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Research Article

The Split Common Fixed Point Problem for ϱ-Strictly Pseudononspreading Mappings

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We introduce and analyze the viscosity approximation algorithm for solving the split common fixed point problem for the strictly pseudononspreading mappings in Hilbert spaces. Our results improve and develop previously discussed feasibility problems and related results.

1. Introduction

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let I denote the identity operator on H. Let H_1 and H_2 be two real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. Given closed convex subsets C and Q of H_1 and H_2, respectively.

The split feasibility problem (SFP) (Censor and Elfving 1994 [1]), modeling phase retrieval problems, is to find a point x^* with the property

$$x^* \in C, \quad Ax^* \in D. \tag{1}$$

Recently, it has been found that the SFP can also be used to model the intensity-modulated radiation therapy [2–8]. A special case of the SFP (1) is the convexly constrained linear problem:

$$Ax = b, \quad x \in C. \tag{2}$$

This problem, due to its applications in many applied disciplines, has extensively been investigated in the literature ever since Landweber [9] introduced his iterative method in 1951.

Note that the split feasibility problem (1) can be formulated as fixed point equation by using the fact

$$x^* = P_C\left(I - \gamma A^*\left(I - P_Q\right)A\right)x^*, \tag{3}$$

where P_C and P_Q are the projections onto C and Q, respectively, $\gamma > 0$ is any positive constant, and A^* denotes the adjoint of A; that is, x^* solves the SFP (1) if and only if x^* solves the fixed point equation (3) (see [10] for more details). This implies that we can use fixed point algorithms to solve SFP.

In 2002, Byrne [2] proposed his CQ algorithm to solve (1). The sequence $\{x_n\}$ is generated by the following iteration scheme:

$$x_{n+1} = P_C\left(I - \gamma A^*\left(I - P_Q\right)A\right)x_n, \quad n \in N, \tag{4}$$

where $\gamma \in (0, 2/\lambda)$, with λ being the spectral radius of the operator A^*A.

The CQ algorithm (4) is a special case of the K-M algorithm. Due to the fixed point formulation (2) of the SFP, Moudafi [11] applied the K-M algorithm to the operator $P_C(I - \gamma A^*(I - P_Q)A)$ to obtain a sequence given by

$$x_{n+1} = \left(1 - \alpha_n\right)x_n - \alpha_n P_C\left(I - \gamma A^*\left(I - P_Q\right)A\right)x_n, \quad n \in N, \tag{5}$$

where $\gamma \in (0, 2/\lambda)$, with λ being the spectral radius of the operator A^*A, and the sequence $\{\alpha_n\}$ satisfies the condition $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n) = +\infty$; he proved weak convergence result of the algorithm (5) in Hilbert spaces.

In 2009, Censor and Segal [12] considered the following algorithm to be solved (1).

Algorithm 1. Initialization: let $x^* \in H_1 = R^n$ be arbitrary.
Iterative step: for $n \in N$ let

$$x_{n+1} = U\left(x_n + \gamma A^*\left((V - I)Ax_n\right)\right), \quad n \in N, \tag{6}$$

where $\gamma \in (0, 2/\lambda)$ with λ being the spectral radius of the operator A^*A and U, V be a single pair of directed operators.

In 2010, Moudafi [13] extended the Algorithm 1 and introduced the following algorithm with weak convergence for the split common fixed point problem.

Algorithm 2. Initialization: let $x^* \in H_1 = R^n$ be arbitrary.
Iterative step: for $n \in N$ let

$$\begin{aligned} u_n &= x_n + \gamma\beta A^*(V - I)Ax_n, \\ x_{n+1} &= (1 - \alpha_n)u_n + \alpha_n U(u_n), \quad n \in N, \end{aligned} \tag{7}$$

where $\beta \in (0, 1)$, $\alpha_n \in (0, 1)$, and $\gamma \in (0, 1/\lambda\beta)$ with λ being the spectral radius of the operator A^*A and U, V be a pair of quasi-nonexpansive operators.

In 2012, Zhao and He [14] continue to consider the split common fixed point problem with quasi-nonexpansive operators and to use the following algorithm to obtain the strong convergence of the viscosity method for solving the split common fixed point problem.

Algorithm 3. Initialization: let $x^* \in H_1 = R^n$ be arbitrary.
Iterative step: for $n \in N$ let

$$\begin{aligned} T &= U\left(I + \gamma A^*(V - I)A\right) \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) \\ &\quad \times \left((1 - \omega_n)x_n + \omega_n Tx_n\right), \quad n \in N, \end{aligned} \tag{8}$$

where $f : H \to H$ is a contractive mapping with constant $\beta \in (0, 1)$, $\alpha_n \in (0, 1)$, and $\gamma \in (0, 1/\lambda)$ with λ being the spectral radius of the operator A^*A and U, V be a pair of quasi-nonexpansive operators.

Motivated and inspired by Censor and Segal [12], Moudafi [11], and Zhao and He [14], we introduce the following relaxed algorithm.

Algorithm 4. Initialization: let $x^* \in H_1 = R^n$ be arbitrary.
Iterative step: for $n \in N$ let

$$\begin{aligned} T &= U\left(I + \gamma A^*(V - I)A\right) \\ x_{n+1} &= \alpha_n \sigma f(x_n) + (I - \mu\alpha_n B) \\ &\quad \times \left((1 - \omega_n)x_n + \omega_n Tx_n\right), \quad n \in N, \end{aligned} \tag{9}$$

where $f : H \to H$ is a contractive mapping with constant $\beta \in (0, 1)$, $B : H \to H$ is η-strongly monotone and boundedly Lipschitzian, $\alpha_n \in (0, 1)$, and $\gamma \in (0, 1/\lambda)$ with λ being the spectral radius of the operator A^*A and U, V be a pair of ϱ_i-strictly pseudononspreading mappings $i = 1, 2$.

This paper establishes the strong convergence of the sequence given by (9) to the unique solution of solving the split common fixed point problem and the following variational inequality problem VIP($\mu B - \sigma f, T$):

$$\text{find } x^* \in \Gamma \text{ such that } \left\langle (\mu B - \sigma f)x^*, v - x^* \right\rangle \geq 0, \quad v \in \Gamma. \tag{10}$$

2. Preliminaries

In this section, we introduce the concepts of contraction mappings, nonexpansive mappings, quasi-nonexpansive mappings, and ϱ-strictly pseudononspreading mappings and some Lemmas.

Assume that C is a nonempty closed and convex subset of Hilbert space H. Recall that the (nearest point or metric) projection from H onto K, that denoted P_C, assigns, to each $x \in H$, the unique point $P_C x \in C$ with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}. \tag{11}$$

Definition 5. A mapping $T : H \to H$ is said to be

(1) contraction, if $\|Tx - Ty\| \leq \beta\|x - y\|$, $\forall x, y \in H$ and $\beta \in (0, 1)$;

(2) nonexpansive, if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in H$;

(3) quasi-nonexpansive, $\|Tx - q\| \leq \|x - q\|$, $\forall (x, q) \in H \times F_{ix}(T)$.

Remark 6. From the Definition 5, It is easy to see that

(i) iterative methods for quasi-nonexpansive mappings have been extensively investigated; see [13–17];

(ii) a nonexpansive mapping is a quasi-nonexpansive mapping.

Following the terminology of Browder and Petryshyn [18], we obtain the following definitions.

Definition 7. A mapping $T : D(T) \subseteq H \to H$ is ϱ-strictly pseudononspreading if there exists $\varrho \in [0, 1)$ such that

$$\begin{aligned} \|Tx - Ty\|^2 &\leq \|x - y\|^2 + \varrho\|x - Tx - (y - Ty)\|^2 \\ &\quad + 2\langle x - Tx, y - Ty\rangle, \end{aligned} \tag{12}$$

for all $x, y \in D(T)$.

Iterative methods for strictly pseudononspreading mapping have been extensively investigated; see [19–23].

Lemma 8 (see [24]). *Let H be a Hilbert spaces, and $f : H \to H$ is a contractive mapping with constant $\beta \in (0, 1)$. $B : H \to H$ is k-Lipschitzian and η-strongly monotone operator with $k > 0$, $\eta > 0$. Then for $0 < \sigma < \mu\eta/\beta$,*

$$\begin{aligned} &\langle x - y, (\mu B - \sigma f)x - (\mu B - \sigma f)y\rangle \\ &\quad \geq (\mu\eta - \sigma\beta)\|x - y\|^2, \quad x, y \in H. \end{aligned} \tag{13}$$

That is, $\mu B - \sigma f$ is strongly monotone with coefficient $\mu\eta - \sigma\beta$.

Lemma 9. *Let H be a real Hilbert space. Then the following well-known results hold: for all $x, y \in H$ and $t \in [0,1]$*

(i) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$;

(ii) $\|tx+(1-t)y\|^2 = t\|x\|^2+(1-t)\|y\|^2-t(1-t)\|x-y\|^2$;

(iii) $\langle x, y\rangle = -(1/2)\|x-y\|^2 + (1/2)\|x\|^2 + (1/2)\|y\|^2$.

Lemma 10. *Let T be a ϱ-strictly pseudononspreading mapping with $\varrho \in (0,1)$, and set $T_\alpha = (1-\alpha)I + \alpha T$, $\alpha \in (\varrho, 1)$. The following properties are reached for each $(x,p) \in H \times F_{ix}(T)$:*

(1) $\langle x-Tx, x-p\rangle \ge ((1-\varrho)/2)\|x-Tx\|^2$ *and* $\langle x-Tx, p-Tx\rangle \le ((1+\varrho)/2)\|x-Tx\|^2$;

(2) $\|T_\alpha x - p\|^2 \le \|x-p\|^2 - \alpha(1-\varrho-\alpha)\|x-Tx\|^2$;

(3) $\langle x-T_\alpha x, x-p\rangle \ge (\alpha(1-\varrho)/2)\|x-Tx\|^2$.

Proof. Note that property (1) is easily deduced from the Lemma 8(iii) and the fact that T is ϱ-strictly pseudononspreading mapping, we obtain

$$\begin{aligned}\langle x-Tx, x-p\rangle &= -\frac{1}{2}\|Tx-p\|^2 + \frac{1}{2}\|x-Tx\|^2 \\ &\quad + \frac{1}{2}\|x-p\|^2 \\ &= -\frac{1}{2}\|Tx-p\|^2 + \frac{1-\varrho}{2}\|x-Tx\|^2 \\ &\quad + \frac{1}{2}\left(\|x-p\|^2 + \varrho\|x-Tx\|^2\right) \\ &\ge -\frac{1}{2}\|Tx-p\|^2 + \frac{1}{2}\|Tx-p\|^2 \\ &\quad + \frac{1-\varrho}{2}\|x-Tx\|^2 \\ &\ge \frac{1-\varrho}{2}\|x-Tx\|^2, \\ \langle x-Tx, p-Tx\rangle &= -\frac{1}{2}\|x-p\|^2 + \frac{1}{2}\|x-Tx\|^2 \\ &\quad + \frac{1}{2}\|Tx-p\|^2 \\ &\le -\frac{1}{2}\|Tx-p\|^2 + \|Tx-p\|^2 \\ &\quad + \left(\frac{1}{2}+\varrho\right)\|Tx-p\|^2 \\ &\le \frac{1+\varrho}{2}\|x-Tx\|^2.\end{aligned} \tag{14}$$

Property (2) is obtained from property (1) and by

$$\begin{aligned}\|T_\alpha x - p\|^2 &= \|x-p\|^2 - 2\alpha\langle x-p, x-Tx\rangle \\ &\quad + \alpha^2\|Tx-x\|^2.\end{aligned} \tag{15}$$

Property (3) is given by $I - T_\alpha = \alpha(I-T)$ and property (1). □

Lemma 11 (see [25]). *Let $\{\mathcal{T}_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\mathcal{T}_{n_j}\}_{j\ge 0}$ of $\{\mathcal{T}_n\}$ which satisfies $\mathcal{T}_{n_j} < \mathcal{T}_{n_j+1}$ for all $j \ge 0$. Also consider the sequence of integers $\{\delta(n)\}_{n\ge n_0}$ defined by*

$$\delta(n) = \max\{k \le n \mid \mathcal{T}_k < \mathcal{T}_{k+1}\}. \tag{16}$$

Then $\{\delta(n)\}_{n\ge n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty}\delta(n) = \infty$, for all $n \ge n_0$; it holds that $\mathcal{T}_{\delta(n)} < \mathcal{T}_{\delta(n)+1}$ and one has

$$\mathcal{T}_n < \mathcal{T}_{\delta(n)+1}. \tag{17}$$

Lemma 12. *Let K be a closed convex subset of a real Hilbert space H, given $x \in H$ and $y \in K$. Then $y = P_K x$ if and only if there holds the inequality*

$$\langle x-y, y-z\rangle \ge 0, \quad \forall z \in K. \tag{18}$$

3. Main Results

In what follows, we will focus our attention on the following general two operator split common fixed point problem in real Hilbert space H:

$$\text{find } x^* \in C \text{ such that } Ax^* \in D, \tag{19}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $U : H_1 \to H_1$ and $V : H_2 \to H_2$ are two ϱ_i-strictly pseudononspreading mappings $i = 1,2$ with nonempty fixed point sets $F_{ix}(U) = C$ and $F_{ix}(V) = Q$, and denote the solution set of the two-operator SCFP by

$$\Gamma = \{y \in C; Ay \in Q\}. \tag{20}$$

On the other hand, $x^* \in \Gamma$ is also unique solution of solving the variational inequality problem VIP$(\mu B - \sigma f, T)$:

$$x^* \in \Gamma, \quad \langle(\mu B - \sigma f)x^*, y - x^*\rangle \ge 0, \quad y \in \Gamma, \tag{21}$$

where $B : H \to H$ is η-strongly monotone and k-Lipschitzian on H with $k > 0$, $\eta > 0$. Let $0 < \mu < 2\eta/k^2$, $0 < \sigma < \mu(\eta - (\mu k^2/2))/\beta = \tau/\beta$.

Before stating our main convergence result, we establish the boundedness of the iterates given by (9).

Lemma 13. *The sequence $\{x_n\}$ is generated by (9), and let $U : H_1 \to H_1$ and $V : H_2 \to H_2$ be two ϱ_i-strictly pseudononspreading mappings on H, $i = 1,2$, and $f : H \to H$ is a contractive mapping with constant $\beta \in (0,1)$, $\{\alpha_n\} \subset (0,1)$ and $0 < \varrho_i < \omega_n < 1/2$, $i = 1,2$. Then $\{x_n\}$ is bounded.*

Proof. Set $T_{\omega_n} = (1-\omega_n)I + \omega_n T$. Then $x_{n+1} = \alpha_n\sigma f(x_n) + (I - \mu\alpha_n B)T_{\omega_n}x_n$.

Taking $y \in \Gamma$, that is, $y \in F_{ix}(U)$ and $Ay \in F_{ix}(V)$. We obtain

$$\begin{aligned}\|x_{n+1}-y\| &= \|\alpha_n\sigma(f(x_n)-f(y))+\alpha_n(\sigma f(y)-\mu By)\\ &\quad+(I-\mu\alpha_n B)(T_{\omega_n}x_n-y)\|\\ &\le \alpha_n\sigma\|f(x_n)-f(y)\|+\alpha_n\|\sigma f(y)-\mu By\|\\ &\quad+(1-\alpha_n\tau)\|T_{\omega_n}x_n-y\|\\ &\le \alpha_n\sigma\beta\|x_n-y\|+\alpha_n\|\sigma f(y)-\mu By\|\\ &\quad+(1-\alpha_n\tau)\|T_{\omega_n}x_n-y\|.\end{aligned} \tag{22}$$

From the definition of T_{ω_n}, we have

$$\begin{aligned}\|T_{\omega_n}x_n-y\|^2 &= \|(1-\omega_n)x_n+\omega_n Tx_n-y\|^2\\ &= \|x_n-y+\omega_n(Tx_n-x_n)\|^2\\ &= \|x_n-y\|^2-2\omega_n\langle x_n-y,x_n-Tx_n\rangle\\ &\quad+\omega_n^2\|Tx_n-x_n\|^2.\end{aligned} \tag{23}$$

On the other hand, we obtain

$$\begin{aligned}\|Tx_n-y\|^2 &= \|U(I+\gamma A^*(V-I)A)x_n-y\|^2\\ &\le \|(I+\gamma A^*(V-I)A)x_n-y\|^2\\ &= \|x_n-y\|^2+\gamma^2\|A^*(V-I)Ax_n\|^2\\ &\quad+2\gamma\langle x_n-y,A^*(V-I)Ax_n-y\rangle\\ &= \|x_n-y\|^2+\gamma^2\langle(V-I)Ax_n,AA^*(V-I)Ax_n\rangle\\ &\quad+2\gamma\langle x_n-y,A^*(V-I)Ax_n\rangle.\end{aligned} \tag{24}$$

According to the definition of λ, we have

$$\begin{aligned}&\gamma^2\langle(V-I)Ax_n,AA^*(V-I)Ax_n\rangle\\ &\quad\le \lambda\gamma^2\langle(V-I)Ax_n,(V-I)Ax_n\rangle\\ &\quad= \lambda\gamma^2\|(V-I)Ax_n\|^2.\end{aligned} \tag{25}$$

Now, by using property (1) of Lemma 9, we obtain

$$\begin{aligned}&2\gamma\langle x_n-y,A^*(V-I)Ax_n\rangle\\ &\quad= 2\gamma\langle A(x_n-y),(V-I)Ax_n\rangle\\ &\quad= 2\gamma\langle A(x_n-y)+(V-I)Ax_n\\ &\qquad-(V-I)Ax_n,(V-I)Ax_n\rangle\\ &\quad= 2\gamma\left(\langle VAx_n-Ay,(V-I)Ax_n\rangle\right.\\ &\qquad\left.-\|(V-I)Ax_n\|^2\right)\\ &\quad\le 2\gamma\left(\frac{1-\varrho_2}{2}\|(V-I)Ax_n\|^2\right.\\ &\qquad\left.-\|(V-I)Ax_n\|^2\right)\\ &\quad\le -\gamma(1-\varrho_2)\|(V-I)Ax_n\|^2.\end{aligned} \tag{26}$$

Combining (24)–(26), we obtain

$$\begin{aligned}\|Tx_n-y\|^2 &= \|x_n-y\|^2+\lambda\gamma^2\|(V-I)Ax_n\|^2\\ &\quad-\gamma(1-\varrho_2)\|(V-I)Ax_n\|^2\\ &= \|x_n-y\|^2-\gamma((1-\varrho_2)-\lambda\gamma)\\ &\quad\times\|(V-I)Ax_n\|^2\\ &\le \|x_n-y\|^2.\end{aligned} \tag{27}$$

From property (i) of Lemma 9 and (23), we get

$$\begin{aligned}\|T_{\omega_n}x_n-y\|^2 &\le \|x_n-y\|^2-(1-\varrho_2)\omega_n\|x_n-Tx_n\|^2\\ &\quad+\omega_n^2\|Tx_n-x_n\|^2\\ &= \|x_n-y\|^2-\omega_n(1-\varrho_2-\omega_n)\\ &\quad\times\|x_n-Tx_n\|^2\\ &\le \|x_n-y\|^2.\end{aligned} \tag{28}$$

Combining (22), (23), and (28), we have

$$\begin{aligned}\|x_{n+1}-y\| &\le \alpha_n\sigma\beta\|x_n-y\|+\alpha_n\|f(y)-\mu By\|\\ &\quad+(1-\alpha_n\tau)\|T_{\omega_n}x_n-y\|\\ &= [1-\alpha_n(\tau-\sigma\beta)]\|x_n-y\|\\ &\quad+\alpha_n\|f(y)-\mu By\|\\ &\le \max\left\{\|x_n-y\|,\frac{1}{\tau-\sigma\beta}\|f(y)-\mu By\|\right\}.\end{aligned} \tag{29}$$

It follows from (29) and induction that

$$\|x_{n+1}-y\| \le \max\left\{\|x_0-y\|,\frac{1}{\tau-\sigma\beta}\|f(y)-\mu By\|\right\}, \tag{30}$$

and hence $\{x_n\}$ is bounded. □

Now we are in position to claim the main convergence result.

Theorem 14. *Given a bounded linear operator $A : H_1 \to H_2$, let $U : H_1 \to H_1$ and $V : H_2 \to H_2$ be two ϱ_i-strictly pseudononspreading mappings, and $i = 1, 2$ with fixed point $F_{ix}(U) = C$ and $F_{ix}(V) = Q$. Assume that $U - I$ and $V - I$ are demiclosed at origin. Let $B : H \to H$ be η-strongly monotone and k-Lipschitzian on H with $k > 0$, $\eta > 0$, and $f : H \to H$ is a contractive mapping with constant $\beta \in (0, 1)$. Assume that $\{x_n\}$ is the sequence given by Algorithm 4 with $\gamma \in (0, 1/\lambda)$, $0 < \varrho_i < \omega_n < 1/2$, and $i = 1, 2$ such that $0 < \liminf_{n\to\infty}\omega_n \le \limsup_{n\to\infty}\omega_n < 1/2$ and $\alpha_n \in (0, 1)$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. If $\Gamma \neq \emptyset$, then the sequence $\{x_n\}$ strongly converges to a split common fixed point $y \in \Gamma$, verifying $y = P_\Gamma(I - \mu B + \sigma f)(y)$ which equivalently solves the following variational inequality problem:*

$$y \in \Gamma, \quad \langle (\mu B - \sigma f) y, x^* - y\rangle \ge 0, \quad x^* \in \Gamma. \tag{31}$$

Proof. Let y be the solution of (31). From (9) we obtain that

$$x_{n+1} - x_n + \alpha_n (\mu B x_n - \sigma f(x_n)) = (I - \mu\alpha_n B)(T_{\omega_n} x_n - x_n), \tag{32}$$

hence

$$\begin{aligned}
&\langle x_{n+1} - x_n + \alpha_n (\mu B x_n - \sigma f(x_n)), x_n - y\rangle \\
&\quad = \langle (I - \mu\alpha_n B)(T_{\omega_n} x_n - x_n), x_n - y\rangle \\
&\quad = (1 - \alpha_n)\langle T_{\omega_n} x_n - x_n, x_n - y\rangle \\
&\qquad + \alpha_n \langle (I - \mu B)(T_{\omega_n} x_n - x_n), x_n - y\rangle \\
&\quad \le (1 - \alpha_n)\langle T_{\omega_n} x_n - x_n, x_n - y\rangle \\
&\qquad + \alpha_n \|(I - \mu B)(T_{\omega_n} x_n - x_n)\| \|x_n - y\| \\
&\quad \le (1 - \alpha_n)\langle T_{\omega_n} x_n - x_n, x_n - y\rangle \\
&\qquad + \alpha_n (1 - \tau)\|T_{\omega_n} x_n - x_n\| \|x_n - y\| \\
&\quad = (1 - \alpha_n)\langle T_{\omega_n} x_n - x_n, x_n - y\rangle \\
&\qquad + \omega_n\alpha_n (1 - \tau)\|Tx_n - x_n\| \|x_n - y\|.
\end{aligned} \tag{33}$$

By (28), we obtain that

$$\begin{aligned}
\langle x_n - T_{\omega_n} x_n, x_n - y\rangle &= \frac{\omega_n^2}{2}\|x_n - T_{\omega_n} x_n\|^2 \\
&\quad + \frac{1}{2}\|x_n - y\|^2 - \frac{1}{2}\|T_{\omega_n} x_n - y\|^2 \\
&\ge \frac{\omega_n^2}{2}\|x_n - Tx_n\|^2 + \frac{1}{2}\|x_n - y\|^2 \\
&\quad - \frac{1}{2}\|x_n - y\|^2 \\
&\quad + \frac{\omega_n}{2}(1 - \varrho_2 - \omega_n)\|x_n - Tx_n\|^2 \\
&= \frac{\omega_n(1 - \varrho_2)}{2}\|x_n - Tx_n\|^2.
\end{aligned} \tag{34}$$

It follows from (33) that

$$\begin{aligned}
&\langle x_{n+1} - x_n + \alpha_n (\mu B x_n - \sigma f(x_n)), x_n - y\rangle \\
&\quad \le -\frac{\omega_n}{2}(1 - \varrho_2)(1 - \alpha_n)\|x_n - Tx_n\|^2 \\
&\qquad + \omega_n\alpha_n (1 - \tau)\|Tx_n - x_n\| \|x_n - y\|
\end{aligned} \tag{35}$$

or equivalently

$$\begin{aligned}
-\langle x_n - x_{n+1}, x_n - y\rangle &\le -\alpha_n \langle (\mu B - \sigma f) x_n, x_n - y\rangle \\
&\quad - \frac{\omega_n}{2}(1 - \varrho_2)(1 - \alpha_n)\|x_n - Tx_n\|^2 \\
&\quad + \omega_n\alpha_n (1 - \tau)\|Tx_n - x_n\| \|x_n - y\|.
\end{aligned} \tag{36}$$

Furthermore, using the classical equality (iii) in Lemma 10 and setting $\mathcal{T}_n = (1/2)\|x_n - y\|^2$, we have

$$\langle x_n - x_{n+1}, x_n - y\rangle = \mathcal{T}_n - \mathcal{T}_{n+1} + \frac{1}{2}\|x_n - x_{n+1}\|^2. \tag{37}$$

So that (36) can be equivalently rewritten as

$$\begin{aligned}
&\mathcal{T}_{n+1} - \mathcal{T}_n - \frac{1}{2}\|x_n - x_{n+1}\|^2 \\
&\quad \le -\alpha_n \langle (\mu B - \sigma f) x_n, x_n - y\rangle \\
&\qquad - \frac{\omega_n}{2}(1 - \varrho_2)(1 - \alpha_n)\|x_n - Tx_n\|^2 \\
&\qquad + \omega_n\alpha_n (1 - \tau)\|Tx_n - x_n\| \|x_n - y\|.
\end{aligned} \tag{38}$$

Now using (32) again, we have

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 = &\|\alpha_n (\sigma f(x_n) - \mu B x_n) \\
&+ (I - \mu\alpha_n B)(T_{\omega_n} x_n - x_n)\|^2.
\end{aligned} \tag{39}$$

Since $B : H \to H$ is η-strongly monotone and k-Lipschitzian on H, hence it is a classical matter to see that

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &\le 2\alpha_n^2\|\sigma f(x_n) - \mu B x_n\|^2 \\
&\quad + 2(1 - \alpha_n\tau)^2\|T_{\omega_n} x_n - x_n\|^2,
\end{aligned} \tag{40}$$

which by $\|T_{\omega_n} x_n - x_n\| = \omega_n\|x_n - Tx_n\|$ yields

$$\begin{aligned}
\frac{1}{2}\|x_{n+1} - x_n\|^2 &\le \alpha_n^2\|\sigma f(x_n) - \mu B x_n\|^2 \\
&\quad + (1 - \alpha_n\tau)^2\omega_n^2\|x_n - Tx_n\|^2.
\end{aligned} \tag{41}$$

Then from (38) and (41), we have

$$\begin{aligned}
&\mathcal{T}_{n+1} - \mathcal{T}_n + \left[\frac{\omega_n}{2}(1 - \varrho_2)(1 - \alpha_n) - \omega_n^2(1 - \alpha_n\tau)^2\right] \\
&\quad \times \|x_n - Tx_n\|^2 \\
&\qquad \le \alpha_n \left[\alpha_n\|\sigma f(x_n) - \mu B x_n\|^2\right. \\
&\qquad\quad - \langle (\mu B - \sigma f) x_n, x_n - y\rangle \\
&\qquad\quad \left. + \omega_n (1 - \tau)\|Tx_n - x_n\| \|x_n - y\|\right].
\end{aligned} \tag{42}$$

The rest of the proof will be divided into two parts.

Case 1. Suppose that there exists n_0 such that $\{\mathcal{T}_n\}_{n\geq n_0}$ is nonincreasing. In this situation, $\{\mathcal{T}_n\}$ is then convergent because it is also nonnegative (hence it is bounded from below), so that $\lim_{n\to\infty}(\mathcal{T}_{n+1}-\mathcal{T}_n)=0$; hence, in light of (42) together with $\lim_{n\to\infty}\alpha_n=0$, the boundedness of $\{x_n\}$ and $0<\liminf_{n\to\infty}\omega_n\leq\limsup_{n\to\infty}\omega_n<1/2$, we obtain

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{43}$$

It also follows from (42) that

$$\begin{aligned}\mathcal{T}_n-\mathcal{T}_{n+1}\geq\alpha_n\big(&-\alpha_n\|\sigma f(x_n)-\mu Bx_n\|^2\\&+\langle(\mu B-\sigma f)x_n,x_n-y\rangle\\&-\omega_n(1-\tau)\|Tx_n-x_n\|\|x_n-y\|\big).\end{aligned} \tag{44}$$

Then, by $\sum_{n=0}^{\infty}\alpha_n=\infty$, we obviously deduce that

$$\begin{aligned}\liminf_{n\to\infty}\alpha_n\big(&-\alpha_n\|\sigma f(x_n)-\mu Bx_n\|^2\\&+\langle(\mu B-\sigma f)x_n,x_n-y\rangle\\&-\omega_n(1-\tau)\|Tx_n-x_n\|\|x_n-y\|\big)\leq 0\end{aligned} \tag{45}$$

or equivalently (as $\alpha_n\|\sigma f(x_n)-\mu Bx_n\|^2\to 0$ and $\lim_{n\to\infty}(\mathcal{T}_{n+1}-\mathcal{T}_n)=0$). From (45), we get

$$\liminf_{n\to\infty}\langle(\mu B-\sigma f)x_n,x_n-y\rangle\leq 0. \tag{46}$$

Moreover, by Lemma 8, we have

$$\begin{aligned}&2(\mu\eta-\sigma\beta)\mathcal{T}_n+\langle(\mu B-\sigma f)y,x_n-y\rangle\\&\quad\leq\langle(\mu B-\sigma f)x_n,x_n-y\rangle,\end{aligned} \tag{47}$$

which by (46) entails

$$\liminf_{n\to\infty}\langle 2(\mu\eta-\sigma\beta)\mathcal{T}_n+(\mu B-\sigma f)y,x_n-y\rangle\leq 0. \tag{48}$$

Hence, recalling that $\lim_{n\to\infty}\mathcal{T}_n$ exists, we equivalently obtain

$$2(\mu\eta-\sigma\beta)\lim_{n\to\infty}\mathcal{T}_n+\liminf_{n\to\infty}\langle(\mu B-\sigma f)y,x_n-y\rangle\leq 0, \tag{49}$$

Namely,

$$2(\mu\eta-\sigma\beta)\lim_{n\to\infty}\mathcal{T}_n\leq-\liminf_{n\to\infty}\langle(\mu B-\sigma f)y,x_n-y\rangle. \tag{50}$$

Now we prove that

$$\liminf_{n\to\infty}\langle(\mu B-\sigma f)y,x_n-y\rangle\leq 0. \tag{51}$$

It follows from (27) and (43) that

$$\begin{aligned}&\gamma((1-\varrho_2)-\lambda\gamma)\|(V-I)Ax_n\|^2\\&\quad\leq\|x_n-y\|^2-\|Tx_n-y\|^2\\&\quad=(\|x_n-y\|-\|Tx_n-y\|)\\&\qquad\times(\|x_n-y\|+\|Tx_n-y\|)\\&\quad=(\|x_n-Tx_n\|)(\|x_n-y\|+\|Tx_n-y\|)\\&\qquad\longrightarrow 0,\quad(n\longrightarrow\infty),\end{aligned} \tag{52}$$

and hence

$$\lim_{n\to\infty}\|(V-I)Ax_n\|=0. \tag{53}$$

Taking $x^*\in\omega_w(x_n)$, from the demiclosedness of $V-I$ at 0, we have

$$V(Ax^*)=Ax^*. \tag{54}$$

Now, by setting $u_n=x_n+\gamma A^*V-IAx_n$, it follows that $x^*\in\omega_w(u_n)$. On the other hand,

$$\begin{aligned}\|U(u_n)-u_n\|&=\|Tx_n-x_n-\gamma A^*V-IAx_n\|\\&\leq\|Tx_n-x_n\|+\gamma\|A^*\|\|V-IAx_n\|\longrightarrow 0,\end{aligned} \tag{55}$$

which, combined with the demiclosedness of $U-I$ at 0, yields

$$Ux^*=x^*. \tag{56}$$

Hence, $x^*\in C$ and $x^*\in\Gamma$. We can take subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{n\to\infty}x_{n_j}=x^*$ and

$$\liminf_{n\to\infty}\langle(\sigma f-\mu B)y,x_n-y\rangle=\lim_{j\to\infty}\left\langle(\sigma f-\mu B)y,x_{n_j}-y\right\rangle, \tag{57}$$

which leads to

$$\liminf_{n\to\infty}\langle(\sigma f-\mu B)y,x_n-y\rangle=\langle(\sigma f-\mu B)y,x^*-y\rangle\leq 0. \tag{58}$$

By (50), we have $\lim_{n\to\infty}\Gamma_n=0$, and hence $\{x_n\}$ converges strongly to y.

Case 2. Suppose that there exists a subsequence $\{\mathcal{T}_{n_k}\}_{k\geq 0}$ of $\{\mathcal{T}_n\}_{n\geq 0}$ such that $\mathcal{T}_{n_k}\leq\mathcal{T}_{n_{k+1}}$ for all $k\geq 0$. In this situation, we consider the sequence of indices $\{\delta(n)\}$ as defined in Lemma 11. It follows that $\mathcal{T}_{\delta(n+1)}-\mathcal{T}_{\delta(n)}>0$, which by (42) amounts to

$$\begin{aligned}&\left[\frac{\omega_n}{2}(1-\varrho_2)(1-\alpha_{\delta(n)})-\omega_n^2(1-\alpha_{\delta(n)}\tau)^2\right]\|x_{\delta(n)}-Tx_{\delta(n)}\|^2\\&\quad\leq\alpha_{\delta(n)}\big[\alpha_{\delta(n)}\|\sigma f(x_{\delta(n)})-\mu Bx_n\|^2\\&\qquad-\langle(\mu B-\sigma f)x_n,x_n-y\rangle\\&\qquad+\omega_n(1-\tau)\|Tx_{\delta(n)}-x_{\delta(n)}\|\|x_{\delta(n)}-y\|\big].\end{aligned} \tag{59}$$

By the boundedness of $\{x_n\}$ and $\lim_{n\to\infty}\alpha_n = 0$, we immediately obtain

$$\lim_{n\to\infty}\left\|x_{\delta(n)} - Tx_{\delta(n)}\right\| = 0. \tag{60}$$

Using (9), we have

$$\begin{aligned}\left\|x_{\delta(n)+1} - x_{\delta(n)}\right\| &\le \alpha_{\delta(n)}\left\|\sigma f\left(x_{\delta(n)}\right) - \mu B x_{\delta(n)}\right\| \\ &\quad + \left|1 - \alpha_{\delta(n)}\tau\right|\left\|T_{\omega_n}x_{\delta(n)} - x_{\delta(n)}\right\| \\ &\le \alpha_{\delta(n)}\left\|\sigma f\left(x_{\delta(n)}\right) - \mu B x_{\delta(n)}\right\| \\ &\quad + \left|1 - \alpha_{\delta(n)}\omega_n\tau\right|\left\|Tx_{\delta(n)} - x_{\delta(n)}\right\|,\end{aligned} \tag{61}$$

which together with (60) and $\lim_{n\to\infty}\alpha_n = 0$ yields

$$\lim_{n\to\infty}\left\|x_{\delta(n)+1} - x_{\delta(n)}\right\| = 0. \tag{62}$$

Similar to Case 1, we have

$$\liminf_{n\to\infty}\left\langle\left(\mu B - \sigma f\right)y, x_{\delta(n)} - y\right\rangle \ge 0. \tag{63}$$

Now by (59) we clearly have

$$\begin{aligned}&\alpha_{\delta(n)}\left\|\sigma f\left(x_{\delta(n)}\right) - \mu B x_{\delta(n)}\right\|^2 + \alpha\left(1-\tau\right) \\ &\quad\times\left\|Tx_{\delta(n)} - x_{\delta(n)}\right\|\left\|x_{\delta(n)} - y\right\| \\ &\quad\ge\left\langle\left(\mu B - \sigma f\right)x_{\delta(n)}, x_{\delta(n)} - y\right\rangle,\end{aligned} \tag{64}$$

which in the light of (47) yields

$$\begin{aligned}&2\left(\mu\eta - \sigma\beta\right)\mathcal{T}_{\delta(n)} + \left\langle\left(\mu B - \sigma f\right)y, x_{\delta(n)} - y\right\rangle \\ &\quad\le \alpha_{\delta(n)}\left\|\sigma f\left(x_{\delta(n)}\right) - \mu B x_{\delta(n)}\right\|^2 \\ &\quad\quad + \alpha\left(1-\tau\right)\left\|Tx_{\delta(n)} - x_{\delta(n)}\right\|\left\|x_{\delta(n)} - y\right\|.\end{aligned} \tag{65}$$

Hence (as $\lim_{n\to\infty}\alpha_{\delta(n)}\|\sigma f(x_{\delta(n)}) - \mu B x_{\delta(n)}\|^2 = 0$ and $\lim_{n\to\infty}\|Tx_{\delta(n)} - x_{\delta(n)}\| = 0$) it follows that

$$2\left(\mu\eta - \sigma\beta\right)\limsup_{n\to\infty}\mathcal{T}_{\delta(n)} \le -\liminf_{n\to\infty}\left\langle\left(\mu B - \sigma f\right)y, x_{\delta(n)} - y\right\rangle. \tag{66}$$

From (59) and (63), we obtain

$$\lim_{n\to\infty}\left\langle\left(\mu B - \sigma f\right)y, x_{\delta(n)} - y\right\rangle \ge 0, \tag{67}$$

which by (60) yields $\limsup_{n\to\infty}\mathcal{T}_{\delta(n)} = 0$, so that $\lim_{n\to\infty}\mathcal{T}_{\delta(n)} = 0$. Combining (62), we have $\lim_{n\to\infty}\mathcal{T}_{\delta(n)+1} = 0$. Then, recalling that $\mathcal{T}_n < \mathcal{T}_{\delta(n)+1}$ (by Lemma 11), we get $\lim_{n\to\infty}\mathcal{T}_n = 0$, so that $x_n \to y$ strongly.

In addition, the variational inequality (50) and (67) can be written as

$$\left\langle\left(I - \mu B + \sigma f\right)y - y, x^* - y\right\rangle \le 0, \quad x^* \in \Gamma. \tag{68}$$

So, by the Lemma 12, it is equivalent to the fixed point equation

$$P_\Gamma\left(I - \mu B + \sigma f\right)y = y. \tag{69}$$

□

4. Application in Other Nonlinear Operators

In order to define our motivations, we recall some definitions of classed of operators as follows

Definition 15. $T : D(T) \subseteq H \to H$ is said to be

(1) nonspreading in [26, 27], if

$$\left\|Tx - Ty\right\|^2 \le \left\|Tx - y\right\|^2 + \left\|Ty - x\right\|^2, \quad \forall x, y \in C, \tag{70}$$

(2) demicontractive in [28], if there exists a constant $\alpha < 1$ such that

$$\begin{aligned}\left\|Tx - q\right\|^2 &\le \left\|x - q\right\|^2 + \alpha\|x - Tx\|^2, \\ &\quad\forall\left(x, q\right) \in H \times F_{ix}\left(T\right).\end{aligned} \tag{71}$$

Remark 16. Iemoto and Takahashi [29] proved that (70) is equivalent to

$$\begin{aligned}\left\|Tx - Ty\right\|^2 &\le \left\|x - y\right\|^2 + 2\left\langle x - Tx, y - Ty\right\rangle, \\ &\quad\forall x, y \in C.\end{aligned} \tag{72}$$

Iterative methods for nonspreading mapping have been extensively investigated; see [30–34].

Remark 17. From the Definition 5 (3), Definition 7, and Definition 15, we have the following facts.

(i) Observe that every nonspreading mapping is 0-strictly pseudononspreading.

(ii) If T is nonspreading mapping and the set of fixed point is nonempty, then T is quasi-nonexpansive mapping.

(iii) Every pseudononspreading mapping with a nonempty fixed point set $F_{ix}(T)$ is demicontractive (see [28]).

Corollary 18. *Given a bounded linear operator $A : H_1 \to H_2$, let $U : H_1 \to H_1$ and $V : H_2 \to H_2$ be two nonspreading mappings with fixed point $F_{ix}(U) = C$ and $F_{ix}(V) = Q$. Assume that $U - I$ and $V - I$ are demiclosed at origin. Let $B : H \to H$ be η-strongly monotone and k-Lipschitzian on H with $k > 0$, $\eta > 0$, and let $f : H \to H$ be a contractive mapping with constant $\beta \in (0,1)$. Let $\{x_n\}$ be the sequence given by (9) with $\gamma \in (0, 1/\lambda)$, $0 < \varrho_i < \omega_n < 1/2$, and $i = 1, 2$ such that $0 < \liminf_{n\to\infty}\omega_n \le \limsup_{n\to\infty}\omega_n < 1/2$ and $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. If $\Gamma \ne \emptyset$, then the sequence $\{x_n\}$ strongly converges to a split common fixed point $x^* \in \Gamma$, verifying $x^* = P_\Gamma(I - \mu B + \sigma f)(x^*)$ which equivalently solves the following variational inequality problem:*

$$x^* \in \Gamma, \quad \left\langle\left(\mu B - \sigma f\right)x^*, y - x^*\right\rangle \ge 0, \quad y \in \Gamma. \tag{73}$$

Proof. Form the proof of the Theorem 14, we can easily certify this theorem by nonspreading mapping (i.e., nonspreading is 0-strictly pseudononspreading). □

From the Remark 17(ii) and the Corollary 18, we have the following corollary.

Corollary 19. *Given a bounded linear operator $A : H_1 \to H_2$, let $U : H_1 \to H_1$ and $V : H_2 \to H_2$ be two quasi-nonexpansives with fixed point $F_{ix}(U) = C$ and $F_{ix}(V) = Q$. Assume that $U - I$ and $V - I$ are demiclosed at origin. Let $B : H \to H$ be η-strongly monotone and k-Lipschitzian on H with $k > 0$, $\eta > 0$, and let $f : H \to H$ be a contractive mapping with constant $\beta \in (0,1)$. Let $\{x_n\}$ be the sequence given by (9) with $\gamma \in (0, 1/\lambda)$, $\omega_n \in (0, 1/2)$ such that $0 < \liminf_{n\to\infty}\omega_n \le \limsup_{n\to\infty}\omega_n < 1/2$ and $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. If $\Gamma \neq \emptyset$, then the sequence $\{x_n\}$ strongly converges to a split common fixed point $x^* \in \Gamma$, verifying $x^* = P_\Gamma f(x^*)$ which equivalently solves the following variational inequality problem:*

$$x^* \in \Gamma, \quad \langle (\mu B - \sigma f) x^*, y - x^* \rangle \ge 0, \quad y \in \Gamma. \tag{74}$$

If $\sigma = \mu = 1$ and $B = I$ in (9), thus $k = \eta = 1$ and $\beta \in (0, 1/2)$, and then we obtain (8) and the following corollary. On the other hand, this corollary was proven by Zhao and He [14].

Corollary 20. *Given a bounded linear operator $A : H_1 \to H_2$, let $U : H_1 \to H_1$ and $V : H_2 \to H_2$ be two quasi-nonexpansives with fixed point $F_{ix}(U) = C$ and $F_{ix}(V) = Q$. Assume that $U - I$ and $V - I$ are demiclosed at origin. Let $f : H \to H$ be a contractive mapping with constant $\beta \in (0, 1/2)$. Let $\{x_n\}$ be the sequence given by (8) with $\gamma \in (0, 1/\lambda)$, $\omega_n \in (0, 1/2)$ such that $0 < \liminf_{n\to\infty}\omega_n \le \limsup_{n\to\infty}\omega_n < 1/2$, and $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. If $\Gamma \neq \emptyset$, then the sequence $\{x_n\}$ strongly converges to a split common fixed point $x^* \in \Gamma$, verifying $x^* = P_\Gamma(I - \mu B + \sigma f)(x^*)$ which equivalently solves the following variational inequality problem:*

$$x^* \in \Gamma, \quad \langle (I - f) x^*, y - x^* \rangle \ge 0, \quad y \in \Gamma. \tag{75}$$

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