know that $T = L^{-\alpha/2}$ is bounded from L^{p_0} to L^{s_0}.

We take $A_r = I - (I - e^{-r^2L})^m$, where $m \ge 1$ is an integer to be chosen. We apply Lemma 13. Note that (66) is (9). Also, (10) follows from (67) after expanding $A_r = I - (I - e^{-r^2L})^m$. Then, we have that $\sum_{j\le 1} g_i(j)$ for $i = 1, 2$ by choosing $2m > n/s_0$. Consequently applying Theorem 1, we conclude that $\|[b, L^{-\alpha/2}]f\|_q \le C\|b\|_{\dot{\Lambda}_\beta}\|f\|_p$. □

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