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Research Article

Investigation of the Equivalent Representation Form of Strongly Damped Nonlinear Oscillators by a Nonlinear Transformation Approach

Alex Elías-Zúñiga and Oscar Martínez-Romero

Departamento de Ingeniería Mecánica, Tecnológico de Monterrey, Monterrey Campus, Eugenio Garza Sada 2501 Sur, 64849 Monterrey, NL, Mexico

Correspondence should be addressed to Alex Elías-Zúñiga; aelias@itesm.mx

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We use a nonlinear transformation method to develop equivalent equations of motion of nonlinear homogeneous oscillatory systems with linear and nonlinear odd damping terms. We illustrate the applicability of our approach by using the equations of motion that arise in many engineering problems and compare their amplitude-time curves with those obtained by the numerical integration solutions of the original equations of motion.

1. Introduction

The dynamics response of some systems can be more precisely described when nonlinear damping terms are used to model their dynamics behaviors. For instance, the elastomeric vibration isolators [1], the motion of a rolling ship subjected to the synchronous beam waves [2], the backlash and impact phenomena [3], and the micromechanical oscillators [4], to say a few, have been modeled by considering nonlinear damping terms. In fact, the dynamic behavior of double-well oscillators in which a nonlinear damping term with a fractional exponent covers the gaps between viscous, dry friction, and turbulent damping phenomena has been used by Litak et al. in [5] to study, by using the Melnikov criterion, the system global homoclinic bifurcation and its transition to chaos. It is evident from the previously mentioned works and references cited therein that the global system dynamics behavior can be accurately described if one is able to identify the order of the nonlinear stiffness and the damping effects that agree with the experimental observations [6]. Of course, the influence of the nonlinear damping terms on the resulting equations of motion increases the difficulty of finding their closed-form solutions.

In this paper, a nonlinear transformation of the damped nonlinear equation

$$\ddot{x} + F(x, \dot{x}) = 0, \qquad x(0) = x_{10}, \qquad \dot{x}(0) = 0, \tag{1}$$

is proposed to obtain its equivalent damped Duffing's equation of motion. Here, we assume that $F(x, \dot{x})$ is the nonconservative system restoring force which could have rational or irrational conservative force terms as well as linear or nonlinear damping terms, and we assume that x_{10} is the initial amplitude. The main motivation to find a nonlinear transformation form of (1) is based on the fact that exact or approximate solutions of damped nonlinear oscillators of the Duffing type can be found in the literature. See, for instance, [7–10] and references cited therein. Therefore, if we can transform (1) into the damped Duffing equation, one could find its dynamical response in an easier way.

In order to achieve such transformation, we first assume that the conservative terms of $F(x, \nu\dot{x})$ can be written in equivalent forms by using, for instance, the Chebyshev polynomials of the first kind [11–15]:

$$f(x) \sum_{n=0}^{N} b_{2n+1}(x_{10}) T_{2n+1}(x), \tag{2}$$

where

$$b_{2n+1} = \frac{2}{\pi}\int_{-1}^{+1}\frac{1}{\sqrt{1-x^2}}f(x)\,T_{2n+1}(x)\,dx. \tag{3}$$

Since the first three Chebyshev polynomials of the first kind are given by

$$T_1(x) = x, \qquad T_3(x) = 4x^3 - 3x, \qquad T_5(x) = 16x^5 - 20x^3 + 5x, \tag{4}$$

thus, the equivalent conservative restoring force $f(x)$ can be written as

$$\begin{aligned} f(x) &\equiv b_1(q)\,T_1(x) + b_3(q)\,T_3(x) + b_5(q)\,T_5(x) \\ &\approx \alpha(x_{10})\,x + \beta(x_{10})\,x^3 + \gamma(x_{10})\,x^5. \end{aligned} \tag{5}$$

One must notice that the coefficients $\alpha(x_{10})$, $\beta(x_{10})$, and $\gamma(x_{10})$ depend on the amplitude of oscillation, x_{10}, and the Chebyshev coefficient terms. Therefore, we assume that the nonlinear differential equation (1) can be replaced by an equivalent equation of the form

$$\frac{d^2x}{dt^2} + f(\nu\dot{x}) + \alpha(A)\,x + \beta(A)\,x^3 + \gamma(A)\,x^5 \approx 0, \tag{6}$$

where $f(\dot{x})$ represents the system damping terms and ν is the damping coefficient. We next use a cubication transformation to write the restoring force term of (6) in the form

$$\begin{aligned} F(x,\dot{x}) = {} & f(\nu\dot{x}) + \alpha(x_{10})\,x + \beta(x_{10})\,x^3 + \gamma(x_{10})\,x^5 \\ \equiv {} & f(\nu_1\dot{x}) + \delta(x_{10})\,x + \epsilon(x_{10})\,x^3. \end{aligned} \tag{7}$$

Here, ν_1, $\delta(x_{10})$, and $\epsilon(x_{10})$ can be found by replacing the terms $f(\nu\dot{x}) + \alpha(x_{10})x + \beta(x_{10})x^3 + \gamma(x_{10})x^5$ by the cubic polynomial $f(\nu_1\dot{x}) + \delta(x_{10})x + \epsilon(x_{10})x^3$ that satisfies

$$\begin{aligned} F_1(\delta,\epsilon,\nu_1) = {} & \int_0^{\sigma}\Big(f(\nu\dot{x}) + \alpha x + \beta x^3 + \gamma x^5 \\ & -f(\nu_1\dot{x}) - \delta x - \epsilon x^3\Big)^2 dx \longrightarrow \min, \end{aligned} \tag{8}$$

$$\begin{aligned} F_2(\delta,\epsilon,\nu_1) = {} & \int_0^{\upsilon}\Big(f(\nu\dot{x}) + \alpha x + \beta x^3 + \gamma x^5 \\ & -f(\nu_1\dot{x}) - \delta x - \epsilon x^3\Big)^2 d\dot{x} \longrightarrow \min, \end{aligned} \tag{9}$$

$$\frac{\partial F_1(\delta,\epsilon,\nu_1)}{\partial\delta} = 0, \qquad \frac{\partial F_1(\delta,\epsilon,\nu_1)}{\partial\epsilon} = 0, \qquad \frac{\partial F_2(\delta,\epsilon,\nu_1)}{\partial\nu_1} = 0. \tag{10}$$

Thus, the cubic-like equivalent representation form of (1) is given as

$$\begin{aligned} & \frac{d^2x}{dt^2} + f(\nu\dot{x}) + \alpha(x_{10})\,x + \beta(x_{10})\,x^3 + \gamma(x_{10})\,x^5 \\ & \quad \equiv \frac{d^2x}{dt^2} + f(\nu_1\dot{x}) + \delta(x_{10})\,x + \epsilon(x_{10})\,x^3 \approx 0. \end{aligned} \tag{11}$$

We next examine the application of our proposed nonlinear transformation approach to obtain approximate solutions of the damped oscillatory systems such as the damped cubic-quintic Duffing equation, the damped general pendulum equation of motion, the damped rational-form elastic term oscillator, and the nonlinear damped cubic term oscillator.

2. The Damped Cubic-Quintic Duffing Equation

It is well known that this equation is used to describe the dynamical behavior of beams resting on an elastic substrate [16], the nonlinear transverse vibration of a hinged-hinged flexible beam subjected to constant excitation at its free end [17], the biological rhythmic movements [18], the propagation of electromagnetic pulses in media with saturable nonlinearity [19], the intermodulation distortion in radio-frequency microelectromechanical systems (MEMS) capacitors [20], the motion of a rolling ship subjected to synchronous beam waves [2], and so forth. This equation has the form

$$\begin{aligned} & \ddot{y} + 2\nu\dot{y} + \alpha y + \beta y^3 + \gamma y^5 = 0, \\ & \text{with } y(0) = y_{10},\ \dot{y}(0) = 0, \end{aligned} \tag{12}$$

where y denotes the displacement of the system, ν is the damping coefficient, and α, β, and γ are the system constant parameters. We next introduce the following change of the variable $x = A/y$ which transforms (13) into an equation of the form

$$\begin{aligned} & \ddot{x} + 2\nu\dot{x} + Ax + Bx^3 + Gx^5 = 0, \\ & \text{with } x(0) = 1,\ \dot{x}(0) = 0. \end{aligned} \tag{13}$$

Here, $A = \alpha$, $B = \beta y_{10}^2$, and $G = \gamma y_{10}^4$. In accordance with our proposed nonlinear transformation approach, we first replace the restoring force $F(x,\dot{x}) = 2\nu\dot{x} + Ax + Bx^3 + Gx^5$ by an equivalent cubic-like polynomial expression by using (8), (9), and (10). This provides the following restoring force expression:

$$F(x,\dot{x}) = 2\nu\dot{x} + Ax + Bx^3 + Gx^5 \equiv 2\nu_1\dot{x} + \delta x + \epsilon x^3. \tag{14}$$

Thus, (13) can be written in an equivalent form as follows:

$$\ddot{x} + 2\nu_1\dot{x} + \delta x + \epsilon x^3 \approx 0, \tag{15}$$

where

$$\delta = \frac{1}{49}\left(49A - 5G\sigma^4\right),$$
$$\epsilon = \frac{1}{189}\left(189B + 190G\sigma^2\right), \tag{16}$$
$$\nu_1 = -\frac{-1323\upsilon\nu + 32G\sigma^5}{1323\upsilon},$$

$$\delta = \frac{1}{21}\left(21A - 25G\sigma^4\right),$$
$$\epsilon = \frac{1}{27}\left(27B + 50G\sigma^2\right), \tag{17}$$
$$\nu_1 = -\frac{-189\upsilon\nu - 32G\sigma^5}{189\upsilon}.$$

Notice that σ and υ are the parameter values that must satisfy (9) and (10). Figure 1 illustrates the numerical integration solutions of (12) and (15) by considering the parameter values $\alpha = 5$, $\beta = 2$, $\gamma = 0.5$, and $\nu = 0.1$ with the initial conditions $y(0) = 1$ and $\dot{y}(0) = 0$. To obtain the smallest root-mean-square error (RMSE) between both numerical solutions, we have fitted the values $\sigma = 0.94$ and $\upsilon = 100000$ in (17). Thus, the parameter values of the damped cubic Duffing equation (15) become $\delta = 4.9601$, $\epsilon = 2.4441$, and $\nu_1 = 0.0999$ with an RMSE value of 0.0111 for the time interval shown in Figure 1. Also, we have computed the RMSE values by considering different system initial conditions as showed in Table 1. As we may see from Table 1, the RMSE values are not bigger than 0.667. As a second example, we now assume the following system parameter values $\alpha = 1$, $\beta = 20$, $\gamma = 0.1$, and $\nu = 0.25$. The RMSE values obtained from the comparison of the numerical integration solutions of (14) and (15) are shown in Table 2. Notice that the maximum RMSE value is now 0.318 when $y_{10} = 10$.

We will next examine the applicability of our proposed approach by deriving the equivalent expression of the damped general pendulum equation.

3. A Damped Pendulum Equation

We now use our nonlinear transformation approach to find the equivalent equation of motion of the damped pendulum equation [21]

$$\frac{d^2 y}{dt^2} + f(y, \dot{y}) = 0, \qquad f(y, \dot{y}) = 2\nu\dot{y} - by + a\sin y, \tag{18}$$

with the initial conditions $y(0) = y_{10}$ and $\dot{y}(0) = 0$. Here, a and b represent the system constant parameter values. First, let us introduce the transformation $x = y/y_{10}$ and re-write (18) as follows:

$$\frac{d^2 x}{dt^2} + 2\nu\dot{x} - bx + \frac{a}{y_{10}}\sin(xy_{10}) = 0, \quad x(0) = 1,\ \dot{x}(0) = 0. \tag{19}$$

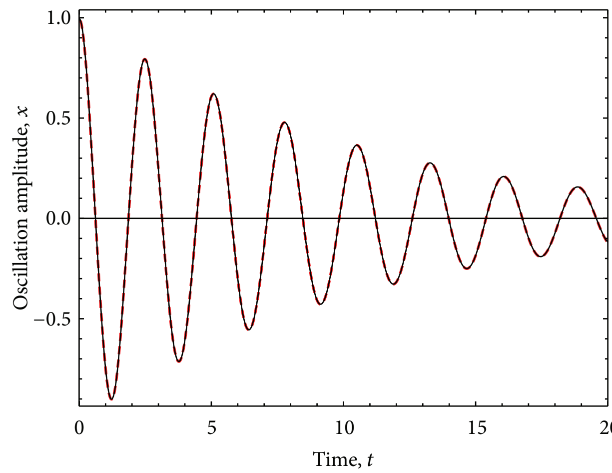

Figure 1: Amplitude-time response curves of (1) and (15). Here, the solid line represents the numerical integration solution of (1), while the dashed line represents the prediction obtained by using the derived equivalent equation of motion by applying our enhanced cubication procedure. The parameter values are $\alpha = 5$, $\beta = 2$, and $\gamma = 0.5$, and $\nu = 0.1$ with $y(0) = 1$ and $\dot{y}(0) = 0$.

We next use our proposed approach and write (19) in its equivalent cubic-like form

$$\frac{d^2 x}{dt^2} + 2\nu_1\dot{x} + \delta x + \epsilon x^3 \approx 0, \tag{20}$$

where

$$\delta = \frac{1}{21}\left(21\alpha - 5\gamma\sigma^4\right), \qquad \epsilon = \frac{1}{9}\left(9\beta + 10\gamma\sigma^2\right),$$
$$\nu_1 = -\frac{-63\upsilon\nu + 2\gamma\sigma^5}{63\upsilon},$$
$$\alpha = \frac{6a}{A^4}\left(y_{10}\left(y_{10}^2 - 80\right)J_1(y_{10}) - 16\left(y_{10}^2 - 20\right)J_2(y_{10})\right) - b,$$
$$\beta = \frac{32a}{y_{10}^4}\left(-y_{10}\left(y_{10}^2 - 60\right)J_1(y_{10}) + 2\left(7y_{10}^2 - 120\right)J_2(y_{10})\right),$$
$$\gamma = \frac{32a}{y_{10}^4}\left(y_{10}\left(y_{10}^2 - 48\right)J_1(y_{10}) - 12\left(y_{10}^2 - 16\right)J_2(y_{10})\right). \tag{21}$$

Here, $J_1(y_{10})$ and $J_2(y_{10})$ are the first and second-order Bessel functions of the first kind, respectively. Once again, σ and υ are the fitting parameters that must satisfy (8) and (9). We next illustrate in Figure 2 the numerical integration solutions of (19) and (20) by considering the system parameter values $a = 3$, $b = 4$, and $\nu = 0.1$. In this particular case, the values of σ, υ, and ν_1 are 0.8, 1000, and 0.099. The initial angular displacement amplitude is assumed to have the value of 175°. As we may see from Figure 2, both solutions are

TABLE 1: Comparison of the numerical integration solutions of the equivalent and the original equations of motion for the parameter values $\alpha = 5$, $\beta = 2$, $\gamma = 0.5$, $\nu = 0.1$, $\sigma = 0.94$, and $\upsilon = 100,000$.

y_{10}	δ	ϵ	ν_1	RMSE
0.1	5	0.02	0.1	0.0
1	4.9601	2.4441	0.0999	0.0111
5	−19.8963	327.586	0.0999	0.5378
10	−393.341	4641.38	0.0999	0.5620

TABLE 2: Comparison of the numerical integration solutions of the equivalent and the original equations of motion for the parameter values $\alpha = 1$, $\beta = 20$, $\gamma = 0.1$, $\nu = 0.25$, $\sigma = 0.773$, and $\upsilon = 100,000$.

y_{10}	δ	ϵ	ν_1	RMSE
0.1	1	0.2	0.25	0.0
1	0.9963	20.060	0.25	0.0004
5	−1.2773	537.545	0.249	0.0676
10	−35.437	2600.73	0.249	0.318

almost the same. In fact, the maximum RMSE value attained is 0.0466 with the computed parameter values $\alpha = 6.9589$, $\beta = -4.3253$, $\gamma = 1.4577$, $\delta = 6.8167$, and $\epsilon = -3.2887$.

To further assess the applicability of our proposed approach, we next derive the equivalent cubic-like representation form of a damped oscillator with a mass attached to two stretched elastic springs.

4. The Damped Nonlinear Oscillator with an Irrational Restoring Force

Oscillators with irrational restoring forces are used to model the oscillations of a mass attached to a stretched wire [22] and are also used to study the dynamical response of vibration isolators or vibration absorbers in two-degree-of-freedom systems [23]. In this case, we assume that, besides the nonlinear irrational restoring force due to the stretched wires, we have a damper attached to the mass system, and hence, the nonlinear differential equation of motion for a single-degree-of-freedom system can be written as

$$\frac{d^2y}{dt^2} + 2\nu\dot{y} + y\left(\lambda_1 + \frac{\lambda_2}{\sqrt{D+y^2}}\right) = 0, \tag{22}$$

where D, ν, λ_1, and λ_2 are the system parameter values. If we introduce the transformation $x = y/y_{10}$, then, (22) becomes

$$\frac{d^2x}{dt^2} + f(x,\dot{x}) = 0,$$
$$f(x,\dot{x}) = 2\nu\dot{x} + x\left(\lambda_1 + \frac{\lambda_2}{\sqrt{D+y_{10}^2x^2}}\right), \tag{23}$$

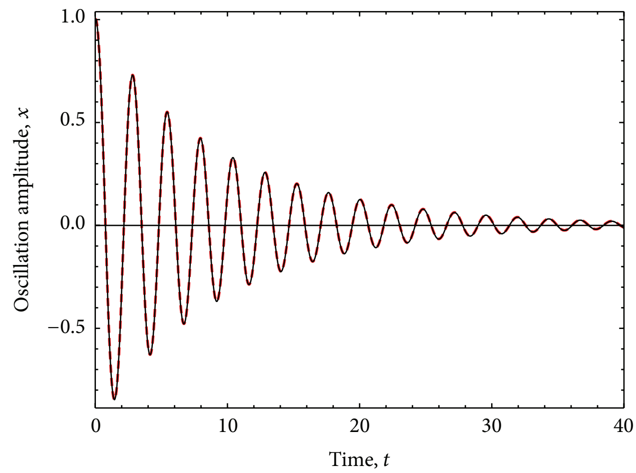

FIGURE 2: Amplitude-time response curves of (18) and (20). Here, the solid line represents the numerical integration solution of (18), while the dashed line represents the prediction obtained by using the derived equivalent equation of motion by applying our enhanced cubication procedure. The parameter values are $a = 3$, $b = 4$, and $\nu = 0.1$ with $y_{10} = 175^\circ$.

with the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$. We next apply our proposed nonlinear transformation approach to get the equivalent representation form of (23) as follows:

$$\frac{d^2x}{dt^2} + 2\nu_1\dot{x} + \delta x + \epsilon x^3 \approx 0, \tag{24}$$

where

$$\delta = \alpha - 0.10204\gamma\sigma^4, \qquad \epsilon = \beta + 1.0052\gamma\sigma^2,$$
$$\nu_1 = \nu - \frac{0.02418\gamma\sigma^5}{\upsilon},$$
$$\alpha = \frac{1}{(3y_{10}^6\pi)} \times \Bigg(3y_{10}^6B\pi + 4C\sqrt{D} \times \Bigg(\left(9y_{10}^4 + 112A^2D + 128D^2\right)E\left(-\frac{y_{10}^2}{D}\right) - \left(57y_{10}^4 + 176y_{10}^2D + 128D^2\right) \times K\left(-\frac{y_{10}^2}{D}\right)\Bigg)\Bigg),$$
$$\beta = \frac{1}{(3y_{10}^6\pi)} \times \Bigg(64C\sqrt{D}\Bigg(-\left(y_{10}^4 + 24y_{10}^2D + 32D^2\right)E\left(-\frac{y_{10}^2}{D}\right) + \left(11y_{10}^4 + 40y_{10}^2D + 32D^2\right) \times K\left(-\frac{y_{10}^2}{D}\right)\Bigg)\Bigg),$$

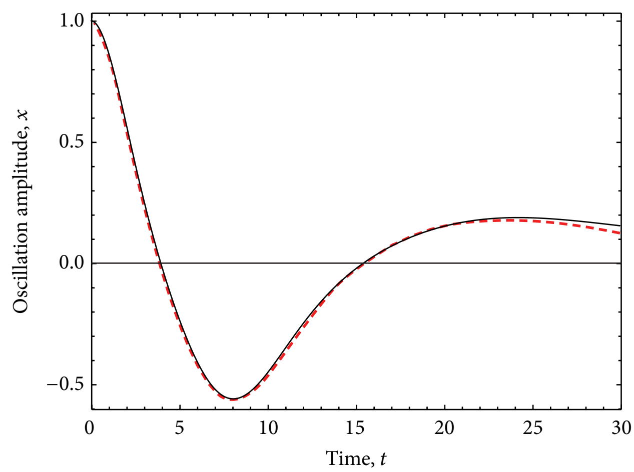

FIGURE 3: Amplitude-time response curves of (22) and (24). Here, the solid line represents the numerical integration solution of (18), while the dashed line represents the prediction obtained by using the derived equivalent equation of motion by applying our enhanced cubication procedure. The parameter values are $D = 1$, $\lambda_1 = 1$, $\lambda_2 = -1$, and $\nu = 0.1$ with $y_{10} = 1$ and $\dot{y}(0) = 0$.

$$\gamma = \frac{1}{(15y_{10}^6\pi)} \times \left(64C\sqrt{D}\left(\left(3y_{10}^4 + 88y_{10}^2D + 128D^2\right)E\left(-\frac{y_{10}^2}{D}\right) - \left(3y_{10}^2 + 8D\right)\left(13y_{10}^2 + 16D\right) \times K\left(-\frac{y_{10}^2}{\mathrm{D}}\right)\right)\right). \tag{25}$$

Here, $K(-y_{10}^2/D)$ and $E(-y_{10}^2/D)$ represent the complete elliptic integrals of the first and second kinds, respectively, with modulus equal to $-y_{10}^2/D$. To assess the accuracy of our equivalent damped equation of motion (24), let us consider the case for which $D = 1$, $\lambda_1 = 1$, $\lambda_2 = -1$, and $\nu = 0.1$. Figure 3 shows the numerical integration solutions of (22) and (24) with $y_{10} = 1$. As we can see from Figure 3, both solutions are almost the same for most of the time interval shown. The computed parameter values are $\delta = 0.016, \epsilon = 0.3169$, $\nu_1 = 0.1$, $\sigma = 0.85$, $\upsilon = 1000$, $\alpha = 0.0097$, $\beta = 0.4091$, and $\gamma = -0.1269$. In this case, the RMSE value is equal to 0.01974. It is easy to show that the RMSE values do not exceed 0.0648 on $y_{10} \in [0.01, 10]$. It is evident from our numerical results that the usage of irrational restoring forces with damping effects can be equivalently described by the damped Duffing oscillator. This could help us in better understanding the influence of irrational forces on the dynamical responses of the vibrational systems with two or more degrees of freedom.

We next derive the equivalent equation of motion of a Duffing oscillator with linear and cubic damped terms.

5. The Duffing Equation with Linear and Cubic Damped Terms

The equation of motion

$$\frac{d^2x}{dt^2} + \nu\dot{x} + Ax + Bx^3 + \kappa\dot{x}^3 = 0, \quad x(0) = x_{10},\ \dot{x}(0) = 0, \tag{26}$$

with linear and cubic damped terms is used to model the dynamical responses of several engineering applications such as the nanomechanical dynamical response of a doubly clamped beam [24], the nonlinear rolling motion of a ship in random beam seas [25], the modeling of nonlinear elastomeric vibration isolators [1], and the generalized damped general pendulum equation [26], among others. In (26), ν and κ represent the magnitude of the linear and nonlinear cubic damped terms, respectively. Of course, the exact solution of (26) is unknown, and thus, the numerical methods or perturbation techniques must be used to obtain its approximate solution.

Trueba and coworkers in [26] used the Melnikov analysis to write the forced version of (26) as the damped Duffing equation

$$\frac{d^2x}{dt^2} + \mu\dot{x} + Ax + Bx^3 = F(t), \quad x(0) = x_{10},\ \dot{x}(0) = 0, \tag{27}$$

in which $F(t)$ is the forcing term, $A = -1$, and

$$\mu = \nu + \frac{12}{35}\kappa. \tag{28}$$

Our aim in this section is to use our proposed approach to develop an equivalent representation form of (26) and compare its theoretical predictions with respect to those of (26) and (27) with $F(t) = 0$.

First, let us assume that the restoring force in (26) can be written in an equivalent form by using the following equations:

$$\begin{aligned} F_1(\delta, \epsilon, \nu_1) &= \int_0^{\sigma}\Big(\nu\dot{x} + Ax + Bx^3 + \kappa\dot{x}^3 - \left(\kappa\left|\nu_1\right| + \nu\right)\dot{x} - \delta x - \epsilon x^3\Big)^2 dx \longrightarrow \min, \\ F_2(\delta, \epsilon, \nu_1) &= \int_0^{\upsilon}\Big(\nu\dot{x} + Ax + Bx^3 + \kappa\dot{x}^3 - \left(\kappa\left|\nu_1\right| + \nu\right)\dot{x} - \delta x - \epsilon x^3\Big)^2 d\dot{x} \longrightarrow \min, \end{aligned} \tag{29}$$

where the coefficients ν_1, δ, and ϵ can be determined by using the expressions (10). Notice that we have introduced a slight modification in the damping coefficient terms of (29)

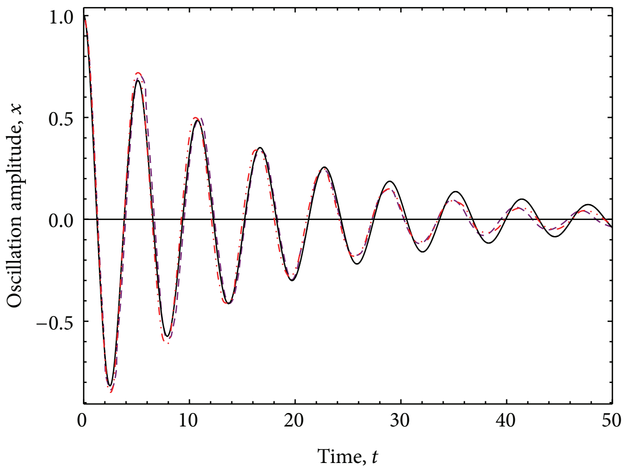

Figure 4: Amplitude-time response curves of (26), (27), and (31). Here, the solid line represents the numerical integration solution of (18), while the red and purple dashed lines represent, respectively, the numerical integration solutions of (27) and (30). The parameter values are $\nu = 0.1$, $A = 1$, $B = 1$, and $\kappa = 0.1$ with $x_{10} = 1$.

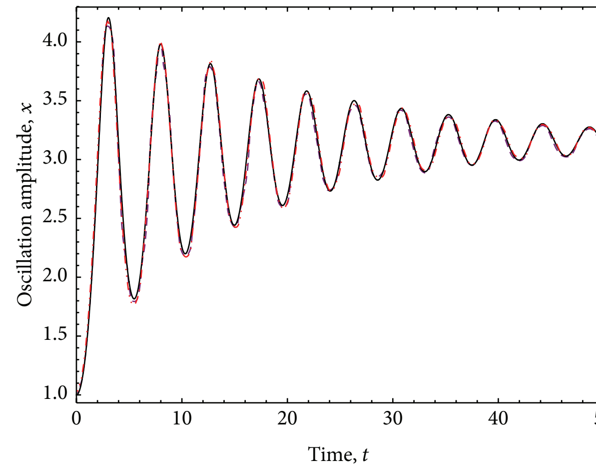

Figure 5: Amplitude-time response curves of (26), (27), and (31). Here, the solid lines represents the numerical integration solution of (18), while the red and purple dashed line represent, respectively, the numerical integration solutions of (27) and (30). The parameter values are $\nu = 0.1$, $A = -1$, $B = 0.1$, and $\kappa = 0.01$ with $x_{10} = 1$.

to take into account the influence of the cubic nonlinear damping term, κ, in the dynamical system response. We then follow our nonlinear transformation method to obtain the equivalent representation form of (26) which is given as

$$\frac{d^2x}{dt^2} + (\kappa |\nu_1| + \nu)\dot{x} + \delta x + \epsilon x^3 \approx 0, \tag{30}$$

$$\delta = \frac{18\kappa\upsilon^3 + A\sigma}{\sigma}, \qquad \epsilon = \frac{B\sigma^3 - 14\kappa\upsilon^3}{\sigma^3}, \qquad \nu_1 = \frac{27\upsilon^2}{5}. \tag{31}$$

To numerically evaluate the accuracy of our proposed equivalent representation form (30), let us consider the following system parameter values $\nu = 0.1$, $A = 1$, $B = 1$, and $\kappa = 0.1$ with $x_{10} = 1$. As we may see from Figure 4, the numerical integration solution of (30) with the estimated parameter values $\sigma = 0.4$, $\upsilon = 0.25$, $\delta = 1.0703$, $\epsilon = 0.6582$, and $\nu_1 = 0.3375$ agrees well with the numerical integration solution of (26). In this case, the computed RMSE value does not exceed 0.0426, while the corresponding RMSE value found by using (27) is about 0.05196. In Figure 4, the solid line and the red and purple dashed lines represent, respectively, the numerical integration solutions of (26), (27), and (30). It is evident from Figure 4 that our developed cubic-like solution exhibits good accuracy when compared with the numerical predictions of (27). To further assess the precision of our equivalent representation form (30), we next consider the system parameter values $\nu = 0.1$, $A = -1$, $B = 0.1$, and $\kappa = 0.01$ with $x_{10} = 1$. In this case, we found that $\sigma = -1.65$, $\upsilon = 0.4$, $\delta = -1.0069$, $\epsilon = 0.1019$, and $\nu_1 = 0.864$, with the computed RMSE values 0.0607 and 0.0313 obtained by using (27) and (30), respectively. Figure 5 illustrates the numerical integration amplitude-time response curves of (26), (27), and (30). As expected, our equivalent equation of motion (30) follows well the numerical solution of (26). This confirms the usefulness of our proposed approach to solve nonlinear oscillators with nonlinear damping terms.

6. Conclusions

In this paper, we have used a nonlinear transformation procedure to obtain equivalent equations of motion of nonlinear oscillators with conservative and dissipative restoring forces. Our solution procedure provides equivalent equations whose numerical predictions follow well the numerical integration solutions of the original equations of motion for small or larger damping coefficient values. To show the feasibility of our proposed approach, we have found the equivalent representation forms of four damped nonlinear oscillators. In fact, we have shown that (15) describes well the damped cubic-quintic Duffing equation since the computed RMSE values shown in Tables 1 and 2 are not bigger than 0.6699 on $y_{10} \in [0.01, 10]$ even for larger nonlinear parameter values. Furthermore, we found that the dynamical behavior of a damped pendulum is accurately described by its equivalent damped Duffing equation (20). In this case, the RMSE value does not exceed 0.0466 for the system parameter values $a = 3$, $b = 4$, and $\nu = 0.1$ with $y_{10} = 175^\circ$. A similar conclusion can be drawn by using the equivalent equation of motion (24) that models a damped nonlinear oscillator with an irrational restoring force term.

Finally, we have shown that the effect of a nonlinear damping dissipative force is equivalent to a linearly damped nonlinear Duffing oscillator with a modified damping coefficient. This is illustrated in the case of the Duffing equation with linear and nonlinear damping terms in which numerical predictions follow well the responses of the original equations of motion. Here, our numerical predictions coincide with those obtained from the Melnikov analysis even at larger nonlinear damping values. It is clear that our proposed approach could be extended not only to include driving forces or dissipative effects with even or fractional nonlinear damping terms but also to obtain equivalent equations of physical system with two or more degrees of freedom.

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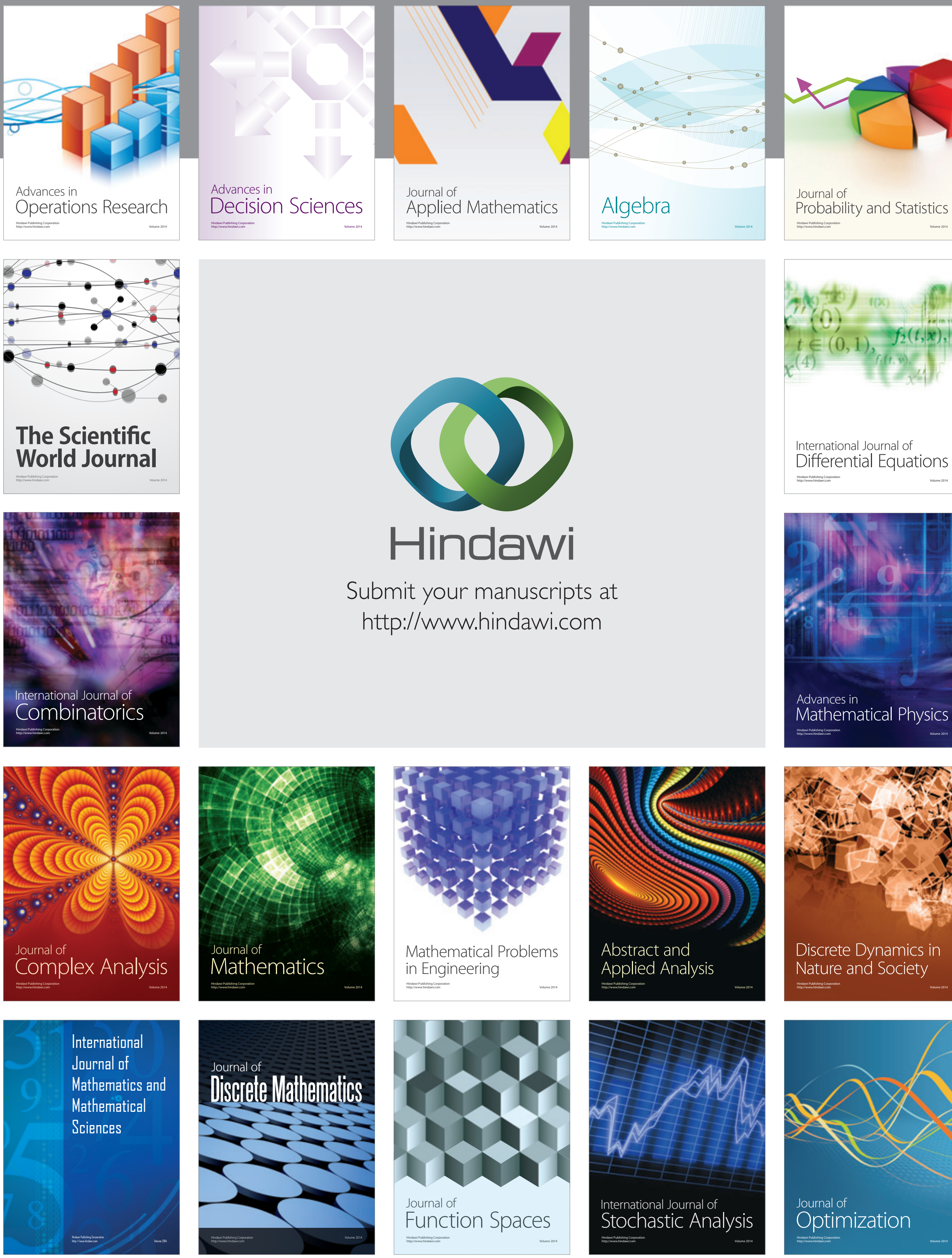
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