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Research Article

New Representations of the Group Inverse of 2×2 Block Matrices

Xiaoji Liu, Qi Yang, and Hongwei Jin

College of Science, Guangxi University for Nationalities, Nanning 530006, China

Correspondence should be addressed to Xiaoji Liu; xiaojiliu72@126.com

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This paper presents a full rank factorization of a 2×2 block matrix without any restriction concerning the group inverse. Applying this factorization, we obtain an explicit representation of the group inverse in terms of four individual blocks of the partitioned matrix without certain restriction. We also derive some important coincidence theorems, including the expressions of the group inverse with Banachiewicz-Schur forms.

1. Introduction

Let $\mathbb{C}^{m\times n}$ denote the set of all $m \times n$ complex matrices. We use $R(A)$, $N(A)$, and $r(A)$ to denote the range, the null space, and the rank of a matrix A, respectively. The Moore-Penrose inverse of a matrix $A \in \mathbb{C}^{m\times n}$ is a matrix $X \in \mathbb{C}^{n\times m}$ which satisfies

$$\begin{array}{ll} (1)\ AXA = A & (2)\ XAX = X \\ (3)\ (AX)^* = AX & (4)\ (XA)^* = XA. \end{array} \tag{1}$$

The Moore-Penrose inverse of A is unique, and it is denoted by $A^\dagger$.

Recall that the group inverse of A is the unique matrix $X \in \mathbb{C}^{m\times m}$ satisfying

$$AXA = A, \qquad XAX = X, \qquad AX = XA. \tag{2}$$

The matrix X is called the group inverse of A and it is denoted by $A^\#$.

Partitioned matrices are very useful in investigating various properties of generalized inverses and hence can be widely used in the matrix theory and have many other applications (see [1–4]). There are various useful ways to write a matrix as the product of two or three other matrices that have special properties. For example, linear algebra texts relate Gaussian elimination to the LU factorization and the Gram-Schmidt process to the QR factorization. In this paper, we consider a factorization based on the full rank factorization of a matrix. Our purpose is to provide an integrated theoretical development of and setting for understanding a number of topics in linear algebra, such as the Moore-Penrose inverse and the group inverse.

A full rank factorization of A is in the form

$$A = F_A G_A, \tag{3}$$

where F_A is of full column rank and G_A is of full row rank. Any choice in (3) is acceptable throughout the paper, although this factorization is not unique.

For a complex matrix $\mathscr{A}$ of the form

$$\mathscr{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{C}^{(m+s)\times(n+t)}, \tag{4}$$

in the case when $m = n$ and A is invertible, the Schur complement of A in $\mathscr{A}$ is defined by $S = D - CA^{-1}B$. Sometimes, we denote the Schur complement of A in $\mathscr{A}$ by $(\mathscr{A}/A)$. Similarly, if $s = t$ and D is invertible, then the Schur complement of D in $\mathscr{A}$ is defined by $T = A - BD^{-1}C$.

In the case when A is not invertible, the generalized Schur complement of A in $\mathscr{A}$ is defined by

$$S = D - CA^\dagger B. \tag{5}$$

Similarly, the generalized Schur complement of D in $\mathscr{A}$ is defined by

$$T = A - BD^\dagger C. \tag{6}$$

The Schur complement and generalized Schur complement have quite important applications in the matrix theory, statistics, numerical analysis, applied mathematics, and so forth.

There are a great deal of works [5–8] for the representations of the generalized inverse of $\mathscr{A}$. Various other generalized inverses have also been researched by a lot of researchers, for example, Burns et al. [6], Marsaglia and Styan [8], Benítez and Thome [9], Cvetković-Ilić et al. [10], Miao [11], Chen et al., and so forth [12] and the references therein. The concept of a group inverse has numerous applications in matrix theory, from convergence to Markov chains and from generalized inverses to matrix equations. Furthermore, the group inverse of block matrix has many applications in singular differential equations, Markov chains iterative methods, and so forth [13–17]. Some results for the group inverse of a 2×2 block matrix (operator) can be found in [18–30]. Most works in the literature concerning representations for the group inverses of partitioned matrices were carried out under certain restrictions on their blocks. Very recently, Yan [31] obtained an explicit representation of the Moore-Penrose inverse in terms of four individual blocks of the partitioned matrix by using the full rank factorization without any restriction. This motivates us to investigate the representations of the group inverse without certain restrictions.

In this paper, we aimed at a new method in giving the representation of the group inverse for the fact that there is no known representation for $\mathscr{A}^{\#}$, $\mathscr{A}^{D}$ with A, B, C, and D arbitrarily. The outline of our paper is as follows. In Section 2, we first present a full rank factorization of $\mathscr{A}$ using previous results by Marsaglia and Styan [8]. Inspired by this factorization, we extend the analysis to obtain an explicit representation of the group inverse of $\mathscr{A}$ without any restriction. Furthermore, we discuss variants special forms with the corresponding consequences, including Banachiewicz-Schur forms and some other extensions as well.

2. Representation of the Group Inverse: General Case

Yan [31] initially considered the representation of the Moore-Penrose inverse of the partitioned matrix by using the full rank factorization technique. The following result is borrowed from [31, Theorem 2.2].

For convenience, we first state some notations which will be helpful throughout the paper:

$$P_\alpha = I - \alpha\alpha^-, \quad Q_\alpha = I - \alpha^-\alpha, \quad \text{where } \alpha^- \in \alpha\{1\}, \tag{7}$$

$$\begin{aligned} S &= D - CA^\dagger B, \qquad E = P_A B, \\ W &= CQ_A, \qquad R = P_W S Q_E. \end{aligned} \tag{8}$$

Let A, E, W, R have the full rank factorizations

$$\begin{aligned} A &= F_A G_A, \qquad E = F_E G_E, \\ W &= F_W G_W, \qquad R = F_R G_R, \end{aligned} \tag{9}$$

respectively; then there is a full rank factorization of the block matrix $\mathscr{A}$:

$$\mathscr{A} = FG = \begin{bmatrix} F_A & 0 & 0 & F_E \\ CG_A^\dagger & F_R & F_W & P_W S G_E^\dagger \end{bmatrix} \begin{bmatrix} G_A & F_A^\dagger B \\ 0 & G_R \\ G_W & F_W^\dagger S \\ 0 & G_E \end{bmatrix}. \tag{10}$$

Now, the Moore-Penrose inverse of $\mathscr{A}$ can be expressed as $\mathscr{A}^\dagger = G^\dagger F^\dagger$. In particular, when A is group inverse, let $S = D - CA^{\#}B$; then the full rank factorization of $\mathscr{A}$ is

$$\mathscr{A} = \begin{bmatrix} F_A & 0 & 0 & F_E \\ CA^{\#}F_A & F_R & F_W & P_W S G_E^\dagger \end{bmatrix} \begin{bmatrix} G_A & G_A A^{\#} B \\ 0 & G_R \\ G_W & F_W^\dagger S \\ 0 & G_E \end{bmatrix}. \tag{11}$$

This motivates us to obtain some new results concerning the group inverse by using the full rank factorization related to the group inverse.

Recall that if a matrix $A \in \mathbb{C}^{n\times n}$ is group inverse (which is true when $\operatorname{ind}(A) = 1$), then $A^{\#}$ can be expressed in terms of $A\{1\}$; that is,

$$A^{\#} = A\left(A^{(1)}\right)^3 A. \tag{12}$$

Particularly, we have

$$A^{\#} = A\left(A^{\dagger}\right)^3 A. \tag{13}$$

The following result follows by using [31, Theorem 3.6] and (13).

Theorem 1. *Let $\mathscr{A}$ be defined by* (4)*; then the group inverse of $\mathscr{A}$ can be expressed as*

$$\begin{aligned} \mathscr{A}^{\#} = {} & \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ & \times \Big[\left(V_5 + V_5 V_3 V_2^* V_1 V_2 - V_4 V_1 V_2\right) V_3 \\ & \qquad \times \begin{bmatrix} W^\dagger & 0 \\ 0 & E^\dagger \end{bmatrix} \left(U_3 U_5 + U_2^* U_1 U_2 U_5 - U_2^* U_1 U_4\right) \\ & \qquad + \left(V_4 - V_5 V_3 V_2^*\right) V_1 \begin{bmatrix} A^\dagger & 0 \\ 0 & R^\dagger \end{bmatrix} \\ & \qquad \times U_1 \left(U_4 - U_2 U_5\right)\Big]^3 \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \end{aligned} \tag{14}$$

where

$$F_1 = \begin{bmatrix} F_A^\dagger & 0 \\ 0 & F_R^\dagger \end{bmatrix}, \qquad F_2 = \begin{bmatrix} F_W^\dagger & 0 \\ 0 & F_E^\dagger \end{bmatrix},$$

$$G_1 = \begin{bmatrix} G_A^{\dagger *} & 0 \\ 0 & G_R^{\dagger *} \end{bmatrix}, \qquad G_2 = \begin{bmatrix} G_W^{\dagger *} & 0 \\ 0 & G_E^{\dagger *} \end{bmatrix},$$

$$U_1 = \begin{bmatrix} X_3^{-1} & -X_3^{-1} H P_W X_2^{-1} X_4 \\ -X_4^* X_2^{-1} P_W H^* X_3^{-1} & X_4 + X_4^* X_2^{-1} P_W H^* X_3^{-1} H P_W X_2^{-1} X_4 \end{bmatrix},$$

$$U_2 = \begin{bmatrix} H & H P_W H_1^* X_1^{-1} \\ I & P_W H_1^* X_1^{-1} \end{bmatrix}, \qquad U_3 = \begin{bmatrix} I & 0 \\ 0 & X_1^{-1} \end{bmatrix},$$

$$U_4 = \begin{bmatrix} I & H \\ 0 & I \end{bmatrix}, \qquad U_5 = \begin{bmatrix} 0 & WW^\dagger \\ EE^\dagger & H_1 P_W \end{bmatrix},$$

$$V_1 = \begin{bmatrix} Y_3^{-1} & -Y_3^{-1} K Q_E Y_2^{-1} Y_4 \\ -Y_4 Y_2^{-1} Q_E K^* Y_3^{-1} & Y_4 + Y_4 Y_2^{-1} Q_E K^* Y_3^{-1} K Q_E Y_2^{-1} Y_4 \end{bmatrix},$$

$$V_2 = \begin{bmatrix} K K_1^* & K \\ K_1^* & 0 \end{bmatrix},$$

$$V_3 = \begin{bmatrix} Y_1^{-1} & -Y_1^{-1} K_1 \\ -K_1^* Y_1^{-1} & I + K_1^* Y_1^{-1} K_1 \end{bmatrix}, \qquad V_4 = \begin{bmatrix} I & 0 \\ K^* & I \end{bmatrix},$$

$$V_5 = \begin{bmatrix} W^\dagger W & 0 \\ K_1^* & E^\dagger E \end{bmatrix}, \tag{15}$$

with

$$H = A^{\dagger *} C^*, \qquad H_1 = E^{\dagger *} S^*, \qquad K = A^\dagger B,$$
$$K_1 = W^\dagger S,$$
$$X_1 = I + H_1 P_W H_1^*, \qquad X_2 = I + P_W H_1^* H_1 P_W,$$
$$X_3 = I + H P_W \left(X_2^{-1} - X_2^{-1} X_4 X_2^{-1} \right) P_W H^*,$$
$$X_4 = \left(R R^\dagger X_2^{-1} R R^\dagger \right)^\dagger, \tag{16}$$
$$Y_1 = I + K_1 Q_E K_1^*, \qquad Y_2 = I + Q_E K_1^* K_1 Q_E,$$
$$Y_3 = I + K Q_E \left(Y_2^{-1} - Y_2^{-1} Y_4 Y_2^{-1} \right) Q_E K^*,$$
$$Y_4 = \left(R^\dagger R Y_2^{-1} R^\dagger R \right)^\dagger.$$

If the $(1,1)$-element A of $\mathscr{A}$ is group inverse, we immediately have Theorem 2 by using the full rank factorization of (11).

Theorem 2. *Let $\mathscr{A}$ be defined by* (4). *Suppose A is group inverse; then the group inverse of $\mathscr{A}$ can be expressed as*

$$\begin{aligned} \mathscr{A}^\# = & \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ & \times \left[\left(V_5 + V_5 V_3 V_2^* V_1 V_2 - V_4 V_1 V_2 \right) V_3 \right. \\ & \quad \times \begin{bmatrix} W^\dagger & 0 \\ 0 & E^\dagger \end{bmatrix} \left(U_3 U_5 + U_2^* U_1 U_2 U_5 - U_2^* U_1 U_4 \right) \\ & \quad + \left(V_4 - V_5 V_3 V_2^* \right) V_1 \begin{bmatrix} A^\dagger & 0 \\ 0 & R^\dagger \end{bmatrix} \\ & \quad \left. \times U_1 \left(U_4 - U_2 U_5 \right) \right]^3 \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \end{aligned} \tag{17}$$

where $H = A^{\#*} C^*$, $K = A^\# B$, and H_1, K_1, U_1, U_2, U_3, U_4, U_5, V_1, V_2, V_3, V_4, V_5, X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4 are the same as those in Theorem 1.

The two representations of $F^\dagger$, $G^\dagger$ (which can be found in [31, Theorem 3.1]),

$$F^\dagger = \begin{bmatrix} F_1 U_1 U_4 - F_1 U_1 U_2 U_5 \\ -F_2 U^* U_1 U_4 + F_2 \left(U_3 + U_2^* U_1 U_2 \right) U_5 \end{bmatrix}, \tag{18}$$

$$G^\dagger = \left[V_4 V_1 G_1^* - V_5 V_3 V_2^* V_1 G_1^*, \ -V_4 V_1 V_2 V_3 G_2^* + V_5 \left(V_3 + V_3 V_2^* V_1 V_2 V_3 \right) G_2^* \right], \tag{19}$$

will be helpful in the proofs of the following results.

Theorem 3. *Let $\mathscr{A}$ be defined by* (4); *then the following statements are true.*

(a) *If E is of full column rank and W is of full row rank, then*

$$\begin{aligned} \mathscr{A}^\# = & \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ & \times \left(\begin{bmatrix} A^\dagger & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} I & -K - K_1 \\ 0 & I \end{bmatrix} \begin{bmatrix} W^\dagger & 0 \\ 0 & E^\dagger \end{bmatrix} \right. \\ & \quad \left. \times \begin{bmatrix} -H^* & I \\ I & 0 \end{bmatrix} \right)^3 \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \end{aligned} \tag{20}$$

(b) *If $E = 0$, $W = 0$, then*

$$\begin{aligned} \mathscr{A}^\# = & \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ & \times \left(\begin{bmatrix} \widetilde{Y} & -\widetilde{Y} K \\ Q_S K^* \widetilde{Y} & I - Q_S K^* \widetilde{Y} K \end{bmatrix} \begin{bmatrix} A^\dagger & 0 \\ 0 & S^\dagger \end{bmatrix} \right. \\ & \quad \left. \times \begin{bmatrix} \widetilde{X} & \widetilde{X} H P_S \\ -H^* \widetilde{X} & I - H^* \widetilde{X} H P_S \end{bmatrix} \right)^3 \\ & \times \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \end{aligned} \tag{21}$$

where $\widetilde{X} = (I + H P_S H^*)^{-1}$ and $\widetilde{Y} = (I + K Q_S K^*)^{-1}$.

Proof. (a) If E is full row rank, then $Q_E = 0$, and hence $R = 0$, $X_1 = I$, $X_2 = I$, $X_3 = I$, and $X_4 = 0$. Thus, V_1, V_2, V_3, V_4, V_5 defined in Theorem 1 can be simplified to

$$V_1 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad V_2 = \begin{bmatrix} K K_1^* & K \\ K_1^* & 0 \end{bmatrix},$$

$$V_3 = \begin{bmatrix} I & -K_1 \\ -K_1^* & I + K_1^* K_1 \end{bmatrix}, \qquad V_4 = \begin{bmatrix} I & 0 \\ K^* & I \end{bmatrix}, \tag{22}$$

$$V_5 = \begin{bmatrix} W^\dagger W & 0 \\ K_1^* & I \end{bmatrix},$$

which imply

$$V_4V_1 = \begin{bmatrix} I & 0 \\ K^* & 0 \end{bmatrix}, \qquad V_2V_3 = \begin{bmatrix} 0 & K \\ K_1^* & -K_1^*K \end{bmatrix},$$
$$V_1V_2V_3 = \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix},$$
$$V_5V_3V_2^*V_1 = \begin{bmatrix} 0 & 0 \\ K^* & 0 \end{bmatrix}, \qquad V_4V_1V_2V_3 = \begin{bmatrix} 0 & K \\ 0 & K^*K \end{bmatrix},$$
$$V_5V_3 = \begin{bmatrix} W^\dagger W & -K_1 \\ 0 & I \end{bmatrix}, \qquad V_5V_3V_2^*V_1V_2V_3 = \begin{bmatrix} 0 & 0 \\ 0 & K^*K \end{bmatrix}. \tag{23}$$

So, (19) is reduced to

$$G^\dagger = \begin{bmatrix} I & I & -K-K_1 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} G_A^\dagger & 0 & 0 \\ 0 & G_W^\dagger & 0 \\ 0 & 0 & G_E^\dagger \end{bmatrix}. \tag{24}$$

When W is full row rank, one gets $P_W = 0$ which implies $R = 0$, $X_1 = I$, $X_2 = I$, $X_3 = I$, and $X_4 = 0$. Thus,

$$U_1 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \qquad U_2 = \begin{bmatrix} H & 0 \\ I & 0 \end{bmatrix}, \qquad U_3 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$
$$U_4 = \begin{bmatrix} I & H \\ 0 & I \end{bmatrix}, \qquad U_5 = \begin{bmatrix} 0 & I \\ EE^\dagger & 0 \end{bmatrix}. \tag{25}$$

Simple computations show that

$$U_1U_4 = \begin{bmatrix} I & H \\ 0 & I \end{bmatrix}, \qquad U_1U_2U_5 = \begin{bmatrix} 0 & H \\ 0 & I \end{bmatrix},$$
$$U_2^*U_1U_4 = \begin{bmatrix} H^* & I+H^*H \\ 0 & 0 \end{bmatrix}, \qquad U_3U_5 = \begin{bmatrix} 0 & I \\ EE^\dagger & 0 \end{bmatrix}, \tag{26}$$
$$U_2^*U_1U_2U_5 = \begin{bmatrix} 0 & I+H^*H \\ 0 & 0 \end{bmatrix}.$$

Now, $F^\dagger$ possesses the following form according to (18):

$$F^\dagger = \begin{bmatrix} F_A^\dagger & 0 & 0 \\ 0 & F_W^\dagger & 0 \\ 0 & 0 & F_E^\dagger \end{bmatrix} \begin{bmatrix} I & 0 \\ -H^* & I \\ I & 0 \end{bmatrix}. \tag{27}$$

Since

$$\mathscr{A}^\dagger = G^\dagger F^\dagger = \begin{bmatrix} A^\dagger & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} I & -K-K_1 \\ 0 & I \end{bmatrix} \times \begin{bmatrix} W^\dagger & 0 \\ 0 & E^\dagger \end{bmatrix} \begin{bmatrix} -H^* & I \\ I & 0 \end{bmatrix}, \tag{28}$$

one gets the expression of $\mathscr{A}^\#$ by using (13):

$$\mathscr{A}^\# = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \left(\begin{bmatrix} A^\dagger & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} I & -K-K_1 \\ 0 & I \end{bmatrix} \begin{bmatrix} W^\dagger & 0 \\ 0 & E^\dagger \end{bmatrix} \times \begin{bmatrix} -H^* & I \\ I & 0 \end{bmatrix} \right)^3 \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{29}$$

(b) If $E = 0$, then $H_1 = 0$, $X_1 = I$, and $X_2 = I$ such that $X_3 = I + HP_WP_RP_WH^*$ and $X_4 = RR^\dagger$. Letting $X = X_3^{-1}$, then

$$U_1 = \begin{bmatrix} X & -XHP_WRR^\dagger \\ -RR^\dagger P_WH^*X & RR^\dagger + RR^\dagger P_WH^*XHP_WRR^\dagger \end{bmatrix},$$
$$U_2 = \begin{bmatrix} H & 0 \\ I & 0 \end{bmatrix}, \qquad U_3 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \tag{30}$$
$$U_4 = \begin{bmatrix} I & H \\ 0 & I \end{bmatrix}, \qquad U_4 = \begin{bmatrix} 0 & WW^\dagger \\ 0 & 0 \end{bmatrix}.$$

By short computations, one gets

$$U_1U_4 = \begin{bmatrix} X & XH\left(I - P_WRR^\dagger\right) \\ -RR^\dagger P_WH^*X & RR^\dagger - RR^\dagger P_WH^*XH\left(I - P_WRR^\dagger\right) \end{bmatrix},$$
$$U_1U_2U_5 = \begin{bmatrix} 0 & XH\left(I - P_WRR^\dagger\right)WW^\dagger \\ 0 & RR^\dagger WW^\dagger - RR^\dagger P_WH^*XH\left(I - P_WRR^\dagger\right)WW^\dagger \end{bmatrix},$$
$$U_2^*U_1U_4 = \begin{bmatrix} \left(I - RR^\dagger P_W\right)H^*X & RR^\dagger + \left(I - RR^\dagger P_W\right)H^*XH\left(I - P_WRR^\dagger\right) \\ 0 & 0 \end{bmatrix},$$
$$U_3U_5 = \begin{bmatrix} 0 & WW^\dagger \\ 0 & 0 \end{bmatrix},$$
$$U_2^*U_1U_2U_5 = \begin{bmatrix} 0 & RR^\dagger WW^\dagger + \left(I - RR^\dagger P_W\right)H^*XH\left(I - P_WRR^\dagger\right) \\ 0 & 0 \end{bmatrix}. \tag{31}$$

Hence,

$$F^\dagger = \begin{bmatrix} F_A^\dagger & 0 & 0 \\ 0 & F_R^\dagger & 0 \\ 0 & 0 & F_W^\dagger \end{bmatrix} \times \begin{bmatrix} X & XH\left(I - P_WRR^\dagger\right)P_W \\ -P_WH^*X & P_W - P_WH^*XH\left(I - P_WRR^\dagger\right)P_W \\ \left(I - RR^\dagger P_W\right)H^*X & I - RR^\dagger P_W \end{bmatrix}. \tag{32}$$

If $W = 0$, then $K_1 = 0$, $Y_1 = I$, and $Y_2 = I$ such that $Y_3 = I + KQ_EQ_RQ_EK^*$ and $Y_4 = R^\dagger R$. Letting $Y = Y_3^{-1}$, then

$$V_1 = \begin{bmatrix} Y & -YKQ_ER^\dagger R \\ -R^\dagger RQ_EK^*Y & R^\dagger R + R^\dagger RQ_EK^*YKQ_ER^\dagger R \end{bmatrix},$$
$$V_2 = \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}, \qquad V_3 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \tag{33}$$
$$V_4 = \begin{bmatrix} I & 0 \\ K^* & I \end{bmatrix}, \qquad V_5 = \begin{bmatrix} 0 & 0 \\ 0 & E^\dagger E \end{bmatrix},$$

which imply

$$V_4V_1 = \begin{bmatrix} Y & -YKQ_ER^\dagger R \\ K^*Y - R^\dagger RQ_EK^*Y & R^\dagger R - \left(I - R^\dagger RQ_E\right)K^*YKQ_ER^\dagger R \end{bmatrix},$$

$$V_1V_2V_3 = \begin{bmatrix} 0 & YK \\ 0 & -R^\dagger RQ_EK^*YK \end{bmatrix},$$

$$V_5V_3V_2^*V_1 = \begin{bmatrix} 0 & 0 \\ E^\dagger EK^*Y & -E^\dagger EK^*YKQ_ER^\dagger R \end{bmatrix},$$

$$V_4V_1V_2V_3 = \begin{bmatrix} 0 & YK \\ 0 & K^*YK - R^\dagger RQ_EK^*YK \end{bmatrix},$$

$$V_5V_3V_2^*V_1V_2V_3 = \begin{bmatrix} 0 & 0 \\ 0 & K^*YK - R^\dagger RQ_EK^*YK \end{bmatrix}. \quad (34)$$

So, (19) is reduced to

$$G^\dagger = \begin{bmatrix} Y & -YKQ_E & -YK \\ Q_RQ_EK^*Y & I - Q_RQ_EK^*YK & I \end{bmatrix} \times \begin{bmatrix} G_A^\dagger & 0 & 0 \\ 0 & G_R^\dagger & 0 \\ 0 & 0 & G_E^\dagger \end{bmatrix}. \quad (35)$$

Then,

$$\mathscr{A}^\dagger = G^\dagger F^\dagger = \begin{bmatrix} \widetilde{Y} & -\widetilde{Y}K \\ Q_SK^*\widetilde{Y} & I - Q_SK^*\widetilde{Y}K \end{bmatrix} \times \begin{bmatrix} A^\dagger & 0 \\ 0 & S^\dagger \end{bmatrix} \begin{bmatrix} \widetilde{X} & \widetilde{X}HP_S \\ -H^*\widetilde{X} & I - H^*\widetilde{X}HP_S \end{bmatrix}. \quad (36)$$

Therefore, we have

$$\mathscr{A}^\# = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \left(\begin{bmatrix} \widetilde{Y} & -\widetilde{Y}K \\ Q_SK^*\widetilde{Y} & I - Q_SK^*\widetilde{Y}K \end{bmatrix} \times \begin{bmatrix} A^\dagger & 0 \\ 0 & S^\dagger \end{bmatrix} \begin{bmatrix} \widetilde{X} & \widetilde{X}HP_S \\ -H^*\widetilde{X} & I - H^*\widetilde{X}HP_S \end{bmatrix} \right)^3 \times \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \quad (37)$$

□

Theorem 4. *Let $\mathscr{A}$ be defined by* (4)*, then the following statements are true.*

(a) *If $E = 0$, $W = 0$, and $R(C) \subset R(S)$, then*

$$\mathscr{A}^\# = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \left(\begin{bmatrix} \widetilde{Y} & -\widetilde{Y}K \\ Q_SK^*\widetilde{Y} & I - Q_SK^*\widetilde{Y}K \end{bmatrix} \begin{bmatrix} A^\dagger & 0 \\ 0 & S^\dagger \end{bmatrix} \times \begin{bmatrix} I & 0 \\ -H^* & I \end{bmatrix} \right)^3 \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad (38)$$

where $\widetilde{Y} = (I + KQ_SK^*)^{-1}$.

(b) *If $E = 0$, $W = 0$, and $R(B^*) \subset R(S^*)$, then*

$$\mathscr{A}^\# = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \left(\begin{bmatrix} I & -K \\ 0 & I \end{bmatrix} \begin{bmatrix} A^\dagger & 0 \\ 0 & S^\dagger \end{bmatrix} \times \begin{bmatrix} \widetilde{X} & \widetilde{X}HP_S \\ -H^*\widetilde{X} & I - H^*\widetilde{X}HP_S \end{bmatrix} \right)^3 \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad (39)$$

where $\widetilde{X} = (I + HP_SH^*)^{-1}$.

(c) *If $E = 0$, $W = 0$, and $R(B) \subset R(A)$, $R(C) \subset R(S)$, $R(B^*) \subset R(S^*)$, $R(C^*) \subset R(A^*)$, then*

$$\mathscr{A}^\# = \begin{bmatrix} A\left(A^\dagger\right)^3 A + AA^\dagger KS^\dagger H^*A^\dagger A & -AA^\dagger K\left(S^\dagger\right)^2 S \\ -S\left(S^\dagger\right)^2 H^*A^\dagger A & S\left(S^\dagger\right)^3 S \end{bmatrix}. \quad (40)$$

Proof. (a) Since $E = 0$ and $W = 0$, by Theorem 3(b), one gets

$$\mathscr{A}^\dagger = \begin{bmatrix} \widetilde{Y} & -\widetilde{Y}K \\ Q_SK^*\widetilde{Y} & I - Q_SK^*\widetilde{Y}K \end{bmatrix} \begin{bmatrix} A^\dagger & 0 \\ 0 & S^\dagger \end{bmatrix} \times \begin{bmatrix} \widetilde{X} & \widetilde{X}HP_S \\ -H^*\widetilde{X} & I - H^*\widetilde{X}HP_S \end{bmatrix}. \quad (41)$$

Since $R(C) \subset R(S)$, that is, $P_SC = 0$, then $\widetilde{X} = I$, then the equality previously mentioned is simplified to

$$\mathscr{A}^\dagger = \begin{bmatrix} \widetilde{Y} & -\widetilde{Y}K \\ Q_SK^*\widetilde{Y} & I - Q_SK^*\widetilde{Y}K \end{bmatrix} \begin{bmatrix} A^\dagger & 0 \\ 0 & S^\dagger \end{bmatrix} \begin{bmatrix} I & 0 \\ -H^* & I \end{bmatrix}. \quad (42)$$

By using $\mathscr{A}^\# = \mathscr{A}(\mathscr{A}^\dagger)^3\mathscr{A}$, we have

$$\mathscr{A}^\# = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \left(\begin{bmatrix} \widetilde{Y} & -\widetilde{Y}K \\ Q_SK^*\widetilde{Y} & I - Q_SK^*\widetilde{Y}K \end{bmatrix} \begin{bmatrix} A^\dagger & 0 \\ 0 & S^\dagger \end{bmatrix} \begin{bmatrix} I & 0 \\ -H^* & I \end{bmatrix} \right)^3 \times \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \quad (43)$$

(b) Similarly to the proof of (a).

(c) Since $E = 0$ and $W = 0$, by Theorem 2(b), one gets

$$\mathscr{A}^\dagger = \begin{bmatrix} \widetilde{Y} & -\widetilde{Y}K \\ Q_SK^*\widetilde{Y} & I - Q_SK^*\widetilde{Y}K \end{bmatrix} \begin{bmatrix} A^\dagger & 0 \\ 0 & S^\dagger \end{bmatrix} \times \begin{bmatrix} \widetilde{X} & \widetilde{X}HP_S \\ -H^*\widetilde{X} & I - H^*\widetilde{X}HP_S \end{bmatrix}. \quad (44)$$

Since $R(B) \subset R(A)$, $R(C) \subset R(S)$, $R(B^*) \subset R(S^*)$, $R(C^*) \subset R(A^*)$, that is, $P_AB = 0$, $CQ_A = 0$, $P_SC = 0$, $BQ_S = 0$, then the previous equality is simplified to

$$\begin{aligned}\mathscr{A}^{\dagger} &= \begin{bmatrix} I & -K \\ 0 & I \end{bmatrix} \begin{bmatrix} A^{\dagger} & 0 \\ 0 & S^{\dagger} \end{bmatrix} \begin{bmatrix} I & 0 \\ -H^* & I \end{bmatrix} \\ &= \begin{bmatrix} A^{\dagger} + KS^{\dagger}H^* & -KS^{\dagger} \\ -S^{\dagger}H^* & S^{\dagger} \end{bmatrix}.\end{aligned} \tag{45}$$

Moreover,

$$\begin{aligned}\mathscr{A}\mathscr{A}^{\dagger} &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A^{\dagger} + KS^{\dagger}H^* & -KS^{\dagger} \\ -S^{\dagger}H^* & S^{\dagger} \end{bmatrix} \\ &= \begin{bmatrix} AA^{\dagger} + AKS^{\dagger}H^* - BS^{\dagger}H^* & -AKS^{\dagger} + BS^{\dagger} \\ CA^{\dagger} + CKS^{\dagger}H^* - DS^{\dagger}H^* & -CKS^{\dagger} + DS^{\dagger} \end{bmatrix} \\ &= \begin{bmatrix} AA^{\dagger} & 0 \\ 0 & SS^{\dagger} \end{bmatrix}, \\ \mathscr{A}^{\dagger}\mathscr{A} &= \begin{bmatrix} A^{\dagger} + KS^{\dagger}H^* & -KS^{\dagger} \\ -S^{\dagger}H^* & S^{\dagger} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ &= \begin{bmatrix} A^{\dagger}A + KS^{\dagger}H^*A - KS^{\dagger}C & K + KS^{\dagger}H^*B - KS^{\dagger}D \\ -S^{\dagger}H^*A + S^{\dagger}C & -S^{\dagger}H^*B + S^{\dagger}D \end{bmatrix} \\ &= \begin{bmatrix} A^{\dagger}A & 0 \\ 0 & S^{\dagger}S \end{bmatrix}.\end{aligned} \tag{46}$$

Therefore,

$$\begin{aligned}\mathscr{A}^{\#} &= \mathscr{A}\mathscr{A}^{\dagger}\mathscr{A}^{\dagger}\mathscr{A}^{\dagger}\mathscr{A} = \begin{bmatrix} AA^{\dagger} & 0 \\ 0 & SS^{\dagger} \end{bmatrix} \\ &\quad \times \begin{bmatrix} A^{\dagger} + KS^{\dagger}H^* & -KS^{\dagger} \\ -S^{\dagger}H^* & S^{\dagger} \end{bmatrix} \begin{bmatrix} A^{\dagger}A & 0 \\ 0 & S^{\dagger}S \end{bmatrix} \\ &= \begin{bmatrix} A(A^{\dagger})^3A + AA^{\dagger}KS^{\dagger}H^*A^{\dagger}A & -AA^{\dagger}K(S^{\dagger})^2S \\ -S(S^{\dagger})^2H^*A^{\dagger}A & S(S^{\dagger})^3S \end{bmatrix}.\end{aligned} \tag{47}$$

□

Theorem 5. *Let $\mathscr{A}$ be defined by* (4)*; let $S = D - CA^{\#}B$ be the Schur complement of D in $\mathscr{A}$; then the following statements are true.*

(a) *If A and S are group inverse, $P_AB = 0$, $CQ_A = 0$, and $P_SC = 0$, then*

$$\begin{aligned}&\mathscr{A}^{\#} \\ &= \begin{bmatrix} A^{\#} + A^{\#}BS^{\#}CA^{\#} & A^{\#}\left(I + BS^{\#}CA^{\#}\right)A^{\#}BP_S - A^{\#}BS^{\#} \\ -S^{\#}CA^{\#} & S^{\#}\left(I - C\left(A^{\#}\right)^2BP_S\right) \end{bmatrix}.\end{aligned} \tag{48}$$

(b) *If A and S are group inverse, $P_AB = 0$, $CQ_A = 0$, $BP_S = 0$, then*

$$\begin{aligned}&\mathscr{A}^{\#} \\ &= \begin{bmatrix} A^{\#} + A^{\#}BS^{\#}CA^{\#} & -A^{\#}BS^{\#} \\ P_SCA^{\#}\left(I + A^{\#}BS^{\#}C\right)A^{\#} - S^{\#}CA^{\#} & \left(I - P_SC\left(A^{\#}\right)^2B\right)S^{\#} \end{bmatrix}\end{aligned} \tag{49}$$

(c) *Let A and S be group inverse; then $P_AB = 0$, $CQ_A = 0$, $P_SC = 0$, and $BP_S = 0$ if and only if*

$$\mathscr{A}^{\#} = \begin{bmatrix} A^{\#} + A^{\#}BS^{\#}CA^{\#} & -A^{\#}BS^{\#} \\ -S^{\#}CA^{\#} & S^{\#} \end{bmatrix}. \tag{50}$$

Proof. (i) If $P_AB = 0$ and $CQ_A = 0$, then E, W, R defined in (8) can be simplified to $E = 0$, $W = 0$; $R = S$ and then there is a full rank factorization

$$\mathscr{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = FG = \begin{bmatrix} F_A & 0 \\ CA^{\#}F_A & F_S \end{bmatrix} \begin{bmatrix} G_A & G_AA^{\#}B \\ 0 & G_S \end{bmatrix} \tag{51}$$

according to (11). Thus,

$$GF = \begin{bmatrix} G_AF_A + G_AKCA^{\#}F_A & G_AKF_S \\ G_SHF_A & G_SF_S \end{bmatrix}, \tag{52}$$

where $H = CA^{\#}$ and $K = A^{\#}B$. Denote by S' the Schur complement of G_SF_S in the partitioned matrix GF. Then,

$$\begin{aligned}S' &= G_AF_A + G_AKHF_A - G_AKF_S(G_SF_S)^{-1}G_SHF_A \\ &= G_AF_A + G_AKHF_A - G_AKSS^{\#}HF_A \\ &= G_AF_A + G_AKP_SHF_A \\ &= G_AF_A + G_AKP_SCA^{\#}F_A \\ &= G_AF_A.\end{aligned} \tag{53}$$

Applying the Banachiewicz-Schur formula, we have

$$\begin{aligned}&(G_AF_A)^{-1} \\ &= \begin{bmatrix} (G_AF_A)^{-1} & -(G_AF_A)^{-1}G_AKF_S(G_SF_S)^{-1} \\ -(G_SF_S)^{-1}G_SHF_A(G_AF_A)^{-1} & (G_SF_S)^{-1}\left(I + G_SHF_A(G_AF_A)^{-1}G_AKF_S(G_SF_S)^{-1}\right) \end{bmatrix} \\ &= \begin{bmatrix} (G_AF_A)^{-1} & -G_AA^{\#}KS^{\#}F_S \\ -G_SS^{\#}HA^{\#}F_A & (G_SF_S)^{-1} + G_SS^{\#}HKS^{\#}F_S \end{bmatrix}.\end{aligned} \tag{54}$$

Simple computations give

$$\begin{aligned} F(GF)^{-1} &= \begin{bmatrix} F_A & 0 \\ HF_A & F_S \end{bmatrix} \\ &\quad \times \begin{bmatrix} (G_A F_A)^{-1} & -G_A A^{\#} K S^{\#} F_S \\ -G_S S^{\#} H A^{\#} F_A & (G_S F_S)^{-1} + G_S S^{\#} H K S^{\#} F_S \end{bmatrix} \\ &= \begin{bmatrix} A^{\#} F_A & -K S^{\#} F_S \\ 0 & S^{\#} F_S \end{bmatrix}, \\ (GF)^{-1} G &= \begin{bmatrix} (G_A F_A)^{-1} & -G_A A^{\#} K S^{\#} F_S \\ -G_S S^{\#} H A^{\#} F_A & (G_S F_S)^{-1} + G_S S^{\#} H K S^{\#} F_S \end{bmatrix} \\ &\quad \times \begin{bmatrix} G_A & G_A K \\ 0 & G_S \end{bmatrix} \\ &= \begin{bmatrix} G_A A^{\#} & G_A A^{\#} K P_S \\ -G_S S^{\#} H & G_S S^{\#} - G_S S^{\#} H K P_S \end{bmatrix}. \end{aligned} \tag{55}$$

Then,

$$\begin{aligned} \mathscr{A}^{\#} &= F(GF)^{-2} G \\ &= \begin{bmatrix} A^{\#} F_A & -K S^{\#} F_S \\ 0 & S^{\#} F_S \end{bmatrix} \begin{bmatrix} G_A A^{\#} & G_A A^{\#} K P_S \\ -G_S S^{\#} H & G_S S^{\#} - G_S S^{\#} H K P_S \end{bmatrix} \\ &= \begin{bmatrix} A^{\#} + A^{\#} B S^{\#} C A^{\#} & A^{\#} \left(I + B S^{\#} C A^{\#} \right) A^{\#} B P_S - A^{\#} B S^{\#} \\ -S^{\#} C A^{\#} & S^{\#} \left(I - C \left(A^{\#} \right)^2 B P_S \right) \end{bmatrix}. \end{aligned} \tag{56}$$

(b) Since $P_A B = 0$ and $C Q_A = 0$, similar as (a), there is a full rank factorization of $\mathscr{A}$ such that

$$\mathscr{A} = FG = \begin{bmatrix} F_A & 0 \\ C A^{\#} F_A & F_S \end{bmatrix} \begin{bmatrix} G_A & G_A A^{\#} B \\ 0 & G_S \end{bmatrix}. \tag{57}$$

We also have

$$GF = \begin{bmatrix} G_A F_A + G_A K C A^{\#} F_A & G_A K F_S \\ G_S H F_A & G_S F_S \end{bmatrix}. \tag{58}$$

By using $B P_S = 0$, one gets the Schur complement of $G_S F_S$ in GF:

$$\begin{aligned} S' &= G_A F_A + G_A A^{\#} B P_S H F_A \\ &= G_A F_A. \end{aligned} \tag{59}$$

Hence,

$$(GF)^{-1} = \begin{bmatrix} (G_A F_A)^{-1} & -G_A A^{\#} K S^{\#} F_S \\ -G_S S^{\#} H A^{\#} F_A & (G_S F_S)^{-1} + G_S S^{\#} H K S^{\#} F_S \end{bmatrix}. \tag{60}$$

Short computations show that

$$\begin{aligned} F(GF)^{-1} &= \begin{bmatrix} F_A & 0 \\ HF_A & F_S \end{bmatrix} \\ &\quad \times \begin{bmatrix} (G_A F_A)^{-1} & -G_A A^{\#} K S^{\#} F_S \\ -G_S S^{\#} H A^{\#} F_A & (G_S F_S)^{-1} + G_S S^{\#} H K S^{\#} F_S \end{bmatrix}, \\ &= \begin{bmatrix} A^{\#} F_A & -K S^{\#} F_S \\ P_S H A^{\#} F_A & -P_S H K S^{\#} F_S + S^{\#} F_S \end{bmatrix}, \\ (GF)^{-1} G &= \begin{bmatrix} (G_A F_A)^{-1} & -G_A A^{\#} K S^{\#} F_S \\ -G_S S^{\#} H A^{\#} F_A & (G_S F_S)^{-1} + G_S S^{\#} H K S^{\#} F_S \end{bmatrix} \\ &\quad \times \begin{bmatrix} G_A & G_A K \\ 0 & G_S \end{bmatrix} \\ &= \begin{bmatrix} G_A A^{\#} & 0 \\ -G_S S^{\#} H & G_S S^{\#} \end{bmatrix}. \end{aligned} \tag{61}$$

Therefore,

$$\begin{aligned} \mathscr{A}^{\#} &= F(GF)^{-2} G \\ &= \begin{bmatrix} A^{\#} F_A & -K S^{\#} F_S \\ P_S H A^{\#} F_A & -P_S H K S^{\#} F_S + S^{\#} F_S \end{bmatrix} \begin{bmatrix} G_A A^{\#} & 0 \\ -G_S S^{\#} H & G_S S^{\#} \end{bmatrix} \\ &= \begin{bmatrix} A^{\#} + A^{\#} B S^{\#} C A^{\#} & -A^{\#} B S^{\#} \\ P_S C A^{\#} \left(I + A^{\#} B S^{\#} C \right) A^{\#} - S^{\#} C A^{\#} & \left(I - P_S C \left(A^{\#} \right)^2 B \right) S^{\#} \end{bmatrix}. \end{aligned} \tag{62}$$

(c) (⇒:) Since $P_A B = 0$, $C Q_A = 0$, $P_S C = 0$, and $B P_S = 0$, according to the proof of (a) and (b), we have

$$\begin{aligned} F(GF)^{-1} &= \begin{bmatrix} F_A & 0 \\ HF_A & F_S \end{bmatrix} \\ &\quad \times \begin{bmatrix} (G_A F_A)^{-1} & -G_A A^{\#} K S^{\#} F_S \\ -G_S S^{\#} H A^{\#} F_A & (G_S F_S)^{-1} + G_S S^{\#} H K S^{\#} F_S \end{bmatrix} \\ &= \begin{bmatrix} A^{\#} F_A & -K S^{\#} F_S \\ 0 & S^{\#} F_S \end{bmatrix}, \\ (GF)^{-1} G &= \begin{bmatrix} (G_A F_A)^{-1} & -G_A A^{\#} K S S^{\#} \\ -G_S S^{\#} H A^{\#} F_A & (G_S F_S)^{-1} + G_S S^{\#} H K S^{\#} F_S \end{bmatrix} \\ &\quad \times \begin{bmatrix} G_A & G_A K \\ 0 & G_S \end{bmatrix} \\ &= \begin{bmatrix} G_A A^{\#} & 0 \\ -G_S S^{\#} H & G_S S^{\#} \end{bmatrix}. \end{aligned} \tag{63}$$

Hence,

$$\mathscr{A}^{\#} = F(GF)^{-1} (GF)^{-1} G = \begin{bmatrix} A^{\#} + A^{\#} B S^{\#} C A^{\#} & -A^{\#} B S^{\#} \\ -S^{\#} C A^{\#} & S^{\#} \end{bmatrix}. \tag{64}$$

(⇐:) By [9, Theorem 2]. □

Analogous to Theorem 5, if define $T = A - BD^{\#}C$ the Schur complement of A in $\mathscr{A}$, one can obtain the following results.

Theorem 6. *Let $\mathscr{A}$ be defined by (4); let $T = A - BD^{\#}C$ be the Schur complement of A in $\mathscr{A}$; then the following statements are true.*

(a) *If D and T are group inverse, $P_D C = 0$, $BQ_D = 0$, $P_T B = 0$, then*

$$\mathscr{A}^{\#} = \begin{bmatrix} T^{\#}\left(I - B(D^{\#})^2 CQ_T\right) & -T^{\#}BD^{\#} \\ -D^{\#}CT^{\#} + D^{\#}\left(I + CT^{\#}BD^{\#}\right)D^{\#}CQ_T & D^{\#} + D^{\#}CT^{\#}BD^{\#} \end{bmatrix}. \quad (65)$$

(b) *If D and T are group inverse, $P_D C = 0$, $BQ_D = 0$, and $CQ_T = 0$, then*

$$\mathscr{A}^{\#} = \begin{bmatrix} \left(I - P_T B(D^{\#})^2 C\right)T^{\#} & -T^{\#}BD^{\#} + P_T BD^{\#}\left(I + D^{\#}CT^{\#}B\right)D^{\#} \\ -D^{\#}CT^{\#} & D^{\#} + D^{\#}CT^{\#}BD^{\#} \end{bmatrix}. \quad (66)$$

(c) *Let D and T be group inverse; then $P_D C = 0$, $BQ_D = 0$, $CQ_T = 0$, and $P_T B = 0$ if and only if*

$$\mathscr{A}^{\#} = \begin{bmatrix} T^{\#} & -T^{\#}BD^{\#} \\ -D^{\#}CT^{\#} & D^{\#} + D^{\#}CT^{\#}BD^{\#} \end{bmatrix}. \quad (67)$$

Proof. The proof is similar to the proof of Theorem 5. □

Combining Theorems 5 and 6, we have the following results.

Theorem 7. *Let $\mathscr{A}$ be defined by (4); let $S = D - CA^{\#}B$, $T = A - BD^{\#}C$ be the Schur complement of D and A in $\mathscr{A}$, respectively. Then the following statements are true.*

(a) *If A, S, D, T are group inverse, $P_A B = 0$, $CQ_A = 0$, $P_S C = 0$, $P_D C = 0$, $BQ_D = 0$, and $P_T B = 0$, then*

$$\begin{aligned}\mathscr{A}^{\#} &= \begin{bmatrix} A^{\#} + A^{\#}BS^{\#}CA^{\#} & A^{\#}\left(I + BS^{\#}CA^{\#}\right)A^{\#}BP_S - A^{\#}BS^{\#} \\ -S^{\#}CA^{\#} & S^{\#}\left(I - C(A^{\#})^2 BP_S\right) \end{bmatrix} \\ &= \begin{bmatrix} T^{\#}\left(I - B(D^{\#})^2 CQ_T\right) & -T^{\#}BD^{\#} \\ -D^{\#}CT^{\#} + D^{\#}\left(I + CT^{\#}BD^{\#}\right)D^{\#}CQ_T & D^{\#} + D^{\#}CT^{\#}BD^{\#} \end{bmatrix}.\end{aligned} \quad (68)$$

(b) *If A, S, D, T are group inverse, $P_A B = 0$, $CQ_A = 0$, $P_S C = 0$, $P_D C = 0$, $BQ_D = 0$, and $CQ_T = 0$, then*

$$\begin{aligned}\mathscr{A}^{\#} &= \begin{bmatrix} A^{\#} + A^{\#}BS^{\#}CA^{\#} & A^{\#}\left(I + BS^{\#}CA^{\#}\right)A^{\#}BP_S - A^{\#}BS^{\#} \\ -S^{\#}CA^{\#} & S^{\#}\left(I - C(A^{\#})^2 BP_S\right) \end{bmatrix} \\ &= \begin{bmatrix} \left(I - P_T B(D^{\#})^2 C\right)T^{\#} & -T^{\#}BD^{\#} + P_T BD^{\#}\left(I + D^{\#}CT^{\#}B\right)D^{\#} \\ -D^{\#}CT^{\#} & D^{\#} + D^{\#}CT^{\#}BD^{\#} \end{bmatrix}.\end{aligned} \quad (69)$$

(c) *If A, S, D, T are group inverse, $P_A B = 0$, $CQ_A = 0$, $BP_S = 0$, $P_D C = 0$, $BQ_D = 0$, and $P_T B = 0$, then*

$$\begin{aligned}\mathscr{A}^{\#} &= \begin{bmatrix} A^{\#} + A^{\#}BS^{\#}CA^{\#} & -A^{\#}BS^{\#} \\ P_S CA^{\#}\left(I + A^{\#}BS^{\#}C\right)A^{\#} - S^{\#}CA^{\#} & \left(I - P_S C(A^{\#})^2 B\right)S^{\#} \end{bmatrix} \\ &= \begin{bmatrix} T^{\#}\left(I - B(D^{\#})^2 CQ_T\right) & -T^{\#}BD^{\#} \\ -D^{\#}CT^{\#} + D^{\#}\left(I + CT^{\#}BD^{\#}\right)D^{\#}CQ_T & D^{\#} + D^{\#}CT^{\#}BD^{\#} \end{bmatrix}.\end{aligned} \quad (70)$$

(d) *If A, S, D, T are group inverse, $P_A B = 0$, $CQ_A = 0$, $BP_S = 0$, $P_D C = 0$, $BQ_D = 0$, and $CQ_T = 0$, then*

$$\begin{aligned}\mathscr{A}^{\#} &= \begin{bmatrix} A^{\#} + A^{\#}BS^{\#}CA^{\#} & -A^{\#}BS^{\#} \\ P_S CA^{\#}\left(I + A^{\#}BS^{\#}C\right)A^{\#} - S^{\#}CA^{\#} & \left(I - P_S C(A^{\#})^2 B\right)S^{\#} \end{bmatrix} \\ &= \begin{bmatrix} \left(I - P_T B(D^{\#})^2 C\right)T^{\#} & -T^{\#}BD^{\#} + P_T BD^{\#}\left(I + D^{\#}CT^{\#}B\right)D^{\#} \\ -D^{\#}CT^{\#} & D^{\#} + D^{\#}CT^{\#}BD^{\#} \end{bmatrix}.\end{aligned} \quad (71)$$

Theorem 8. *Let $\mathscr{A}$ be defined by (4); let $S = D - CA^{\#}B$, $T = A - BD^{\#}C$ be the Schur complement of D and A in $\mathscr{A}$, respectively. Then*

$$\begin{aligned}\mathscr{A}^{\#} &= \begin{bmatrix} A^{\#} + A^{\#}BS^{\#}CA^{\#} & -A^{\#}BS^{\#} \\ -S^{\#}CA^{\#} & S^{\#} \end{bmatrix} \\ &= \begin{bmatrix} T^{\#} & -T^{\#}BD^{\#} \\ -D^{\#}CT^{\#} & D^{\#} + D^{\#}CT^{\#}BD^{\#} \end{bmatrix}\end{aligned} \quad (72)$$

if and only if one of the following conditions holds

$$\text{(a)}\ P_A B = 0, \quad P_D C = 0, \quad P_S C = 0, \quad CQ_A = 0, \quad BQ_D = 0, \quad BQ_S = 0, \quad (73)$$

$$\text{(b)}\ P_A B = 0, \quad P_D C = 0, \quad P_T B = 0, \quad CQ_A = 0, \quad BQ_D = 0, \quad CQ_T = 0. \quad (74)$$

Proof. (a) Using Theorem 6(c) and Theorem 7(c), we conclude that

$$P_A B = 0, \quad CQ_A = 0, \quad P_S C = 0, \quad BQ_S = 0, \quad P_D C = 0, \quad BQ_D = 0, \quad CQ_T = 0, \quad P_T B = 0, \quad (75)$$

if and only if

$$\begin{aligned}\mathscr{A}^{\#} &= \begin{bmatrix} A^{\#} + A^{\#}BS^{\#}CA^{\#} & -A^{\#}BS^{\#} \\ -S^{\#}CA^{\#} & S^{\#} \end{bmatrix} \\ &= \begin{bmatrix} T^{\#} & -T^{\#}BD^{\#} \\ -D^{\#}CT^{\#} & D^{\#} + D^{\#}CT^{\#}BD^{\#} \end{bmatrix}.\end{aligned} \quad (76)$$

Now, we only need to prove (73) is equivalent to (75). Denote $T' = A^{\#} + A^{\#}BS^{\#}CA^{\#}$. Then

$$\begin{aligned}
TT' &= \left(A - BD^{\#}C\right)\left(A^{\#} + A^{\#}BS^{\#}CA^{\#}\right) \\
&= AA^{\#} + AA^{\#}BS^{\#}CA^{\#} - BD^{\#}CA^{\#} - BD^{\#}CA^{\#}BS^{\#}CA^{\#} \\
&= AA^{\#} + BS^{\#}CA^{\#} - BD^{\#}CA^{\#} - BD^{\#}(D-S)S^{\#}CA^{\#} \\
&= AA^{\#}, \\
T'T &= \left(A^{\#} + A^{\#}BS^{\#}CA^{\#}\right)\left(A - BD^{\#}C\right) \\
&= A^{\#}A - A^{\#}BD^{\#}C - A^{\#}BS^{\#}CA^{\#}A - A^{\#}BS^{\#}CA^{\#}BD^{\#}C \\
&= A^{\#}A - A^{\#}BD^{\#}C - A^{\#}BS^{\#}C - A^{\#}BS^{\#}(D-S)D^{\#}C \\
&= A^{\#}A.
\end{aligned} \tag{77}$$

Moreover, we have

$$\begin{aligned}
TT'T &= AA^{\#}\left(A - BD^{\#}C\right) = A - BD^{\#}C = T, \\
T'TT' &= A^{\#}A\left(A^{\#} + A^{\#}BS^{\#}CA^{\#}\right) = A^{\#} + A^{\#}BS^{\#}CA^{\#} = T'.
\end{aligned} \tag{78}$$

Thus, $T' = T^{\#}$. Hence, $T^{\#}T = A^{\#}A$ and $TT^{\#} = AA^{\#}$. Now, we get $P_AB = P_TB = 0$ and $CQ_A = CQ_T = 0$, which means (73) implying (75). Obviously, (75) implies (73). So, (73) is equivalent to (75).

(b) The proof is similar to (a). □

3. Applications to the Solution of a Linear System

In this section, we will give an application of the previous results above to the solution of a linear system. Using generalized Schur complement, we can split a larger system into two small linear systems by the following steps.

Let

$$\mathscr{A}x = y \tag{79}$$

be a linear system. Applying the block Gaussian elimination to the system, we have

$$\begin{bmatrix} A & B \\ 0 & D - CA^{\dagger}B \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 - CA^{\dagger}y_1 \end{bmatrix}. \tag{80}$$

Hence, we get

$$\begin{aligned}
Ax_1 + Bx_2 &= y_1; \\
Sx_2 &= y_2 - CA^{\dagger}y_1.
\end{aligned} \tag{81}$$

That is,

$$\begin{aligned}
Ax_1 &= y_1 - Bx_2, \\
Sx_2 &= y_2 - CA^{\dagger}y_1.
\end{aligned} \tag{82}$$

Now, the solution of system (79) can be obtained by the two small linear systems previously mentioned. In that case, the operation can be significantly simplified. We will also notice that the Moore-Penrose inverse of A can be replaced by other generalized inverses, such as the group inverse, the Drazin inverse and generalized inverse of A or even the ordinary inverse A^{-1}.

In the following, we will give the group inverse solutions of the linear system.

Theorem 9. *Let*

$$\mathscr{A}x = y \tag{83}$$

be a linear system. Suppose $\mathscr{A}$ satisfies all the conditions of Theorem 5 (c), partitioning x and y as

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{84}$$

which have appropriate sizes with $\mathscr{A}$. If $y \in R(\mathscr{A})$, then the solution $x = \mathscr{A}^{\#}y$ of linear system (79) *can be expressed as*

$$\begin{aligned}
x_1 &= A^{\#}\left(y_1 - Bx_2\right), \\
x_2 &= S^{\#}\left(y_2 - CA^{\#}y_1\right),
\end{aligned} \tag{85}$$

where $S = D - CA^{\#}B$.

Proof. Since $y \in R(\mathscr{A})$, we conclude that $x = \mathscr{A}^{\#}y$ is the solution of linear system (79). By Theorem 5 (c), we can get the following:

$$\begin{aligned}
x = \mathscr{A}^{\#}y &= \begin{bmatrix} A^{\#} + A^{\#}BS^{\#}CA^{\#} & -A^{\#}BS^{\#} \\ -S^{\#}CA^{\#} & S^{\#} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\
&= \begin{bmatrix} A^{\#}y_1 + A^{\#}BS^{\#}CA^{\#}y_1 - A^{\#}BS^{\#}y_2 \\ S^{\#}\left(y_2 - CA^{\#}y_1\right) \end{bmatrix} \\
&= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
\end{aligned} \tag{86}$$

Now, it is easy to see that the solution $x = \mathscr{A}^{\#}y$ can be expressed as

$$\begin{aligned}
x_1 &= A^{\#}\left(y_1 - Bx_2\right), \\
x_2 &= S^{\#}\left(y_2 - CA^{\#}y_1\right),
\end{aligned} \tag{87}$$

which are also the group inverse solutions of the two small linear systems of (82), respectively. □

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References

[1] F. J. Hall, "Generalized inverses of a bordered matrix of operators," *SIAM Journal on Applied Mathematics*, vol. 29, pp. 152–163, 1975.

[2] F. J. Hall, "The Moore-Penrose inverse of particular bordered matrices," *Australian Mathematical Society Journal A*, vol. 27, no. 4, pp. 467–478, 1979.

[3] F. J. Hall and R. E. Hartwig, "Further results on generalized inverses of partitioned matrices," *SIAM Journal on Applied Mathematics*, vol. 30, no. 4, pp. 617–624, 1976.

[4] S. K. Mitra, "Properties of the fundamental bordered matrix used in linear estimation," in *Statistics and Probability, Essays in Honor of C.R. Rao*, G. Kallianpur, Ed., pp. 504–509, North Holland, New York, NY, USA, 1982.

[5] J. K. Baksalary and G. P. H. Styan, "Generalized inverses of partitioned matrices in Banachiewicz-Schur form," *Linear Algebra and its Applications*, vol. 354, pp. 41–47, 2002.

[6] F. Burns, D. Carlson, E. Haynsworth, and T. Markham, "Generalized inverse formulas using the Schur complement," *SIAM Journal on Applied Mathematics*, vol. 26, pp. 254–259, 1974.

[7] D. S. Cvetković-Ilić, "A note on the representation for the Drazin inverse of 2×2 block matrices," *Linear Algebra and its Applications*, vol. 429, no. 1, pp. 242–248, 2008.

[8] G. Marsaglia and G. P. H. Styan, "Rank conditions for generalized inverses of partitioned matrices," *Sankhyā A*, vol. 36, no. 4, pp. 437–442, 1974.

[9] J. Benítez and N. Thome, "The generalized Schur complement in group inverses and $(k+1)$-potent matrices," *Linear and Multilinear Algebra*, vol. 54, no. 6, pp. 405–413, 2006.

[10] D. S. Cvetković-Ilić, J. Chen, and Z. Xu, "Explicit representations of the Drazin inverse of block matrix and modified matrix," *Linear and Multilinear Algebra*, vol. 57, no. 4, pp. 355–364, 2009.

[11] J. M. Miao, "General expressions for the Moore-Penrose inverse of a 2×2 block matrix," *Linear Algebra and its Applications*, vol. 151, pp. 1–15, 1991.

[12] J. Chen, Z. Xu, and Y. Wei, "Representations for the Drazin inverse of the sum $P + Q + R + S$ and its applications," *Linear Algebra and its Applications*, vol. 430, no. 1, pp. 438–454, 2009.

[13] S. L. Campbell, C. D. Meyer, Jr., and N. J. Rose, "Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients," *SIAM Journal on Applied Mathematics*, vol. 31, no. 3, pp. 411–425, 1976.

[14] R. Hartwig, X. Li, and Y. Wei, "Representations for the Drazin inverse of a 2×2 block matrix," *SIAM Journal on Matrix Analysis and Applications*, vol. 27, no. 3, pp. 757–771, 2005.

[15] A. da Silva Soares and G. Latouche, "The group inverse of finite homogeneous QBD processes," *Stochastic Models*, vol. 18, no. 1, pp. 159–171, 2002.

[16] Y. Wei and H. Diao, "On group inverse of singular Toeplitz matrices," *Linear Algebra and its Applications*, vol. 399, pp. 109–123, 2005.

[17] Y. Wei, "On the perturbation of the group inverse and oblique projection," *Applied Mathematics and Computation*, vol. 98, no. 1, pp. 29–42, 1999.

[18] J. Benítez, X. Liu, and T. Zhu, "Additive results for the group inverse in an algebra with applications to block operators," *Linear and Multilinear Algebra*, vol. 59, no. 3, pp. 279–289, 2011.

[19] C. Bu, M. Li, K. Zhang, and L. Zheng, "Group inverse for the block matrices with an invertible subblock," *Applied Mathematics and Computation*, vol. 215, no. 1, pp. 132–139, 2009.

[20] C. Bu, K. Zhang, and J. Zhao, "Some results on the group inverse of the block matrix with a sub-block of linear combination or product combination of matrices over skew fields," *Linear and Multilinear Algebra*, vol. 58, no. 7-8, pp. 957–966, 2010.

[21] C. Bu, J. Zhao, and K. Zhang, "Some results on group inverses of block matrices over skew fields," *Electronic Journal of Linear Algebra*, vol. 18, pp. 117–125, 2009.

[22] C. Bu, J. Zhao, and J. Zheng, "Group inverse for a class 2×2 block matrices over skew fields," *Applied Mathematics and Computation*, vol. 204, no. 1, pp. 45–49, 2008.

[23] C. G. Cao, "Some results of group inverses for partitioned matrices over skew fields," *Journal of Natural Science of Heilongjiang University*, vol. 18, no. 3, pp. 5–7, 2001 (Chinese).

[24] X. Chen and R. E. Hartwig, "The group inverse of a triangular matrix," *Linear Algebra and its Applications*, vol. 237/238, pp. 97–108, 1996.

[25] C. Cao and J. Li, "Group inverses for matrices over a Bezout domain," *Electronic Journal of Linear Algebra*, vol. 18, pp. 600–612, 2009.

[26] C. Cao and J. Li, "A note on the group inverse of some 2×2 block matrices over skew fields," *Applied Mathematics and Computation*, vol. 217, no. 24, pp. 10271–10277, 2011.

[27] C. Cao and X. Tang, "Representations of the group inverse of some 2×2 block matrices," *International Mathematical Forum*, vol. 31, pp. 1511–1517, 2006.

[28] M. Catral, D. D. Olesky, and P. van den Driessche, "Graphical description of group inverses of certain bipartite matrices," *Linear Algebra and its Applications*, vol. 432, no. 1, pp. 36–52, 2010.

[29] P. Patrício and R. E. Hartwig, "The $(2, 2, 0)$ group inverse problem," *Applied Mathematics and Computation*, vol. 217, no. 2, pp. 516–520, 2010.

[30] J. Zhou, C. Bu, and Y. Wei, "Group inverse for block matrices and some related sign analysis," *Linear and Multilinear Algebra*, vol. 60, no. 6, pp. 669–681, 2012.

[31] Z. Yan, "New representations of the Moore-Penrose inverse of 2×2 block matrices," *Linear Algebra and its Applications*, 2013.