Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 375094, 9 pages
http://dx.doi.org/10.1155/2013/375094

Research Article

A Numerical Comparison for a Discrete HIV Infection of $CD4^+$ T-Cell Model Derived from Nonstandard Numerical Scheme

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Received 8 August 2012; Revised 31 October 2012; Accepted 14 November 2012

Academic Editor: Juan Torregrosa

A nonstandard numerical scheme has been constructed and analyzed for a mathematical model that describes HIV infection of $CD4^+$ T cells. This new discrete system has the same stability properties as the continuous model and, particularly, it preserves the same local asymptotic stability properties. Linearized Stability Theory and Schur-Cohn criteria are used for local asymptotic stability of this discrete time model. This proposed nonstandard numerical scheme is compared with the classical explicit Euler and fourth order Runge-Kutta methods. To show the efficiency of this numerical scheme, the simulated results are given in tables and figures.

1. Introduction

Mathematical models are used not only in the natural sciences and engineering disciplines, but also in the social sciences. The differential equations in these mathematical models are usually nonlinear autonomous differential equation systems which have only time-independent parameters. It is not always possible to find the exact solutions of the nonlinear models that have at least two ordinary differential equations. It is sometimes more useful to find numerical solutions of this type systems in order to programme easily and visualize the results. Numerous methods can be used to obtain the numerical solutions of differential equations. By applying a numerical method to a continuous differential equation system, it becomes a difference equation system, in other words discrete time system. While applying these numerical methods, it is necessary that the new difference equation system should provide the positivity conditions and exhibit the same quantitative behaviours of continuous system such as stability, bifurcation, and chaos. It is well known that some traditional and explicit schemes such as forward Euler and Runge-Kutta are unsuccessful at generating oscillation, bifurcations, chaos, and false steady states, despite using adaptative step size [1–6]. For forward Euler's method, if the step size h is chosen small enough and the positivity conditions are satisfied, it is seen that local asymptotic stability for a fixed point is saved while in some special cases Hopf bifurcation cannot be seen. Instead of classical methods, nonstandard finite difference scheme (NFDS) can be alternatively used to obtain more qualitative results and to remove numerical instabilities. These schemes are developed for compensating the weaknesses that may be caused by standard difference methods, for example, numerical instabilities. Also, the dynamic consistency could be presented well by NFDS [7]. The most important advantage of this scheme is that, choosing a convenient denominator function instead of the step size h, better results can be obtained. If the step size h is chosen small enough, the obtained results do not change significantly but if h gets larger this advantage comes into focus.

The NFDS modeling procedures were given in 1989 by Mickens [8]. It removes the problems discussed above by using the suitable denominator function $\phi = h + O(h^2)$. The papers [2, 8–16] show how to choose the denominator function and apply this scheme to many models. Micken's method can be summarized by using [13] as follows.

Let us consider the following ordinary differential equation:

$$\frac{dx}{dt} = F(x, \lambda), \tag{1}$$

where λ is a parameter. The simplest nonstandard finite difference schemes are

$$t \longrightarrow t_n = hn, \qquad x(t) \longrightarrow x_n, \qquad F(x) \longrightarrow F(x_n),$$
$$\frac{dx}{dt} \longrightarrow \frac{x_{n+1} - x_n}{\phi}, \tag{2}$$

where ϕ depends on the step size $\Delta t = h$ and satisfies $\phi = h + O(h^2)$. It should be chosen

$$\phi = \frac{1 - e^{-Rh}}{R}, \tag{3}$$

where R is calculated from a knowledge of the fixed points of (1), that is,

$$F(\widetilde{x}) = 0. \tag{4}$$

Assume that the last equation has I-real solutions and denote by

$$\{\widetilde{x}_i;\ i = 1, 2, \ldots, I\}. \tag{5}$$

Now define R_i as

$$R_i = \left.\frac{dF}{dx}\right|_{x=\widetilde{x}_i}, \tag{6}$$

and take R as

$$R = \operatorname{Max}\{|R_i|;\ i = 1, 2, \ldots, I\}. \tag{7}$$

In this paper, an NFDS scheme is applied to human immunodeficiency virus (HIV), which has spread rapidly around the world in recent years and thus it gains importance. In the last decade, the published papers about the epidemiology of HIV are less in number and they are not detailed enough. A few of these models were simulated using numerical methods such as Runge-Kutta or Euler methods. However, explicit methods are generally known to exhibit contrived chaos whenever the discretization parameters exceed certain values [17, 18].

A model about HIV infection of $CD4^+$ T cells was presented by Perelson and Nelson [19, 20]. This model is given as follows:

$$\frac{dT}{dt} = p - \alpha T + rT\left(1 - \frac{T + I}{T_{\max}}\right) - kVT,$$
$$\frac{dI}{dt} = kVT - \beta I, \tag{8}$$
$$\frac{dV}{dt} = N\beta I - \gamma V,$$

where $T(t)$, $I(t)$, $V(t)$ denote the concentration of $CD4^+$ T cells, the concentration of infected $CD4^+$ T cells by the HIV viruses, and free HIV virus particles, respectively. $k > 0$ is the infection rate. Each infected $CD4^+$ T cell is assumed to produce N virus particles during its life time [21]. $T_{\max}$ is the maximum level of $CD4^+$ T cell population density in the body. $rT(1 - (T + I)/T_{\max})$ is logistic equation, where r is the average specific T-cell growth rate [19]. p, α, β, and γ are positive constants and p is the source of $CD4^+$ T cells from precursors, α is the death rate of $CD4^+$ T cells, β is the death rate of infected cells, and finally γ is the viral clearance rate constant [22–24]. Nelson et al. focused on other models of HIV-1 infection in [25, 26]. These models deal with dynamics occurring after drug treatment. They analyzed the delay differential equation models of HIV-1 infection. Initially they give a standard model of HIV and then afterwards they give delay model of HIV. They analyze the model and give some lemma and proofs. Culshaw and Ruan consider a delay differential equation model of HIV infection of $CD4^+$ T cells [27].

This paper is organized as follows: in Section 2, in order to obtain explicit solutions of (8), first the model is discretizated in a nonstandard form and this discrete model provides the positivity conditions. In Section 3, some lemmas and Linearized Stability Theorem are given for the local asymptotic stability of the discrete time systems. In Section 4, the theorical results obtained in former section are compared with the other numerical methods and the simulated results are given.

2. Discretization of the Model

The nonlinear differential equation system (1) will be discretizated as follows:

$$\begin{aligned} T(t) &\longrightarrow T_n, \\ I(t) &\longrightarrow I_{n+1}, \\ V(t) &\longrightarrow V_{n+1}, \\ T^2(t) &\longrightarrow T_{n+1}T_n, \\ T(t)I(t) &\longrightarrow T_{n+1}I_n, \\ T(t)V(t) &\longrightarrow T_{n+1}V_n. \end{aligned} \tag{9}$$

If T_{n+1}, I_{n+1}, and V_{n+1} are explicitly solve from (8), the following iterations will be obtained:

$$T_{n+1} = \frac{\left(1 + (r - \alpha)\phi_1(h, \alpha)\right)T_n + \phi_1(h, \alpha)\,p}{1 + \phi_1(h, \alpha)\left(kV_n + r\left(T_n + I_n\right)/T_{\max}\right)},$$
$$I_{n+1} = \frac{I_n + \phi_2(h, \beta)\,kV_nT_{n+1}}{1 + \beta\phi_2(h, \beta)}, \tag{10}$$
$$V_{n+1} = \frac{V_n + \phi_3(h, \gamma)\,N\beta I_{n+1}}{1 + \phi_3(h, \gamma)\,\gamma},$$

where denominator functions are chosen as

$$\phi_1(h,\alpha) = \frac{e^{(r-\alpha)h}-1}{r-\alpha},$$
$$\phi_2(h,\beta) = \frac{e^{\beta h}-1}{\beta}, \tag{11}$$
$$\phi_3(h,\gamma) = \frac{e^{\gamma h}-1}{\gamma}.$$

Detailed information about how to find different nonlocal terms to different denominator functions can be read in [9, 10, 13, 15]. Let $r_1, r_2, r_3 \geq 0$ and $p, k, r, T_{\max} > 0$. In order to obtain positive iterations T_{n+1}, I_{n+1}, and V_{n+1} we have to require $r-\alpha > 0$ or if $r-\alpha < 0$ then $\phi_1 > 1/(T_n(\alpha-1)+p)$. If we take the numerical values and initial conditions in [21], for each nonnegative initial conditions r_1, r_2, and r_3, the iterations T_n, I_n, and V_n and consequently T_{n+1}, I_{n+1}, and V_{n+1} are also nonnegative.

3. Stability Analysis of the Model

Some useful lemmas and a theorem should be given for local asymptotic stability of discrete systems. Especially, it is necessary to investigate Schur-Cohn criteria which deal with coefficient matrix of the linearized system as follows:

(i) $\det B < 1$,

(ii) $1 - \operatorname{tr} B + \det B > 0$,

(iii) $1 + \operatorname{tr} B + \det B > 0$,

where B and $\operatorname{tr} B$ denote coefficient matrix of the linearized system and trace of the matrix, respectively. One can find information in [13, 28–31] about the usage of Schur-Cohn criteria which do not need many process as in continuous models.

The following lemmas and theorem given in citejury, citejodar are relevant to the roots of characteristic polynomials.

Lemma 1. *For the quadratic equation $\lambda^2 - a\lambda + b = 0$ the roots satisfy $|\lambda_i| < 1$, $i = 1, 2$, if and only if the following conditions are satisfied:*

(i) $b < 1$,

(ii) $1 - a + b > 0$,

(iii) $1 + a + b > 0$.

Lemma 2 (Jury conditions, Schur-Cohn criteria, $n = 3$). *Suppose the characteristic polynomial $p(\lambda)$ is given by $p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$. The solutions λ_i, $i = 1, 2, 3$, of $p(\lambda) = 0$ satisfy $|\lambda_i| < 1$ if the following three conditions are held:*

(i) $p(1) = 1 + a_1 + a_2 + a_3 > 0$,

(ii) $(-1)^3 p(-1) = 1 - a_1 + a_2 - a_3 > 0$,

(iii) $1 - (a_3)^2 > |a_2 - a_3 a_1|$.

Theorem 3 (the linearized stability theorem). *Let $\tilde{x}$ be an equilibrium point of the difference equation*

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \tag{12}$$

where the function F is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\tilde{x}$. Then the following statements are true.

(1) *If all the roots of the characteristic polynomial have absolute value less then one, then the equilibrium point $\tilde{x}$ is locally asymptotically stable.*

(2) *If at least one root of the characteristic polynomial has absolute value greater than one, then the equilibrium point $\tilde{x}$ is unstable.*

Equilibrium points of (8) are found as follows:

$$X_1^* = (T_1^*, I_1^*, V_1^*) = \left(-\frac{T_{\max}\left(\alpha - r + \sqrt{(\alpha-r)^2 + 4rp/T_{\max}}\right)}{2r}, 0, 0\right),$$
$$X_2^* = (T_2^*, I_2^*, V_2^*) = \left(\frac{T_{\max}\left(r - \alpha + \sqrt{(\alpha-r)^2 + 4rp/T_{\max}}\right)}{2r}, 0, 0\right),$$
$$X_3^* = (T_3^*, I_3^*, V_3^*) = \left(\frac{\gamma}{kN}, -\frac{\tau}{N}, -\frac{\beta\tau}{\gamma}\right), \tag{13}$$

where

$$\tau = \frac{-pT_{\max}k^2N^2 + \alpha\gamma T_{\max}kN - r\gamma T_{\max}kN + r\gamma^2}{k(r\gamma + kN\beta T_{\max})}. \tag{14}$$

Only fixed points X_2^* and X_3^* have real biological meaning: the uninfected steady state $X_2^* = (T_2^*, I_2^*, V_2^*)$ and the (positive) infected steady state $X_3^* = (T_3^*, I_3^*, V_3^*)$ [21, 22]. Firstly, let us examine the fixed point X_2^*. Equation (10) is rewritten as follows:

$$f = \frac{(1 + (r-\alpha)\phi_1(h,\alpha))T_n + \phi_1(h,\alpha)p}{1 + \phi_1(h,\alpha)(kV_n + r(T_n + I_n)/T_{\max})},$$
$$g = \frac{I_n + \phi_2(h,\beta)kV_nT_{n+1}}{1 + \beta\phi_2}, \tag{15}$$
$$h = \frac{V_n + \phi_3(h,\gamma)N\beta I_{n+1}}{1 + \phi_3(h,\gamma)\gamma}.$$

By using these equations, Jacobian matrix will be found:

$$J(T_n, I_n, V_n) = \begin{bmatrix} f_{T_n} & f_{I_n} & f_{V_n} \\ g_{T_n} & g_{I_n} & g_{V_n} \\ h_{T_n} & h_{I_n} & h_{V_n} \end{bmatrix}, \tag{16}$$

where

$$f_{T_n} = \frac{\eta}{\omega} - \frac{(\eta T_n + \phi_1 p)\phi_1 r}{\omega^2 T_{\max}},$$

$$f_{I_n} = -\frac{(\eta T_n + \phi_1 p)\phi_1 r}{\omega^2 T_{\max}},$$

$$f_{V_n} = -\frac{(\eta T_n + \phi_1 p)\phi_1 k}{\omega^2},$$

$$g_{T_n} = \frac{k\phi_2 V_n \eta}{(1+\phi_2\beta)\omega} - \frac{k\phi_2 V_n (\eta T_n + \phi_1 p)\phi_1 r}{(1+\phi_2\beta)\omega^2 T_{\max}},$$

$$g_{I_n} = \frac{1}{1+\beta\phi_2} - \frac{k\phi_2 V_n (\eta T_n + \phi_1 p)\phi_1 r}{(1+\phi_2\beta)\omega^2 T_{\max}},$$

$$g_{V_n} = -\frac{k^2\phi_2 V_n (\eta T_n + \phi_1 p)\phi_1}{(1+\phi_2\beta)\omega^2} + \frac{k\phi_2 (\eta T_n + \phi_1 p)}{(1+\phi_2\beta)\omega},$$

$$h_{T_n} = \frac{\phi_3 N\beta k\phi_2 V_n \eta}{(1+\phi_2\beta)(1+\phi_3\gamma)\omega} - \frac{\phi_3 N\beta k\phi_2 V_n (\eta T_n + \phi_1 p)\phi_1 r}{(1+\phi_2\beta)(1+\phi_1\gamma)\omega^2 T_{\max}},$$

$$h_{I_n} = \frac{\phi_3 N\beta \left(1 - k\phi_2 V_n (\eta T_n + \phi_1 p)\phi_1 r/\omega^2 T_{\max}\right)}{(1+\phi_2\beta)(1+\phi_3\gamma)},$$

$$h_{V_n} = \frac{1}{1+\phi_3\gamma} + \frac{\phi_3 N\beta}{(1+\phi_2\beta)(1+\phi_3\gamma)} \cdot \left(\frac{-k^2\phi_2 (\eta T_n + \phi_1 p) V_n \phi_1}{\omega^2} + \frac{k\phi_2 (\eta T_n + \phi_1 p)}{\omega}\right),$$

$$\eta = (1 + (r-\alpha)\phi_1),$$

$$\omega = \left(1 + \phi_1 \left(kV_n + \frac{r(T_n + I_n)}{T_{\max}}\right)\right). \tag{17}$$

Firstly, let us find Jacobian matrix of (10) around X_2^* to analyze the stability of this fixed point. We obtain

$$J(T_2^*, I_2^*, V_2^*) = \begin{bmatrix} \frac{\eta\zeta T_{\max} - \chi\phi_1 r}{\zeta^2 T_{\max}} & \frac{-\phi_1 r\chi}{\zeta^2 T_{\max}} & \frac{-\phi_1 k\chi}{\zeta^2} \\ 0 & \frac{1}{1+\beta\phi_2} & \frac{\phi_2 k\chi}{(1+\beta\phi_2)\zeta} \\ 0 & \frac{\phi_3 N\beta}{(1+\beta\phi_2)(1+\gamma\phi_3)} & \frac{\zeta(1+\beta\phi_2) + \phi_3 N\beta\phi_2 k\chi}{(1+\beta\phi_2)(1+\gamma\phi_3)\zeta} \end{bmatrix}, \tag{18}$$

where

$$\zeta = \left(1 + \frac{\phi_1 r T_2^*}{T_{\max}}\right), \qquad \chi = \eta T_2^* + \phi_1 p. \tag{19}$$

To analyze the stability of X_2^*, we need to find eigenvalues [32]

$$\left(\lambda - \frac{\eta\zeta T_{\max} - \chi\phi_1 r}{\zeta^2 T_{\max}}\right)\left(\lambda^2 - a\lambda + b\right) = 0, \tag{20}$$

where

$$a = \text{Trace } B, \qquad b = \text{Det } B,$$

$$B = \begin{bmatrix} \frac{1}{1+\beta\phi_2} & \frac{\phi_2 k\chi}{(1+\beta\phi_2)\zeta} \\ \frac{\phi_3 N\beta}{(1+\beta\phi_2)(1+\gamma\phi_3)} & \frac{\zeta(1+\beta\phi_2) + \phi_3 N\beta\phi_2 k\chi}{(1+\beta\phi_2)(1+\gamma\phi_3)\zeta} \end{bmatrix}. \tag{21}$$

The first eigenvalue is

$$\lambda_1 = \frac{\eta\zeta T_{\max} - \chi\phi_1 r}{\zeta^2 T_{\max}}. \tag{22}$$

We can find the other eigenvalues from

$$f(\lambda) = \lambda^2 - a\lambda + b = 0, \tag{23}$$

where

$$a = \frac{\zeta(1+\gamma\phi_3) + \zeta(1+\beta\phi_2) + \phi_3 N\beta\phi_2 k\chi}{\zeta(1+\gamma\phi_3)(1+\beta\phi_2)}, \qquad b = \frac{1}{(1+\gamma\phi_3)(1+\beta\phi_2)}. \tag{24}$$

By considering Lemma 1, when $|\lambda_i| < 1, i = 1, 2$, the following conditions are satisfied and then the fixed point X_2^* is locally asymptotic stable

(i) $b = 1/(1+\gamma\phi_3)(1+\beta\phi_2) < 1$,

(ii) $f(-1) = 1 + a + b = (\zeta((1+\gamma\phi_3)(1+\beta\phi_2) + 3 + \gamma\phi_3 + \beta\phi_2) + k\phi_3 N\beta\phi_2\chi)/\zeta(1+\beta\phi_2)(1+\gamma\phi_3) > 0$,

(iii) $f(1) = 1 - a + b = (\zeta((1+\gamma\phi_3)(1+\beta\phi_2) - 1 - \gamma\phi_3 - \beta\phi_2) - k\phi_3 N\beta\phi_2\chi)/\zeta(1+\beta\phi_2)(1+\gamma\phi_3) > 0$.

Finally, let us examine the fixed point X_3^*. Jacobian matrix around the fixed point X_3^* is obtained as follows:

$$J\left(T_3^*, I_3^*, V_3^*\right) = \begin{bmatrix} \dfrac{\eta\vartheta T_{\max} - \varrho\phi_1 r}{\vartheta^2 T_{\max}} & -\dfrac{\varrho\phi_1 r}{\vartheta^2 T_{\max}} & -\dfrac{\varrho\phi_1 k}{\vartheta^2} \\ \dfrac{k\phi_2 V_3^*\left(\eta\vartheta T_{\max} - \varrho\phi_1 r\right)}{\vartheta^2\left(1+\phi_2\beta\right)T_{\max}} & \dfrac{\vartheta^2 T_{\max} - k\phi_2\varrho\phi_1 r V_3^*}{\vartheta^2 T_{\max}\left(1+\phi_2\beta\right)} & \dfrac{k\phi_2\varrho\left(\vartheta - V_3^*\phi_1 k\right)}{\vartheta^2\left(1+\phi_2\beta\right)} \\ \dfrac{\left(\phi_3 N\beta k\phi_2 V_3^*\right)\left(\eta\vartheta T_{\max} - \varrho\phi_1 r\right)}{\vartheta^2 T_{\max}\left(1+\phi_2\beta\right)\left(1+\phi_3\gamma\right)} & \dfrac{\phi_3 N\beta\left(\vartheta^2 T_{\max} - k\phi_2\varrho\phi_1 r V_3^*\right)}{\vartheta^2 T_{\max}\left(1+\phi_2\beta\right)\left(1+\phi_3\gamma\right)} & \dfrac{\vartheta^2\left(1+\phi_2\beta\right) + \phi_3 N\beta k\phi_2\varrho\left(-kV_3^*\phi_1 + \vartheta\right)}{\vartheta^2\left(1+\phi_2\beta\right)\left(1+\phi_3\gamma\right)} \end{bmatrix}, \tag{25}$$

where

$$\vartheta = \left(1 + \phi_1\left(kV_3^* + \frac{r\left(T_3^* + I_3^*\right)}{T_{\max}}\right)\right), \qquad \varrho = \left(\eta T_3^* + \phi_1 p\right). \tag{26}$$

By considering Lemma 2, we write the characteristic polynomial of $J(T_3^*, I_3^*, V_3^*)$ as follows:

$$p\left(\lambda\right) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3, \tag{27}$$

where

$$\begin{aligned} a_1 =& -\left(\left(1+\gamma\phi_3\right)\left(\theta_1 + \theta_2\left(1+\beta\phi_2\right)\right) + \vartheta^2 T_{\max}\left(1+\beta\phi_2\right)\right. \\ &\left. + T_{\max}\phi_2\phi_3 N\beta k\varrho\theta_3\right) T_{\max}\left(1+\gamma\phi_3\right) \times \left(\vartheta^4\right)^{-1} \\ a_2 =& \frac{\left(\theta_1\theta_2 + \phi_2\theta_2 kV_3^*\phi_1 r\varrho\right)\left(1+\gamma\phi_3\right)}{T_{\max}^2\vartheta^4\left(1+\phi_2\beta\right)\left(1+\gamma\phi_3\right)} \\ &+ \frac{T_{\max}\vartheta^2\left(\theta_1 + \theta_2\left(1+\phi_2\beta\right)\right)}{T_{\max}^2\vartheta^4\left(1+\phi_2\beta\right)\left(1+\gamma\phi_3\right)} \\ &+ \frac{\phi_3 N\beta k\phi_2\varrho T_{\max}\theta_2\left(V_3^* k\phi_1 + \theta_3\right)}{T_{\max}^2\vartheta^4\left(1+\phi_2\beta\right)\left(1+\gamma\phi_3\right)}, \\ a_3 =& -\frac{\theta_2}{\vartheta^4\left(1+\gamma\phi_3\right)\left(1+\beta\phi_2\right)T_{\max}^2}, \end{aligned} \tag{28}$$

where

$$\begin{aligned} \theta_1 &= \vartheta^2 T_{\max} - k\phi_2 V_3^*\varrho\phi_1 r, \\ \theta_2 &= \eta\vartheta T_{\max} - \varrho\phi_1 r, \\ \theta_3 &= \vartheta - V_3^*\phi_1 k, \\ \theta_4 &= \vartheta^2\left(1+\phi_2\beta\right)\left(1+\phi_3\gamma\right). \end{aligned} \tag{29}$$

If Lemma 2 is satisfied, we can say that the fixed point X_3^* is locally asymptotically stable. Finally, it is important to say that the stability depends on time step size h as it can be seen in Jacobian.

4. Numerical Results

In this section, we will use the values and the initial conditions in [21]. These values are given as follows:

$$\begin{aligned} p &= 0.1, \quad \alpha = 0.02, \quad \beta = 0.3, \\ \gamma &= 2.4, \quad k = 0.0027, \\ T_{\max} &= 1500, \quad N = 10, \quad r_1 = 0.1, \\ r_2 &= 0, \quad r_3 = 0.1, \quad h = 0.01. \end{aligned} \tag{30}$$

$R_0 = kNT_2^*/\gamma$ is the basic reproduction number. Wang and Li [21] present that if the basic reproduction number $R_0 \le 1$, the HIV infection is cleared from the T-cell population; if $R_0 > 1$, the HIV infection persists. In this section we will calculate R_0 and see whether X_2^* is locally asymptotically stable or not for different values of r. And by using the criterion given in Section 3, we will check the validity of the results. For the fixed point X_3^*, we will use Lemma 2, and we will also conclude whether fixed point X_3^* is asymptotically stable or not.

4.1. Analysis of the Fixed Point X_2^*. For $r = 0.05$, firstly let us calculate the basic reproduction number

$$R_0 = \frac{kNT_2^*}{\gamma} = 10.16236213 > 1. \tag{31}$$

The first eigenvalue is

$$\lambda_1 = 0.9996978319. \tag{32}$$

From (23), other eigenvalues are found as follows:

$$\begin{aligned} \left|\lambda_2\right| &= 1.015636510, \\ \left|\lambda_3\right| &= 0.9583755908. \end{aligned} \tag{33}$$

From Lemma 1, we see that

(i) $b = 0.9733612405 < 1$,

(ii) $f(-1) = 3.947373342 > 0$,

(iii) $f(1) = -0.6508605 \times 10^{-3} < 0$.

Therefore, the fixed point X_2^* is unstable for $r = 0.05$.

For $r = 0.8$, the basic reproduction number is;

$$R_0 = 16.45456718 > 1, \tag{34}$$

and the first eigenvalue is

$$\lambda_1 = 0.9922289848. \tag{35}$$

From (23),

$$\begin{aligned} |\lambda_2| &= 1.022738962, \\ |\lambda_3| &= 0.9517201132. \end{aligned} \tag{36}$$

So according to Lemma 1,

(i) $b = 0.9733612405 < 1$,

(ii) $f(-1) = 3.947820316 > 0$,

(iii) $f(1) = -0.10978345 \times 10^{-2} < 0$.

As a result, the fixed point X_2^* is unstable for $r = 0.8$.

For $r = 3$; the basic reproduction number is

$$R_0 = 16.76287750 > 1. \tag{37}$$

The first eigenvalue is

$$\lambda_1 = 0.9706383389, \tag{38}$$

and other eigenvalues are found as follows:

$$\begin{aligned} |\lambda_2| &= 1.023053068, \\ |\lambda_3| &= 0.9514279092. \end{aligned} \tag{39}$$

By Lemma 1,

(i) $b = 0.9733612413 < 1$,

(ii) $f(-1) = 3.947842218 > 0$,

(iii) $f(1) = -0.11197357 \times 10^{-2} < 0$.

Therefore, X_2^* fixed point is unstable for $r = 3$. For $r = 0.001$, the basic reproduction number is

$$R_0 = 0.5919960938 < 1. \tag{40}$$

We obtain the eigenvalues as $\lambda_1 = 0.999809947$, $\lambda_2 = 0.9972049810$, and by Lemma 1, $\lambda_3 = 0.9760894290$:

(i) $b = 0.9733612405 < 1$,

(ii) $f(-1) = 3.9446655650 > 0$,

(iii) $f(1) = 0.668305 \times 10^{-4} > 0$.

So, the fixed point X_2^* is locally asymptotically stable for $r = 0.001$.

Table 1: Qualitative results of the fixed point X_3^* for different time step sizes, $r = 0.05$, $t = 0$–5000.

h	Euler	Runge-Kutta	NFDS
0.001	Convergence	Convergence	Convergence
0.01	Convergence	Convergence	Convergence
0.1	Convergence	Convergence	Convergence
0.5	Divergence	Convergence	Convergence
1	Divergence	Divergence	Convergence
10	Divergence	Divergence	Convergence
100	Divergence	Divergence	Convergence

Table 2: Qualitative results of the fixed point X_2^* for different time step sizes, $r = 0.001$, $t = 0$–500.

h	Euler	Runge-Kutta	NFDS
0.001	Convergence	Convergence	Convergence
0.01	Convergence	Convergence	Convergence
0.1	Convergence	Convergence	Convergence
0.5	Convergence	Convergence	Convergence
1	Divergence	Convergence	Convergence
10	Divergence	Divergence	Convergence
100	Divergence	Divergence	Convergence

Table 3: Stability results of the fixed points X_2^* For different r values.

r	R_0	Stability
0.001	0.591960938	stable
0.01	0.1117598344	stable
0.02	0.9742785788	stable
0.021	1.433991904	unstable
0.04	8.493379921	unstable
0.05	10.16236213	unstable
0.8	16.45456718	unstable

4.2. Analysis of the Fixed Point X_3^*. For $r = 0.05$, let us find characteristic polynomial of $J(T_3^*, I_3^*, V_3^*)$:

$$\begin{aligned} p(\lambda) = {} & \lambda^3 - 2.973320340\lambda^2 + 2.946641816\lambda \\ & - 0.9733214565. \end{aligned} \tag{41}$$

By using Lemma 2,

(i) $p(1) = 0.20 \times 10^{-7} > 0$,

(ii) $(-1)^3 p(-1) = 7.893283612 > 0$,

(iii) $\left.\begin{matrix} 1-(a_3)^2 = 0.526453423 \times 10^{-1} \\ |a_2 - a_3 a_1| = 0.52645332 \times 10^{-1} \end{matrix}\right\} 1 - (a_3)^2 > |a_2 - a_3 a_1|$.

Therefore the fixed point X_3^* is locally asymptotically stable for $r = 0.05$. For $r = 0.8$, let us find characteristic polynomial of $J(T_3^*, I_3^*, V_3^*)$:

$$\begin{aligned} p(\lambda) = {} & \lambda^3 - 2.972874374\lambda^2 + 2.945765581\lambda \\ & - 0.9728906881. \end{aligned} \tag{42}$$

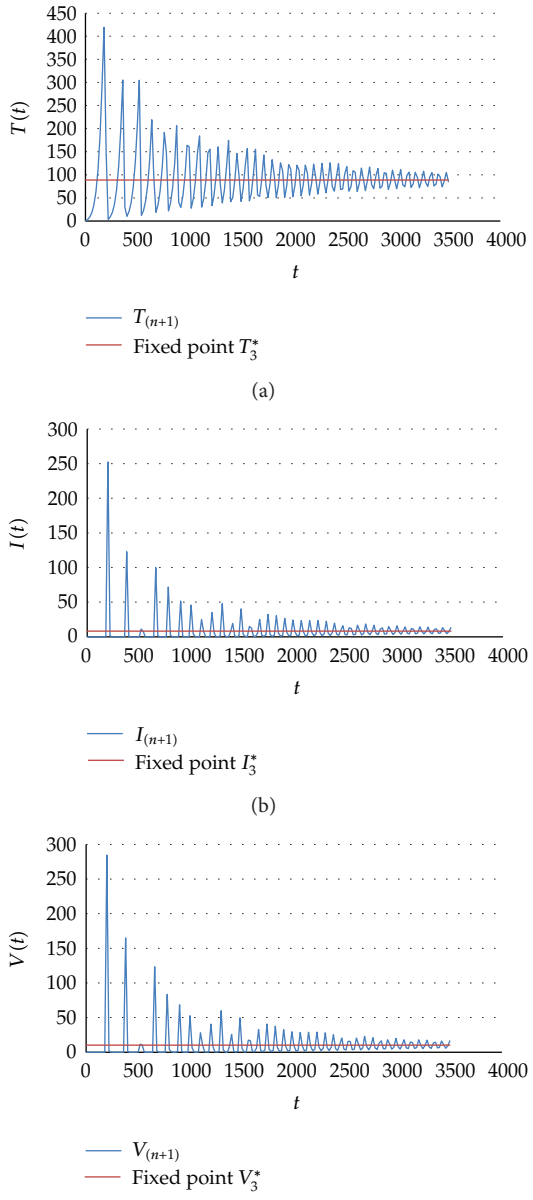

FIGURE 1: NFDS solutions for $T(t)$, $I(t)$, and $V(t)$, $r = 0.05$.

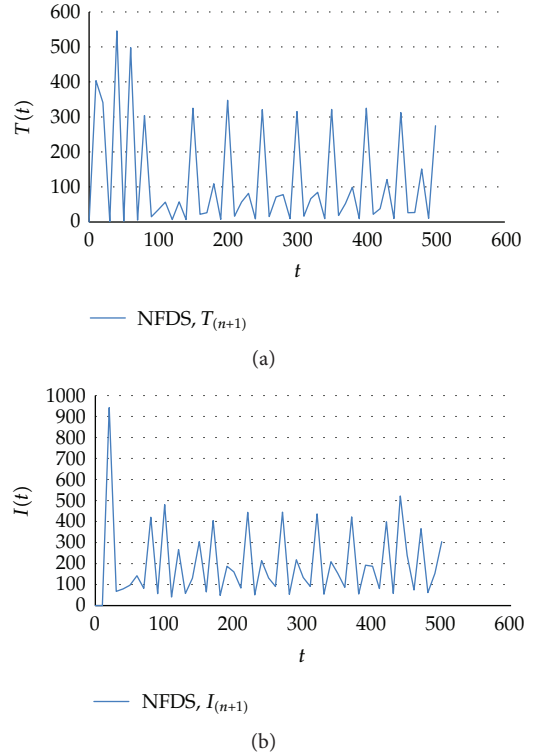

FIGURE 2: NFDS solutions for $T(t)$ and $I(t)$, $r = 0.8$.

From Lemma 2,

(i) $p(1) = 0.519 \times 10^{-6} > 0$,

(ii) $(-1)^3 p(-1) = 7.891530643 > 0$,

(iii) $\left.\begin{matrix} 1-(a_3)^2 = 0.534837090 \times 10^{-1} \\ |a_2 - a_3 a_1| = 0.53483786 \times 10^{-1} \end{matrix}\right\} 1 - (a_3)^2 \not> |a_2 - a_3 a_1|$.

We obtain that the fixed point X_3^* is unstable for $r = 0.8$. For $r = 3$, let us find characteristic polynomial of $J(T_3^*, I_3^*, V_3^*)$:

$$p(\lambda) = \lambda^3 - 2.971566609\lambda^2 + 2.943214150\lambda - 0.9716455797. \quad (43)$$

By using Lemma 2,

(i) $p(1) = 0.1961 \times 10^{-5} > 0$,

(ii) $(-1)^3 p(-1) = 7.886426339 > 0$,

(iii) $\left.\begin{matrix} 1-(a_3)^2 = 0.559048674 \times 10^{-1} \\ |a_2 - a_3 a_1| = .55904590 \times 10^{-1} \end{matrix}\right\} 1 - (a_3)^2 > |a_2 - a_3 a_1|$.

We have the fixed point X_3^* locally asymptotically stable for $r = 3$.

5. Conclusions

In general, it is too hard to analyze the stability of nonlinear three-dimensional systems. In this paper, by using the proposed NFDS scheme, nonlinear ordinary differential equation system which describes HIV infection of $CD4^+$ T cells, is discretizated and the behaviour of the model is investigated. It is seen that the local asymptotic stability results of the fixed points X_2^* and X_3^* of the discrete time system satisfying the positivity condition are the same as in [21]. In Tables 1 and 2, for different step size h and for different r values, the qualitative stability results, obtained by NFDS, of the fixed point X_3^* and X_2^* are respectively compared to classical methods such as forward Euler and Runge-Kutta. The fixed points X_2^* and X_3^* are locally asymptotically stable for the values r given in these two tables. If step size h is chosen small enough, the results of the proposed NFDS are similar with the results of the other two numerical methods. But if the step size is chosen larger, the efficiency of NFDS is clearly seen. In Table 3, stability results for fixed point X_2^* are

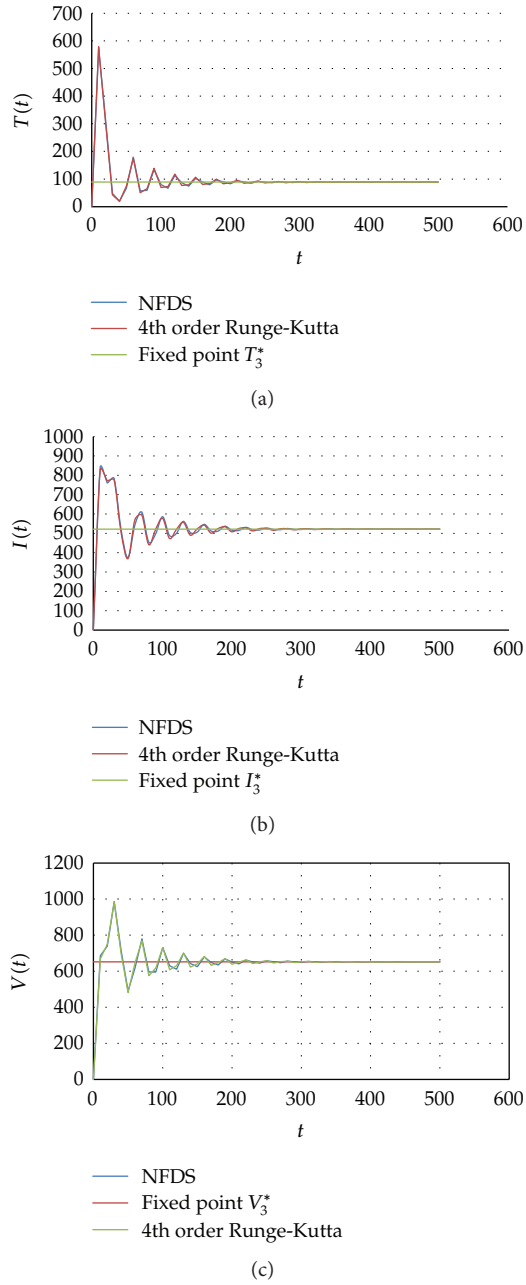

FIGURE 3: Comparison with NFDS and 4th order Runge-Kutta solutions for $I(t)$, $V(t)$, and $T(t)$, $r = 3$.

given for different r values. It is shown in [21] that if $R_0 > 1$, X_2^* is unstable and HIV infection persist in T-cell population. If $0.093453 < r < 1.9118$, then X_3^* is unstable. So in case of $r = 0.8$, neither X_2^* nor X_3^* are stable (Figure 2). In Figures 1 and 3, the NFDS solutions of T, I and V converges to fixed point X_3^* as simulated for $r = 0.05$ and $r = 3$, respectively. Also in Figure 3, Runge-Kutta and proposed NFDS scheme are compared graphically. All the numerical calculations and simulations are performed by using Maple programme. In conclusion, the efficiency of the proposed NFDS scheme is investigated and compared with other numerical methods.

Acknowledgments

The authors would like to thank Professor Apostolos Hadjidimos and Professor Michael N. Vrahatis for their valuable suggestions during the preparation of this paper. Also, the authors are grateful to the reviewers for their constructive comments, which have enhanced the paper.

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