

## Research Article

# Positive Almost Periodic Solutions for a Discrete Competitive System Subject to Feedback Controls

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This paper concerns a discrete competitive system subject to feedback controls. By using Lyapunov function and some preliminary lemmas, the existence and uniformly asymptotic stability of unique positive almost periodic solution of the system are investigated. Numerical simulations suggest the feasibility of our theoretical results.

## 1. Introduction

Many real world phenomena are studied through discrete mathematical models involving difference equations which are more suitable than the continuous ones when the populations have nonoverlapping generations. On the other hand, discrete models can also provide efficient computational models of continuous models for numerical stimulations; therefore, the studies of dynamic systems governed by difference equations have attracted more attention from scholars. Many good results concerned with discrete systems are deliberated (see [1–7] in detail).

Recently, in [1] we consider the following discrete two-species competitive almost system

$$\begin{aligned}x_1(n+1) &= x_1(n) \exp \left[ r_1(n) - a_1(n) x_1(n) - \frac{c_2(n) x_2(n)}{1 + x_2(n)} \right], \\x_2(n+1) &= x_2(n) \exp \left[ r_2(n) - a_2(n) x_2(n) - \frac{c_1(n) x_1(n)}{1 + x_1(n)} \right], \\n &= 0, 1, 2, \dots\end{aligned}\tag{1}$$

Here  $x_i(n)$  stand for the densities of species  $x_i$  at the  $n$ th generation,  $r_i(n)$  represent the natural growth rates of species  $x_i$  at the  $n$ th generation,  $a_i(n)$  are the intraspecific effects of the  $n$ th generation of species  $x_i$  on own population, and

$c_i(n)$  measure the interspecific effects of the  $n$ th generation of species  $x_i$  on species  $x_j$  ( $i, j = 1, 2; i \neq j$ ). The coefficients  $\{r_i(n)\}$ ,  $\{a_i(n)\}$ , and  $\{c_i(n)\}$  are bounded positive almost periodic sequences. We established a criterion for the existence and uniformly asymptotic stability of unique positive almost periodic solution of system (1) (see [1]).

Note that ecosystems in the real world are often disturbed by outside continuous forces. In the language of control, we call the disturbance functions control variables and they can be regarded as feedback controls. For more discussions on this direction, we can refer to [8–14] and the references cited therein. Motivated by the above ideas we can establish the discrete two-species competitive almost system with feedback controls

$$\begin{aligned}x_1(n+1) &= x_1(n) \exp \left[ r_1(n) - a_1(n) x_1(n) - \frac{c_2(n) x_2(n)}{1 + x_2(n)} \right. \\&\quad \left. - e_1(n) u_1(n) \right], \\x_2(n+1) &= x_2(n) \exp \left[ r_2(n) - a_2(n) x_2(n) - \frac{c_1(n) x_1(n)}{1 + x_1(n)} \right. \\&\quad \left. - e_2(n) u_2(n) \right],\end{aligned}$$

$$\begin{aligned}\Delta u_1(n) &= -b_1(n)u_1(n) + d_1(n)x_1(n), \\ \Delta u_2(n) &= -b_2(n)u_2(n) + d_2(n)x_2(n), \\ n &= 0, 1, 2, \dots,\end{aligned}\quad (2)$$

where  $u_1$  and  $u_2$  represent control variables and  $\Delta u_1(n) = u_1(n+1) - u_1(n)$ ,  $\Delta u_2(n) = u_2(n+1) - u_2(n)$  are the forward difference operators.  $\{b_i(n)\}$ ,  $\{d_i(n)\}$  and  $\{e_i(n)\}$  are bounded positive almost periodic sequences, where  $0 < b_i(n) < 1$ ,  $i = 1, 2$ . To belong to the direction of [1], in this contribution, we continue to discuss the effect of feedback controls and establish a criterion for the existence and uniformly asymptotic stability of unique positive almost periodic solution of system (2).

The rest of this paper is organized as follows. In the next section, we introduce some notations, definitions, and lemmas which are available for our main results. Sufficient conditions for the existence and uniformly asymptotic stability of unique positive almost periodic solution of system (2) are established in Section 3. In Section 4, we carry out numerical simulations to substantiate our analytical results.

## 2. Preliminaries

In this section, we give some notations, definitions, and lemmas which will be useful for the later sections.

$\mathbb{R}$ ,  $\mathbb{R}^+$ ,  $\mathbb{Z}$ , and  $\mathbb{Z}^+$  denote the sets of real numbers, non-negative real numbers, integers, and nonnegative integers, respectively.  $\mathbb{R}^4$  and  $\mathbb{R}^k$  denote the cone of 4-dimensional and  $k$ -dimensional real Euclidean space, respectively. For an almost periodic sequence  $\{g(n)\}$  defined on  $\mathbb{Z}^+$ , the notations below will be used

$$g^U = \sup_{n \in \mathbb{Z}^+} \{g(n)\}, \quad g^L = \inf_{n \in \mathbb{Z}^+} \{g(n)\}. \quad (3)$$

**Definition 1** (see [4]). A sequence  $x : \mathbb{Z} \rightarrow \mathbb{R}^k$  is called an almost periodic sequence if the  $\varepsilon$ -translation set of  $x$ ,

$$E\{\varepsilon, x\} := \{\tau \in \mathbb{Z} : |x(n+\tau) - x(n)| < \varepsilon, \forall n \in \mathbb{Z}\}, \quad (4)$$

is a relatively dense set in  $\mathbb{Z}$  for all  $\varepsilon > 0$ ; that is, for any given  $\varepsilon > 0$ , there exists an integer  $l(\varepsilon) > 0$  such that each discrete interval of length  $l(\varepsilon)$  contains a  $\tau = \tau(\varepsilon) \in E\{\varepsilon, x\}$  such that

$$|x(n+\tau) - x(n)| < \varepsilon, \quad \forall n \in \mathbb{Z}. \quad (5)$$

$\tau$  is called the  $\varepsilon$ -translation number of  $x(n)$ .

**Definition 2** (see [4]). Let  $f : \mathbb{Z} \times \mathbb{D} \rightarrow \mathbb{R}^k$ , where  $\mathbb{D}$  is an open set in  $\mathbb{R}^k$ .  $f(n, x)$  is said to be almost periodic in  $n$  uniformly for  $x \in \mathbb{D}$ , or uniformly almost periodic for short, if for any  $\varepsilon > 0$  and any compact set  $\mathbb{S}$  in  $\mathbb{D}$ , there exists a positive integer  $l(\varepsilon, \mathbb{S})$  such that any interval of length  $l(\varepsilon, \mathbb{S})$  contains a integer  $\tau$  for which

$$|f(n+\tau, x) - f(n, x)| < \varepsilon, \quad (6)$$

for all  $n \in \mathbb{Z}$  and all  $x \in \mathbb{S}$ .  $\tau$  is called the  $\varepsilon$ -translation number of  $f(n, x)$ .

**Lemma 3** (see [4]).  $\{x(n)\}$  is an almost periodic sequence if and only if for any sequence  $\{t'_k\} \subset \mathbb{Z}$  there exists a subsequence  $\{t_k\} \subset \{t'_k\}$  such that  $x(n+t_k)$  converges uniformly on  $n \in \mathbb{Z}$  as  $k \rightarrow \infty$ . Furthermore, the limit sequence is also an almost periodic sequence.

Consider the following almost periodic difference system

$$x(n+1) = f(n, x(n)), \quad n \in \mathbb{Z}^+, \quad (7)$$

where  $f : \mathbb{Z}^+ \times \mathbb{S}_B \rightarrow \mathbb{R}^k$ ,  $\mathbb{S}_B = \{x \in \mathbb{R}^k : \|x\| < B\}$ , and  $f(n, x)$  is almost periodic in  $n$  uniformly for  $x \in \mathbb{S}_B$  and is continuous in  $x$ . The product system of (7) is the following system:

$$x(n+1) = f(n, x(n)), \quad y(n+1) = f(n, y(n)), \quad (8)$$

and Zhang [5] established the following result.

**Lemma 4** (see [5]). Suppose that there exists a Lyapunov function  $V(n, x, y)$  defined for  $n \in \mathbb{Z}^+$ ,  $\|x\| < B$ ,  $\|y\| < B$  satisfying the following conditions:

- (i)  $a(\|x - y\|) \leq V(n, x, y) \leq b(\|x - y\|)$ , where  $a, b \in K$  with  $K = \{\alpha \in C(\mathbb{R}^+, \mathbb{R}^+) : \alpha(0) = 0 \text{ and } \alpha \text{ is increasing}\}$ ;
- (ii)  $|V(n, x_1, y_1) - V(n, x_2, y_2)| \leq L(\|x_1 - x_2\| + \|y_1 - y_2\|)$ , where  $L > 0$  is a constant;
- (iii)  $\Delta V_{(8)}(n, x, y) \leq -\gamma V(n, x, y)$ , where  $0 < \gamma < 1$  is a constant and

$$\Delta V_{(8)}(n, x, y) = V(n+1, f(n, x), f(n, y)) - V(n, x, y). \quad (9)$$

Moreover, if there exists a solution  $\varphi(n)$  of system (7) such that  $\|\varphi(n)\| \leq B^* < B$  for  $n \in \mathbb{Z}^+$ , then there exists a unique uniformly asymptotically stable almost periodic solution  $p(n)$  of system (7) which satisfies  $\|p(n)\| \leq B^*$ . In particular, if  $f(n, x)$  is periodic of period  $\omega$ , then there exists a unique uniformly asymptotically stable periodic solution of system (7) of period  $\omega$ .

## 3. Main Results

We first give the following two propositions which are useful for our main results.

**Proposition 5.** Any positive solution  $(x_1(n), x_2(n), u_1(n), u_2(n))$  of system (2) satisfies

$$\begin{aligned}\limsup_{n \rightarrow +\infty} x_1(n) &\leq x_1^*, & \limsup_{n \rightarrow +\infty} x_2(n) &\leq x_2^*, \\ \limsup_{n \rightarrow +\infty} u_1(n) &\leq u_1^*, & \limsup_{n \rightarrow +\infty} u_2(n) &\leq u_2^*,\end{aligned}\quad (10)$$

where

$$\begin{aligned}x_1^* &= \frac{\exp(r_1^U - 1)}{a_1^L}, & x_2^* &= \frac{\exp(r_2^U - 1)}{a_2^L}, \\ u_1^* &= \frac{x_1^* d_1^U}{b_1^L}, & u_2^* &= \frac{x_2^* d_2^U}{b_2^L}.\end{aligned}\quad (11)$$

*Proof.* We first prove that  $\limsup_{n \rightarrow +\infty} x_1(n) \leq x_1^*$ ; to do so, we consider Cases (1) and (2).

*Case (1).* Suppose that there exists an  $n_0 \in \mathbb{Z}^+$  such that  $x_1(n_0 + 1) \geq x_1(n_0)$ ; it follows from the first equation of system (2) that

$$r_1(n_0) - a_1(n_0)x_1(n_0) - \frac{c_2(n_0)x_2(n_0)}{1+x_2(n_0)} - e_1(n_0)u_1(n_0) \geq 0, \quad (12)$$

which implies

$$x_1(n_0) \leq \frac{r_1(n_0)}{a_1(n_0)} \leq \frac{r_1^U}{a_1^L} \leq \frac{\exp(r_1^U - 1)}{a_1^L} = x_1^*. \quad (13)$$

Hence,

$$\begin{aligned} x_1(n_0 + 1) &= x_1(n_0) \exp \left[ r_1(n_0) - a_1(n_0)x_1(n_0) \right. \\ &\quad \left. - \frac{c_2(n_0)x_2(n_0)}{1+x_2(n_0)} - e_1(n_0)u_1(n_0) \right] \\ &\leq x_1(n_0) \exp [r_1^U - a_1^L x_1(n_0)] \\ &= \frac{r_1^U}{a_1^L} \cdot \frac{a_1^L}{r_1^U} x_1(n_0) \exp \left[ r_1^U \left( 1 - \frac{a_1^L}{r_1^U} x_1(n_0) \right) \right] \\ &\leq \frac{r_1^U}{a_1^L} \cdot \frac{\exp(r_1^U - 1)}{r_1^U} = \frac{\exp(r_1^U - 1)}{a_1^L} = x_1^*, \end{aligned} \quad (14)$$

where we use the fact that  $\max_{x \in \mathbb{R}^+} x \exp[r(1-x)] = \exp(r-1)/r$  for  $r > 0$ .

It is claimed that  $x_1(n) \leq x_1^*$  for all  $n \geq n_0$ . By way of contradiction, assume that there is a  $q_0 > n_0$  such that  $x_1(q_0) > x_1^*$ , then  $q_0 \geq n_0 + 2$ . Set  $\tilde{q}_0 = \min\{q_0 : q_0 \geq n_0 + 2, x_1(q_0) > x_1^*\}$ ; that is to say,  $x_1(\tilde{q}_0) > x_1^*$  and  $\tilde{q}_0 \geq n_0 + 2$ ; then  $x_1(\tilde{q}_0) > x_1^* \geq x_1(\tilde{q}_0 - 1)$ . It is easy to see that  $x_1(\tilde{q}_0) \leq x_1^*$  from the above argument, which is a contradiction. Therefore,  $x_1(n) \leq x_1^*$  for all  $n \geq n_0$ , then  $\limsup_{n \rightarrow +\infty} x_1(n) \leq x_1^*$ . This proves the claim.

*Case (2).* Suppose that  $x_1(n) > x_1(n+1)$  for all  $n \in \mathbb{Z}^+$ . In particular,  $\lim_{n \rightarrow +\infty} x_1(n)$  exists, denoted by  $\bar{x}_1$ . We will prove that  $\bar{x}_1 \leq x_1^*$  by way of contradiction, if  $\bar{x}_1 > x_1^*$ , then by taking limit in the first equation in system (2) we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} \left[ r_1(n) - a_1(n)x_1(n) - \frac{c_2(n)x_2(n)}{1+x_2(n)} - e_1(n)u_1(n) \right] \\ = 0. \end{aligned} \quad (15)$$

Notice that  $r_1^U/a_1^L \leq \exp(r_1^U - 1)/a_1^L = x_1^* < \bar{x}_1$ , so we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} \left[ r_1(n) - a_1(n)x_1(n) - \frac{c_2(n)x_2(n)}{1+x_2(n)} - e_1(n)u_1(n) \right] \\ \leq \lim_{n \rightarrow +\infty} [r_1(n) - a_1(n)x_1(n)] \\ \leq r_1^U - a_1^L \bar{x}_1 < 0, \end{aligned} \quad (16)$$

which is a contradiction. This proves the claim. Hence,  $\limsup_{n \rightarrow +\infty} x_1(n) \leq x_1^*$ .

Similar to  $x_1(n)$ , we can prove that  $\limsup_{n \rightarrow +\infty} x_2(n) \leq x_2^*$ .

Next, we prove  $\limsup_{n \rightarrow +\infty} u_1(n) \leq u_1^*$ . For any  $\varepsilon > 0$ , there exists a large enough integer  $l_0 \in \mathbb{Z}^+$  such that  $x_i(n) \leq x_i^* + \varepsilon$ ,  $i = 1, 2$  for all  $n \geq l_0$ . We have from the third equation of system (2) that

$$\begin{aligned} u_1(n) &= \prod_{i=0}^{n-1} (1 - b_1(i)) \left[ u_1(0) + \sum_{i=0}^{n-1} \frac{d_1(i)x_1(i)}{\prod_{j=0}^i (1 - b_1(j))} \right] \\ &\leq (1 - b_1^L)^n (u_1(0) + \vartheta_1) + d_1^U (x_1^* + \varepsilon) \\ &\quad \times \sum_{i=l_0}^{n-1} \prod_{j=i+1}^{n-1} (1 - b_1(j)) \\ &\leq (1 - b_1^L)^n (u_1(0) + \vartheta_1) + d_1^U (x_1^* + \varepsilon) \sum_{i=l_0}^{n-1} (1 - b_1^L)^{n-i-1}, \end{aligned} \quad (17)$$

where  $\vartheta_1 = \sum_{i=0}^{l_0-1} (d_1(i)x_1(i)/\prod_{j=0}^i (1 - b_1(j)))$ . Since  $0 < b_1^L < 1$ , we can find a positive number  $s$  such that  $1 - b_1^L = e^{-s}$ , then by using Stolz's theorem, we obtain that

$$\lim_{n \rightarrow +\infty} \sum_{i=l_0}^{n-1} (1 - b_1^L)^{n-i-1} = \lim_{n \rightarrow +\infty} \frac{\sum_{i=l_0}^{n-1} e^{s(i+1)}}{e^{sn}} = \frac{1}{1 - e^{-s}} = \frac{1}{b_1^L}. \quad (18)$$

Thus  $\limsup_{n \rightarrow +\infty} u_1(n) \leq (x_1^* + \varepsilon)d_1^U/b_1^L$ . Since  $\varepsilon$  is arbitrary,  $\limsup_{n \rightarrow +\infty} u_1(n) \leq u_1^*$  is valid. Analogously, we can prove  $\limsup_{n \rightarrow +\infty} u_2(n) \leq u_2^*$ . Hence, the proof of Proposition 5 is complete.  $\square$

**Proposition 6.** *If the following inequalities*

$$r_1^L - c_2^U - e_1^U u_1^* > 0, \quad r_2^L - c_1^U - e_2^U u_2^* > 0 \quad (19)$$

*hold. Then any positive solution  $(x_1(n), x_2(n), u_1(n), u_2(n))$  of system (2) satisfies*

$$\begin{aligned} \liminf_{n \rightarrow +\infty} x_1(n) &\geq x_{1*}, & \liminf_{n \rightarrow +\infty} x_2(n) &\geq x_{2*}, \\ \liminf_{n \rightarrow +\infty} u_1(n) &\geq u_{1*}, & \liminf_{n \rightarrow +\infty} u_2(n) &\geq u_{2*}, \end{aligned} \quad (20)$$

where

$$\begin{aligned} x_{1*} &= \frac{r_1^L - c_2^U - e_1^U u_1^*}{a_1^U} \exp [r_1^L - a_1^U x_1^* - c_2^U - e_1^U u_1^*], \\ x_{2*} &= \frac{r_2^L - c_1^U - e_2^U u_2^*}{a_2^U} \exp [r_2^L - a_2^U x_2^* - c_1^U - e_2^U u_2^*], \quad (21) \\ u_{1*} &= \frac{x_{1*} d_1^L}{b_1^U}, & u_{2*} &= \frac{x_{2*} d_2^L}{b_2^U}. \end{aligned}$$

*Proof.* For any small enough  $\varepsilon > 0$ , which satisfies  $r_1^L - c_2^U - e_1^U(u_1^* + \varepsilon) > 0$ , according to Proposition 5, there exists  $n^* \in \mathbb{Z}^+$  such that

$$\begin{aligned} x_1(n) &\leq x_1^* + \varepsilon, & x_2(n) &\leq x_2^* + \varepsilon, \\ u_1(n) &\leq u_1^* + \varepsilon, & u_2(n) &\leq u_2^* + \varepsilon, \end{aligned} \quad (22)$$

for  $n \geq n^*$ .

We first present Cases (1) and (2) to prove that  $\liminf_{n \rightarrow +\infty} x_1(n) \geq x_{1*}$ .

*Case (1).* Assuming that there exists a positive integer  $n_0 \geq n^*$  such that  $x_1(n_0 + 1) \leq x_1(n_0)$ , we have from the first equation of system (2) that

$$\begin{aligned} x_1(n_0 + 1) &= x_1(n_0) \exp \left[ r_1(n_0) - a_1(n_0)x_1(n_0) \right. \\ &\quad \left. - \frac{c_2(n_0)x_2(n_0)}{1 + x_2(n_0)} - e_1(n_0)u_1(n_0) \right] \\ &\geq x_1(n_0) \exp \left[ r_1^L - a_1^U x_1(n_0) - c_2^U - e_1^U(u_1^* + \varepsilon) \right]. \end{aligned} \quad (23)$$

Therefore,

$$r_1^L - a_1^U x_1(n_0) - c_2^U - e_1^U(u_1^* + \varepsilon) \leq 0, \quad (24)$$

which implies that

$$x_1(n_0) \geq \frac{r_1^L - c_2^U - e_1^U(u_1^* + \varepsilon)}{a_1^U} > 0. \quad (25)$$

Then

$$\begin{aligned} x_1(n_0 + 1) &= x_1(n_0) \exp \left[ r_1(n_0) - a_1(n_0)x_1(n_0) \right. \\ &\quad \left. - \frac{c_2(n_0)x_2(n_0)}{1 + x_2(n_0)} - e_1(n_0)u_1(n_0) \right] \\ &\geq \frac{r_1^L - c_2^U - e_1^U(u_1^* + \varepsilon)}{a_1^U} \\ &\quad \times \exp \left[ r_1^L - a_1^U(x_1^* + \varepsilon) - c_2^U - e_1^U(u_1^* + \varepsilon) \right]. \end{aligned} \quad (26)$$

Hence  $x_1(n_0 + 1) \geq x_{1\varepsilon}$ , where

$$\begin{aligned} x_{1\varepsilon} &= \frac{r_1^L - c_2^U - e_1^U(u_1^* + \varepsilon)}{a_1^U} \exp \left[ r_1^L - a_1^U(x_1^* + \varepsilon) \right. \\ &\quad \left. - c_2^U - e_1^U(u_1^* + \varepsilon) \right]. \end{aligned} \quad (27)$$

We claim that  $x_1(n) \geq x_{1\varepsilon}$  for  $n \geq n_0$ . By way of contradiction, assume that there exists a  $q_0 \geq n_0$  such that  $x_1(q_0) < x_{1\varepsilon}$ , then  $q_0 \geq n_0 + 2$ . Let  $\bar{q}_0 = \min\{q_0 : q_0 \geq n_0 + 2, x_1(q_0) < x_{1\varepsilon}\}$ , that is,  $x_1(\bar{q}_0) < x_{1\varepsilon}$  and  $\bar{q}_0 \geq n_0 + 2$ , then

$x_1(\bar{q}_0) < x_{1\varepsilon} \leq x_1(\bar{q}_0 - 1)$ , and the above argument produces that  $x_1(\bar{q}_0) \geq x_{1\varepsilon}$ , which is a contradiction. Thus,  $x_1(n) \geq x_{1\varepsilon}$  for all  $n \geq n_0$ ; since  $\varepsilon$  can be sufficiently small, it obtains that  $\liminf_{n \rightarrow +\infty} x_1(n) \geq x_{1*}$ . This proves the claim.

*Case (2).* We assume that  $x_1(n + 1) > x_1(n)$  for all  $n \in \mathbb{Z}^+$ . Then  $\lim_{n \rightarrow +\infty} x_1(n)$  exists, denoted by  $\underline{x}_1$ . We claim that

$$\underline{x}_1 \geq \frac{r_1^L - c_2^U - e_1^U(u_1^* + \varepsilon)}{a_1^U}. \quad (28)$$

By way of contradiction, assume that  $\underline{x}_1 < (r_1^L - c_2^U - e_1^U(u_1^* + \varepsilon))/a_1^U$ . Taking limit in the first equation in system (2) yields

$$\begin{aligned} \lim_{n \rightarrow +\infty} \left[ r_1(n) - a_1(n)x_1(n) - \frac{c_2(n)x_2(n)}{1 + x_2(n)} - e_1(n)u_1(n) \right] \\ = 0. \end{aligned} \quad (29)$$

However,

$$\begin{aligned} \lim_{n \rightarrow +\infty} \left[ r_1(n) - a_1(n)x_1(n) - \frac{c_2(n)x_2(n)}{1 + x_2(n)} - e_1(n)u_1(n) \right] \\ \geq \lim_{n \rightarrow +\infty} [r_1(n) - a_1(n)x_1(n) - c_2(n) - e_1(n)u_1(n)] \\ \geq r_1^L - a_1^U \underline{x}_1 - c_2^U - e_1^U(u_1^* + \varepsilon) > 0, \end{aligned} \quad (30)$$

which is a contradiction. It implies that  $\underline{x}_1 \geq (r_1^L - c_2^U - e_1^U(u_1^* + \varepsilon))/a_1^U$ . By the fact that  $\min_{x \in \mathbb{R}^+} \{[\exp(x - 1)]/x\} = 1$ , we obtain that  $x_1^* = \exp(r_1^U - 1)/a_1^L \geq r_1^U/a_1^L \geq r_1^L/a_1^U$ , which means  $x_1^* + \varepsilon > r_1^L/a_1^U$ . From (27), we know that  $\underline{x}_1 \geq (r_1^L - c_2^U - e_1^U(u_1^* + \varepsilon))/a_1^U \geq x_{1\varepsilon}$ . Therefore,  $\liminf_{n \rightarrow +\infty} x_1(n) \geq x_{1\varepsilon}$ .

Since  $\varepsilon$  can be sufficiently small, we have

$$\begin{aligned} \liminf_{n \rightarrow +\infty} x_1(n) &\geq x_{1*} \\ &= \frac{r_1^L - c_2^U - e_1^U u_1^*}{a_1^U} \exp \left[ r_1^L - a_1^U x_1^* - c_2^U - e_1^U u_1^* \right]. \end{aligned} \quad (31)$$

By a similar argument, we can prove that

$$\begin{aligned} \liminf_{n \rightarrow +\infty} x_2(n) &\geq x_{2*} \\ &= \frac{r_2^L - c_1^U - e_2^U u_2^*}{a_2^U} \exp \left[ r_2^L - a_2^U x_2^* - c_1^U - e_2^U u_2^* \right]. \end{aligned} \quad (32)$$

Now we prove that  $\liminf_{n \rightarrow +\infty} u_1(n) \geq u_{1*}$ . For any small enough  $\varepsilon > 0$ , there exists a positive integer  $l_0$  such that  $x_i(n) \geq x_{i*} - \varepsilon > 0$  for  $n \geq l_0$ .

By the third equation of system (2), we obtain that

$$\begin{aligned}
 u_1(n) &= \prod_{i=0}^{n-1} (1 - b_1(i)) \left[ u_1(0) + \sum_{i=0}^{n-1} \frac{d_1(i) x_1(i)}{\prod_{j=0}^i (1 - b_1(j))} \right] \\
 &\geq (1 - b_1^U)^n (u_1(0) + \varrho_1) + d_1^L (x_{1*} - \varepsilon) \\
 &\quad \times \sum_{i=l_0}^{n-1} \prod_{j=i+1}^{n-1} (1 - b_1(j)) \\
 &\geq (1 - b_1^U)^n (u_1(0) + \varrho_1) \\
 &\quad + d_1^L (x_{1*} - \varepsilon) \sum_{i=l_0}^{n-1} (1 - b_1^U)^{n-i-1},
 \end{aligned} \tag{33}$$

where  $\varrho_1 = \sum_{i=0}^{l_0-1} (d_1(i) x_1(i) / \prod_{j=0}^i (1 - b_1(j)))$ . Since  $0 < b_1^U < 1$ , we can find a positive number  $t$  such that  $1 - b_1^U = e^{-t}$ , then by Stolz's theorem, we have

$$\lim_{n \rightarrow +\infty} \sum_{i=l_0}^{n-1} (1 - b_1^U)^{n-i-1} = \lim_{n \rightarrow +\infty} \frac{\sum_{i=l_0}^{n-1} e^{t(i+1)}}{e^{tn}} = \frac{1}{1 - e^{-t}} = \frac{1}{b_1^U}. \tag{34}$$

Thus  $\liminf_{n \rightarrow +\infty} u_1(n) \geq (x_{1*} - \varepsilon) d_1^L / b_1^U$ , by the arbitrary of  $\varepsilon$ ,  $\liminf_{n \rightarrow +\infty} u_1(n) \geq u_{1*}$  is valid. The conclusion about  $u_2(n)$  can be obtained in a similar way. Thus the proof of Proposition 6 is complete.  $\square$

Denote

$$\begin{aligned}
 \Omega &= \{(x_1, x_2, u_1, u_2) \mid x_{i*} \leq x_i(n) \leq x_i^*, \\
 &\quad u_{i*} \leq u_i(n) \leq u_i^* (i = 1, 2)\}.
 \end{aligned} \tag{35}$$

From the proofs of Propositions 5 and 6, we know that the set  $\Omega$  is an invariant set of system (2) under the assumptions in (19).

**Theorem 7.** *If the assumptions in (19) are satisfied, then  $\Omega \neq \emptyset$ .*

*Proof.* We have from system (2) that

$$\begin{aligned}
 x_i(n) &= x_i(0) \exp \sum_{m=0}^{n-1} \left[ r_i(m) - a_i(m) x_i(m) \right. \\
 &\quad \left. - \frac{c_j(m) x_j(m)}{1 + x_j(m)} - e_i(m) u_i(m) \right], \\
 u_i(n) &= u_i(0) - \sum_{m=0}^{n-1} [b_i(m) u_i(m) - d_i(m) x_i(m)],
 \end{aligned} \tag{36}$$

for  $i, j = 1, 2, i \neq j$ . Based on Propositions 5 and 6, any solution  $(x_1(n), x_2(n), u_1(n), u_2(n))$  of system (2) satisfies (10)

and (20). Thus, for any  $\varepsilon > 0$ , there exists  $p_0$  large enough such that

$$\begin{aligned}
 x_{i*} - \varepsilon &\leq x_i(n) \leq x_i^* + \varepsilon, \\
 u_{i*} - \varepsilon &\leq u_i(n) \leq u_i^* + \varepsilon, \\
 \forall n &\geq p_0, \quad i = 1, 2.
 \end{aligned} \tag{37}$$

Setting to  $\{t_k\}$  be any positive integer valued sequence such that  $t_k \rightarrow +\infty$  as  $k \rightarrow +\infty$ , it is easy to show that there exists a subsequence of  $\{t_k\}$  still denoted by  $\{t_k\}$ , such that  $x_i(n + t_k) \rightarrow x_i^*(n), u_i(n + t_k) \rightarrow u_i^*(n)$  uniformly in  $n$  on any finite subset  $D$  of  $\mathbb{Z}^+$  as  $k \rightarrow +\infty$ , where  $D = \{\alpha_1, \alpha_2, \dots, \alpha_l\}, \alpha_h \in \mathbb{Z}^+ (h = 1, 2, \dots, l)$  and  $l$  is a finite number.

In reality, for any finite subset  $D \subset \mathbb{Z}^+, t_k + \alpha_h > p_0, h = 1, 2, \dots, l$ , when  $k$  is large enough. Thus,  $x_{i*} - \varepsilon \leq x_i(n + t_k) \leq x_i^* + \varepsilon, u_{i*} - \varepsilon \leq u_i(n + t_k) \leq u_i^* + \varepsilon$ , which mean that  $\{x_i(n + t_k)\}, \{u_i(n + t_k)\}$  are uniformly bounded for  $k$  large enough.

Now, for  $\alpha_1 \in D$ , we can select a subsequence  $\{t_k^{(1)}\}$  of  $\{t_k\}$  such that  $\{x_i(\alpha_1 + t_k^{(1)})\}, \{u_i(\alpha_1 + t_k^{(1)})\}$  uniformly converge on  $\mathbb{Z}^+$  for  $k$  large enough.

Similarly, for  $\alpha_2 \in D$ , we can also select a subsequence  $\{t_k^{(2)}\}$  of  $\{t_k^{(1)}\}$  such that  $\{x_i(\alpha_2 + t_k^{(2)})\}, \{u_i(\alpha_2 + t_k^{(2)})\}$  uniformly converge on  $\mathbb{Z}^+$  for  $k$  large enough.

Repeating the above process, for  $\alpha_l \in D$ , we choose a subsequence  $\{t_k^{(l)}\}$  of  $\{t_k^{(l-1)}\}$  such that  $\{x_i(\alpha_l + t_k^{(l)})\}, \{u_i(\alpha_l + t_k^{(l)})\}$  uniformly converge on  $\mathbb{Z}^+$  for  $k$  large enough.

Then we choose the sequence  $\{t_k^{(l)}\}$  which is a subsequence of  $\{t_k\}$  still denoted by  $\{t_k\}$ ; for all  $n \in D$ , we have  $x_i(n + t_k) \rightarrow x_i^*(n), u_i(n + t_k) \rightarrow u_i^*(n)$  uniformly in  $n \in D$  as  $k \rightarrow +\infty$ . Therefore, the conclusion is true by the arbitrary of  $D$ .

Consider the almost periodicity of  $\{r_i(n)\}, \{a_i(n)\}, \{b_i(n)\}, \{c_i(n)\}, \{d_i(n)\}$ , and  $\{e_i(n)\}, i = 1, 2$ , for the above sequence  $\{t_k\}, t_k \rightarrow +\infty$  as  $k \rightarrow +\infty$ , there exists a subsequence still denoted by  $\{t_k\}$  such that

$$\begin{aligned}
 r_i(n + t_k) &\longrightarrow r_i(n), & a_i(n + t_k) &\longrightarrow a_i(n), \\
 b_i(n + t_k) &\longrightarrow b_i(n), & c_i(n + t_k) &\longrightarrow c_i(n), \\
 d_i(n + t_k) &\longrightarrow d_i(n), & e_i(n + t_k) &\longrightarrow e_i(n),
 \end{aligned} \tag{38}$$

as  $k \rightarrow +\infty$  uniformly on  $\mathbb{Z}^+$ .

For any  $\beta \in \mathbb{Z}^+$ , we can presume that  $t_k + \beta \geq p_0$  for  $k$  large enough. Let  $n \in \mathbb{Z}^+$ , by an inductive argument of system (2) from  $t_k + \beta$  to  $n + t_k + \beta$  which results in

$$\begin{aligned}
 x_i(n + t_k + \beta) &= x_i(t_k + \beta) \\
 &\times \exp \sum_{m=t_k+\beta}^{n+t_k+\beta-1} \left[ r_i(m) - a_i(m) x_i(m) \right. \\
 &\quad \left. - \frac{c_j(m) x_j(m)}{1 + x_j(m)} - e_i(m) u_i(m) \right],
 \end{aligned}$$

$$u_i(n+t_k+\beta) = u_i(t_k+\beta) - \sum_{m=t_k+\beta}^{n+t_k+\beta-1} [b_i(m)u_i(m) - d_i(m)x_i(m)], \quad (39)$$

for  $i, j = 1, 2, i \neq j$ , so it derives that

$$\begin{aligned} x_i(n+t_k+\beta) &= x_i(t_k+\beta) \exp \sum_{m=\beta}^{n+\beta-1} \left[ r_i(m+t_k) - a_i(m+t_k) \right. \\ &\quad \times x_i(m+t_k) \\ &\quad - \frac{c_j(m+t_k)x_j(m+t_k)}{1+x_j(m+t_k)} \\ &\quad \left. - e_i(m+t_k)u_i(m+t_k) \right], \\ u_i(n+t_k+\beta) &= u_i(t_k+\beta) - \sum_{m=\beta}^{n+\beta-1} [b_i(m+t_k)u_i(m+t_k) \\ &\quad - d_i(m+t_k)x_i(m+t_k)]. \end{aligned} \quad (40)$$

Letting  $k \rightarrow +\infty$ , one has

$$\begin{aligned} x_i^*(n+\beta) &= x_i^*(\beta) \exp \sum_{m=\beta}^{n+\beta-1} \left[ r_i(m) - a_i(m)x_i^*(m) \right. \\ &\quad - \frac{c_j(m)x_j^*(m)}{1+x_j^*(m)} \\ &\quad \left. - e_i(m)u_i^*(m) \right], \\ u_i^*(n+\beta) &= u_i^*(\beta) - \sum_{m=\beta}^{n+\beta-1} [b_i(m)u_i^*(m) \\ &\quad - d_i(m)x_i^*(m)]. \end{aligned} \quad (41)$$

By the arbitrary of  $\beta$ , we can easily see that  $(x_1^*(n), x_2^*(n), u_1^*(n), u_2^*(n))$  is a solution of system (2) on  $\mathbb{Z}^+$ , and

$$\begin{aligned} 0 &< x_{i*} - \varepsilon \leq x_i^*(n) \leq x_i^* + \varepsilon, \\ 0 &< u_{i*} - \varepsilon \leq u_i^*(n) \leq u_i^* + \varepsilon, \\ \forall n \in \mathbb{Z}^+, \quad i &= 1, 2. \end{aligned} \quad (42)$$

Since  $\varepsilon$  is an arbitrarily small positive number, we obtain that

$$\begin{aligned} 0 &< x_{i*} \leq x_i^*(n) \leq x_i^*, \\ 0 &< u_{i*} \leq u_i^*(n) \leq u_i^*, \\ \forall n \in \mathbb{Z}^+, \quad i &= 1, 2. \end{aligned} \quad (43)$$

So  $\Omega \neq \emptyset$ . This completes the proof.  $\square$

The following theorem concerns the existence and uniformly asymptotical stability of unique positive almost periodic solution of system (2).

**Theorem 8.** Suppose the inequalities in (19) are satisfied; furthermore,  $0 < \gamma < 1$ , where  $\gamma = \min\{s_{ij}, s_{ij}^*\}$  and

$$\begin{aligned} s_{ij} &= 2a_i^L x_{i*} - (a_i^U)^2 (x_i^*)^2 - (d_i^U)^2 (x_i^*)^2 \\ &\quad - \frac{a_i^U c_j^U x_i^* x_j^*}{(1+x_{j*})^2} - \frac{c_j^U x_j^*}{(1+x_{j*})^2} \\ &\quad - a_i^U e_i^U x_i^* - e_i^U - d_i^U (1-b_i^L) x_i^* - \frac{a_j^U c_i^U x_i^* x_j^*}{(1+x_{i*})^2} \\ &\quad - \frac{c_i^U x_i^*}{(1+x_{i*})^2} - \frac{e_j^U c_i^U x_i^*}{(1+x_{i*})^2} - \frac{(c_i^U)^2 (x_i^*)^2}{(1+x_{i*})^4}, \\ s_{ij}^* &= 2b_i^L - a_i^U e_i^U x_i^* - d_i^U (1-b_i^L) x_i^* \\ &\quad - e_i^U - \frac{e_i^U c_j^U x_j^*}{(1+x_{j*})^2} - (e_i^U)^2 - (b_i^U)^2, \end{aligned} \quad (44)$$

$i, j = 1, 2, i \neq j$ , then there exists a unique uniformly asymptotically stable positive almost periodic solution of system (2) which is bounded by  $\Omega$  for all  $n \in \mathbb{Z}^+$ .

*Proof.* We first make the change of variables

$$y_1(n) = \ln x_1(n), \quad y_2(n) = \ln x_2(n), \quad (45)$$

then system (2) can be reformulated as

$$\begin{aligned} y_i(n+1) &= y_i(n) + r_i(n) - a_i(n) e^{y_i(n)} \\ &\quad - \frac{c_j(n) e^{y_j(n)}}{1 + e^{y_j(n)}} - e_i(n) u_i(n), \\ \Delta u_i(n) &= -b_i(n) u_i(n) + d_i(n) e^{y_i(n)}, \\ i &= 1, 2, \quad i \neq j. \end{aligned} \quad (46)$$

By Theorem 7, it is easy to see that there exists a bounded solution  $(y_1(n), y_2(n), u_1(n), u_2(n))$  of system (46) satisfying

$$\begin{aligned} \ln x_{i*} &\leq y_i(n) \leq \ln x_i^*, \quad u_{i*} \leq u_i(n) \leq u_i^*, \\ i &= 1, 2, \quad n \in \mathbb{Z}^+. \end{aligned} \quad (47)$$



Then  $|y_i(n)| \leq A_i$ ,  $|u_i(n)| \leq B_i$ , where  $A_i = \max\{|\ln x_{i*}|, |\ln x_i^*|\}$ ,  $B_i = \max\{|u_{i*}|, |u_i^*|\}$ , and  $i = 1, 2$ . Let  $\|(y_1(n), y_2(n), u_1(n), u_2(n))\| = \sum_{i=1}^2 \{|y_i(n)| + |u_i(n)|\}$ , where  $(y_1(n), y_2(n), u_1(n), u_2(n)) \in \mathbb{R}^4$ .

The following associate product system of system (46) can be expressed as

$$\begin{aligned} y_i(n+1) &= y_i(n) + r_i(n) - a_i(n) e^{y_i(n)} \\ &\quad - \frac{c_j(n) e^{y_j(n)}}{1 + e^{y_j(n)}} - e_i(n) u_i(n), \\ \Delta u_i(n) &= -b_i(n) u_i(n) + d_i(n) e^{y_i(n)}, \\ z_i(n+1) &= z_i(n) + r_i(n) - a_i(n) e^{z_i(n)} \\ &\quad - \frac{c_j(n) e^{z_j(n)}}{1 + e^{z_j(n)}} - e_i(n) v_i(n), \\ \Delta v_i(n) &= -b_i(n) v_i(n) + d_i(n) e^{z_i(n)}, \\ i &= 1, 2, \quad i \neq j. \end{aligned} \quad (48)$$

Suppose that

$$\begin{aligned} Y &= (y_1(n), y_2(n), u_1(n), u_2(n)), \\ Z &= (z_1(n), z_2(n), v_1(n), v_2(n)) \end{aligned} \quad (49)$$

are any two solutions of system (46) defined on  $\mathbb{S}$ , then

$$\|Y\| \leq B, \quad \|Z\| \leq B, \quad (50)$$

where  $B = \sum_{i=1}^2 \{A_i + B_i\}$ , and

$$\begin{aligned} \mathbb{S} &= \{(y_1(n), y_2(n), u_1(n), u_2(n)) \mid \ln x_{i*} \leq y_i(n) \leq \ln x_i^*, \\ &\quad u_{i*} \leq u_i(n) \leq u_i^*, i = 1, 2, n \in \mathbb{Z}^+\}. \end{aligned} \quad (51)$$

Let us construct the following Lyapunov function defined on  $\mathbb{Z}^+ \times \mathbb{S} \times \mathbb{S}$ :

$$V(n, Y, Z) = \sum_{i=1}^2 \{(y_i(n) - z_i(n))^2 + (u_i(n) - v_i(n))^2\}. \quad (52)$$

Obviously,  $\|Y - Z\| = \sum_{i=1}^2 \{|y_i(n) - z_i(n)| + |u_i(n) - v_i(n)|\}$  is equivalent to  $\|Y - Z\|_* = \{\sum_{i=1}^2 [(y_i(n) - z_i(n))^2 + (u_i(n) - v_i(n))^2]\}^{1/2}$ ; that is, there exist two constants  $D_1 > 0$ ,  $D_2 > 0$ , such that

$$D_1 \|Y - Z\| \leq \|Y - Z\|_* \leq D_2 \|Y - Z\|. \quad (53)$$

Consequently,

$$(D_1 \|Y - Z\|)^2 \leq V(n, Y, Z) \leq (D_2 \|Y - Z\|)^2. \quad (54)$$

Denote  $a, b \in C(\mathbb{R}^+, \mathbb{R}^+)$ ,  $a(x) = D_1^2 x^2$ ,  $b(x) = D_2^2 x^2$ , thus the condition (i) of Lemma 4 is satisfied.

In addition, for any  $(n, Y, Z), (n, \bar{Y}, \bar{Z}) \in \mathbb{Z}^+ \times \mathbb{S} \times \mathbb{S}$ , we find that

$$\begin{aligned} &|V(n, Y, Z) - V(n, \bar{Y}, \bar{Z})| \\ &= \left| \sum_{i=1}^2 \{(y_i(n) - z_i(n))^2 + (u_i(n) - v_i(n))^2\} \right. \\ &\quad \left. - \sum_{i=1}^2 \{(\bar{y}_i(n) - \bar{z}_i(n))^2 + (\bar{u}_i(n) - \bar{v}_i(n))^2\} \right| \\ &\leq \sum_{i=1}^2 |(y_i(n) - z_i(n))^2 - (\bar{y}_i(n) - \bar{z}_i(n))^2| \\ &\quad + \sum_{i=1}^2 |(u_i(n) - v_i(n))^2 - (\bar{u}_i(n) - \bar{v}_i(n))^2| \\ &= \sum_{i=1}^2 \{ |(y_i(n) - z_i(n)) + (\bar{y}_i(n) - \bar{z}_i(n))| \\ &\quad \cdot |(y_i(n) - z_i(n)) - (\bar{y}_i(n) - \bar{z}_i(n))| \} \\ &\quad + \sum_{i=1}^2 \{ |(u_i(n) - v_i(n)) + (\bar{u}_i(n) - \bar{v}_i(n))| \\ &\quad \cdot |(u_i(n) - v_i(n)) - (\bar{u}_i(n) - \bar{v}_i(n))| \} \\ &\leq \sum_{i=1}^2 \{ (|y_i(n)| + |z_i(n)| + |\bar{y}_i(n)| + |\bar{z}_i(n)|) \\ &\quad \cdot (|y_i(n) - \bar{y}_i(n)| + |z_i(n) - \bar{z}_i(n)|) \} \\ &\quad + \sum_{i=1}^2 \{ (|u_i(n)| + |v_i(n)| + |\bar{u}_i(n)| + |\bar{v}_i(n)|) \\ &\quad \cdot (|u_i(n) - \bar{u}_i(n)| + |v_i(n) - \bar{v}_i(n)|) \} \\ &\leq \lambda \left\{ \sum_{i=1}^2 \{ |y_i(n) - \bar{y}_i(n)| + |u_i(n) - \bar{u}_i(n)| \} \right. \\ &\quad \left. + \sum_{i=1}^2 \{ |z_i(n) - \bar{z}_i(n)| + |v_i(n) - \bar{v}_i(n)| \} \right\} \\ &= \lambda \{ \|Y - \bar{Y}\| + \|Z - \bar{Z}\| \}, \end{aligned} \quad (55)$$

where  $\bar{Y} = (\bar{y}_1(n), \bar{y}_2(n), \bar{u}_1(n), \bar{u}_2(n))$ ,  $\bar{Z} = (\bar{z}_1(n), \bar{z}_2(n), \bar{v}_1(n), \bar{v}_2(n))$ , and  $\lambda = 4 \max\{A_i, B_i\}$  ( $i = 1, 2$ ). Hence, the condition (ii) of Lemma 4 is satisfied.

At last, we calculate the  $\Delta V(n, Y, Z)$  of  $V(n, Y, Z)$  along the solutions of system (48) and obtain that

$$\begin{aligned} \Delta V_{(48)}(n, Y, Z) \\ = V(n+1, Y, Z) - V(n, Y, Z) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^2 \left\{ (y_i(n+1) - z_i(n+1))^2 + (u_i(n+1) - v_i(n+1))^2 \right\} \\
&\quad - \sum_{i=1}^2 \left\{ (y_i(n) - z_i(n))^2 + (u_i(n) - v_i(n))^2 \right\} \\
&= \sum_{i=1}^2 \left\{ (y_i(n+1) - z_i(n+1))^2 - (y_i(n) - z_i(n))^2 \right. \\
&\quad \left. + (u_i(n+1) - v_i(n+1))^2 - (u_i(n) - v_i(n))^2 \right\} \\
&= \sum_{i=1}^2 \left\{ \left[ (y_i(n) - z_i(n)) - a_i(n) (e^{y_i(n)} - e^{z_i(n)}) \right. \right. \\
&\quad \left. - e_i(n) (u_i(n) - v_i(n)) \right. \\
&\quad \left. - c_j(n) \left( \frac{e^{y_j(n)}}{1 + e^{y_j(n)}} - \frac{e^{z_j(n)}}{1 + e^{z_j(n)}} \right) \right]^2 \\
&\quad - (y_i(n) - z_i(n))^2 + \left[ (1 - b_i(n)) (u_i(n) - v_i(n)) \right. \\
&\quad \left. + d_i(n) (e^{y_i(n)} - e^{z_i(n)}) \right]^2 \\
&\quad \left. - (u_i(n) - v_i(n))^2 \right\} \\
&= \sum_{i=1}^2 \left\{ -2a_i(n) (y_i(n) - z_i(n)) (e^{y_i(n)} - e^{z_i(n)}) \right. \\
&\quad - 2e_i(n) (y_i(n) - z_i(n)) (u_i(n) - v_i(n)) \\
&\quad - 2c_j(n) (y_i(n) - z_i(n)) \left( \frac{e^{y_j(n)}}{1 + e^{y_j(n)}} - \frac{e^{z_j(n)}}{1 + e^{z_j(n)}} \right) \\
&\quad + a_i^2(n) (e^{y_i(n)} - e^{z_i(n)})^2 + 2a_i(n) e_i(n) \\
&\quad \times (u_i(n) - v_i(n)) (e^{y_i(n)} - e^{z_i(n)}) + 2a_i(n) c_j(n) \\
&\quad \times \left( \frac{e^{y_j(n)}}{1 + e^{y_j(n)}} - \frac{e^{z_j(n)}}{1 + e^{z_j(n)}} \right) (e^{y_i(n)} - e^{z_i(n)}) \\
&\quad + e_i^2(n) (u_i(n) - v_i(n))^2 + 2e_i(n) c_j(n) \\
&\quad \times \left( \frac{e^{y_j(n)}}{1 + e^{y_j(n)}} - \frac{e^{z_j(n)}}{1 + e^{z_j(n)}} \right) (u_i(n) - v_i(n)) \\
&\quad + c_j^2(n) \left( \frac{e^{y_j(n)}}{1 + e^{y_j(n)}} - \frac{e^{z_j(n)}}{1 + e^{z_j(n)}} \right)^2 \\
&\quad + b_i(n) (b_i(n) - 2) (u_i(n) - v_i(n))^2 + 2d_i(n) \\
&\quad \times (1 - b_i(n)) (e^{y_i(n)} - e^{z_i(n)}) (u_i(n) - v_i(n)) \\
&\quad \left. + d_i^2(n) (e^{y_i(n)} - e^{z_i(n)})^2 \right\}. \tag{56}
\end{aligned}$$

By the mean-value theorem, one has

$$e^{y_i(n)} - e^{z_i(n)} = \theta_i(n) (y_i(n) - z_i(n)),$$

$$\frac{e^{y_j(n)}}{1 + e^{y_j(n)}} - \frac{e^{z_j(n)}}{1 + e^{z_j(n)}} = \frac{\sigma_j(n)}{(1 + \sigma_j(n))^2} (y_j(n) - z_j(n)), \tag{57}$$

$i, j = 1, 2, \quad i \neq j,$

where  $\theta_i(n)$  and  $\sigma_j(n)$  lie between  $e^{y_i(n)}$  and  $e^{z_i(n)}$  and  $e^{y_j(n)}$  and  $e^{z_j(n)}$ , respectively. Then  $x_{i*} \leq \theta_i(n) \leq x_i^*$ ,  $x_{j*} \leq \sigma_j(n) \leq x_j^*$ ,  $i, j = 1, 2, i \neq j, n \in \mathbb{Z}^+$ . Substituting (57) into (56), one has

$$\begin{aligned}
&\Delta V_{(48)}(n, Y, Z) \\
&= \sum_{i=1}^2 \left\{ -2a_i(n) \theta_i(n) (y_i(n) - z_i(n))^2 \right. \\
&\quad - 2e_i(n) (y_i(n) - z_i(n)) (u_i(n) - v_i(n)) \\
&\quad - \frac{2c_j(n) \sigma_j(n)}{(1 + \sigma_j(n))^2} (y_i(n) - z_i(n)) (y_j(n) - z_j(n)) \\
&\quad + a_i^2(n) \theta_i^2(n) (y_i(n) - z_i(n))^2 \\
&\quad + 2a_i(n) e_i(n) \theta_i(n) (y_i(n) - z_i(n)) (u_i(n) - v_i(n)) \\
&\quad + \frac{2a_i(n) c_j(n) \theta_i(n) \sigma_j(n)}{(1 + \sigma_j(n))^2} (y_i(n) - z_i(n)) \\
&\quad \times (y_j(n) - z_j(n)) + e_i^2(n) (u_i(n) - v_i(n))^2 \\
&\quad + \frac{2e_i(n) c_j(n) \sigma_j(n)}{(1 + \sigma_j(n))^2} (y_j(n) - z_j(n)) \\
&\quad \times (u_i(n) - v_i(n)) + \frac{c_j^2(n) \sigma_j^2(n)}{(1 + \sigma_j(n))^4} (y_j(n) - z_j(n))^2 \\
&\quad + b_i(n) (b_i(n) - 2) (u_i(n) - v_i(n))^2 \\
&\quad + 2d_i(n) (1 - b_i(n)) \theta_i(n) (y_i(n) - z_i(n)) \\
&\quad \times (u_i(n) - v_i(n)) + d_i^2(n) \theta_i^2(n) (y_i(n) - z_i(n))^2 \left. \right\} \\
&= \sum_{i=1}^2 \left\{ (-2a_i(n) \theta_i(n) + a_i^2(n) \theta_i^2(n) + d_i^2(n) \theta_i^2(n)) \right. \\
&\quad \times (y_i(n) - z_i(n))^2 + 2 \left( \frac{a_i(n) c_j(n) \theta_i(n) \sigma_j(n)}{(1 + \sigma_j(n))^2} \right. \\
&\quad \left. \left. - \frac{c_j(n) \sigma_j(n)}{(1 + \sigma_j(n))^2} \right) \right.
\end{aligned}$$



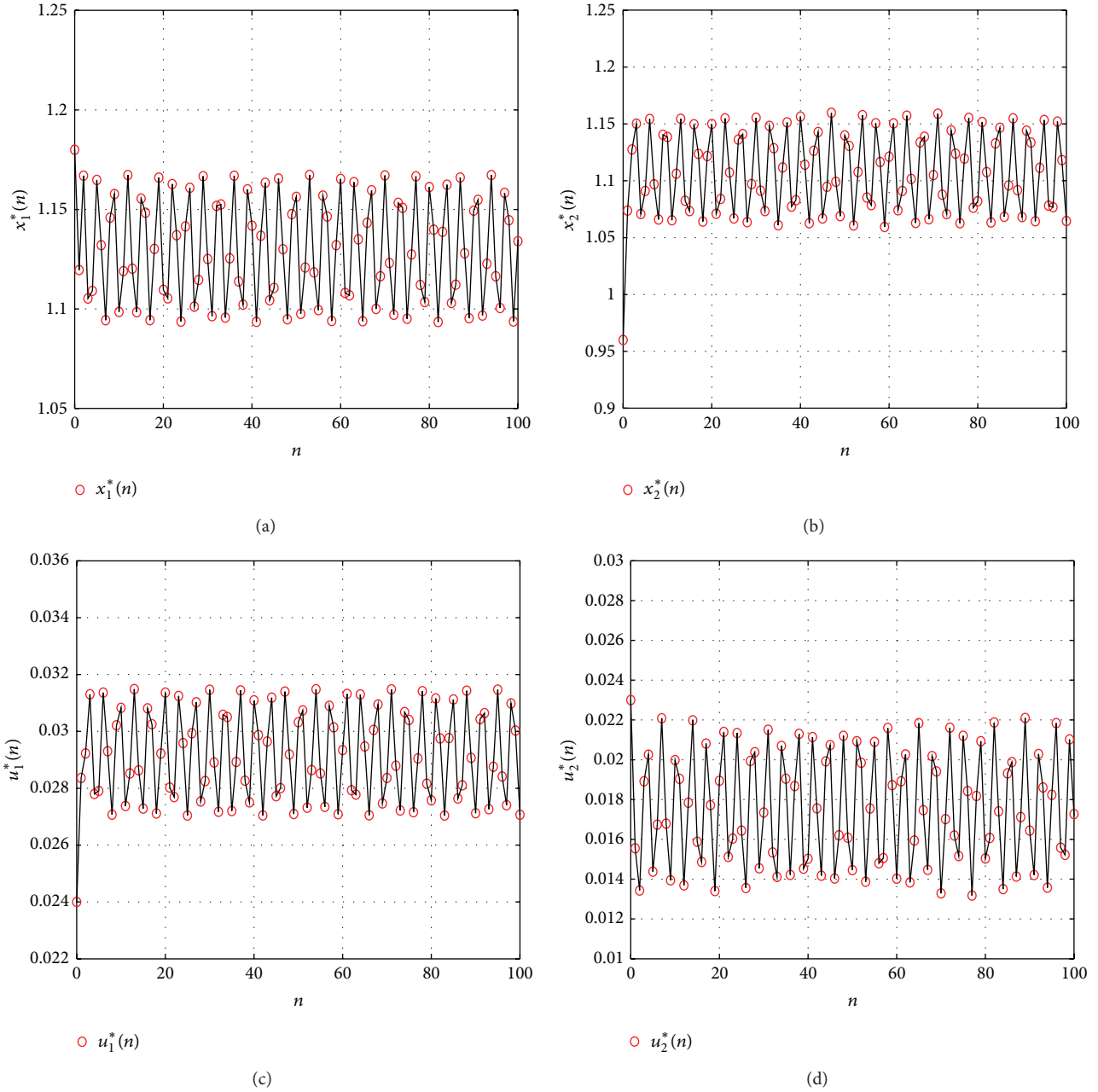


FIGURE 1: Positive almost periodic solution of system (62). (a)–(d) Time-series of  $x_1^*(n)$ ,  $x_2^*(n)$ ,  $u_1^*(n)$ , and  $u_2^*(n)$  with initial values  $x_1^*(0) = 1.18$ ,  $x_2^*(0) = 0.96$ ,  $u_1^*(0) = 0.024$ , and  $u_2^*(0) = 0.023$  for  $n \in [0, 100]$ , respectively.

$$\begin{aligned}
 & \times (y_i(n) - z_i(n)) (y_j(n) - z_j(n)) \\
 & + 2 (a_i(n) e_i(n) \theta_i(n) + d_i(n) (1 - b_i(n)) \\
 & \quad \times \theta_i(n) - e_i(n)) (y_i(n) - z_i(n)) (u_i(n) - v_i(n)) \\
 & + \frac{2e_i(n) c_j(n) \sigma_j(n)}{(1 + \sigma_j(n))^2} (y_j(n) - z_j(n)) \\
 & \times (u_i(n) - v_i(n)) + \frac{c_j^2(n) \sigma_j^2(n)}{(1 + \sigma_j(n))^4} (y_j(n) - z_j(n))^2 \\
 & + \left( e_i^2(n) + b_i(n) (b_i(n) - 2) \right) (u_i(n) - v_i(n))^2 \Big\} \\
 & \leq \sum_{i=1}^2 \left\{ \left( -2a_i(n) \theta_i(n) + a_i^2(n) \theta_i^2(n) + d_i^2(n) \theta_i^2(n) \right) \right. \\
 & \quad \times (y_i(n) - z_i(n))^2 \\
 & \quad \left. + 2 \left( \frac{a_i(n) c_j(n) \theta_i(n) \sigma_j(n)}{(1 + \sigma_j(n))^2} - \frac{c_j(n) \sigma_j(n)}{(1 + \sigma_j(n))^2} \right) \right\}
 \end{aligned}$$

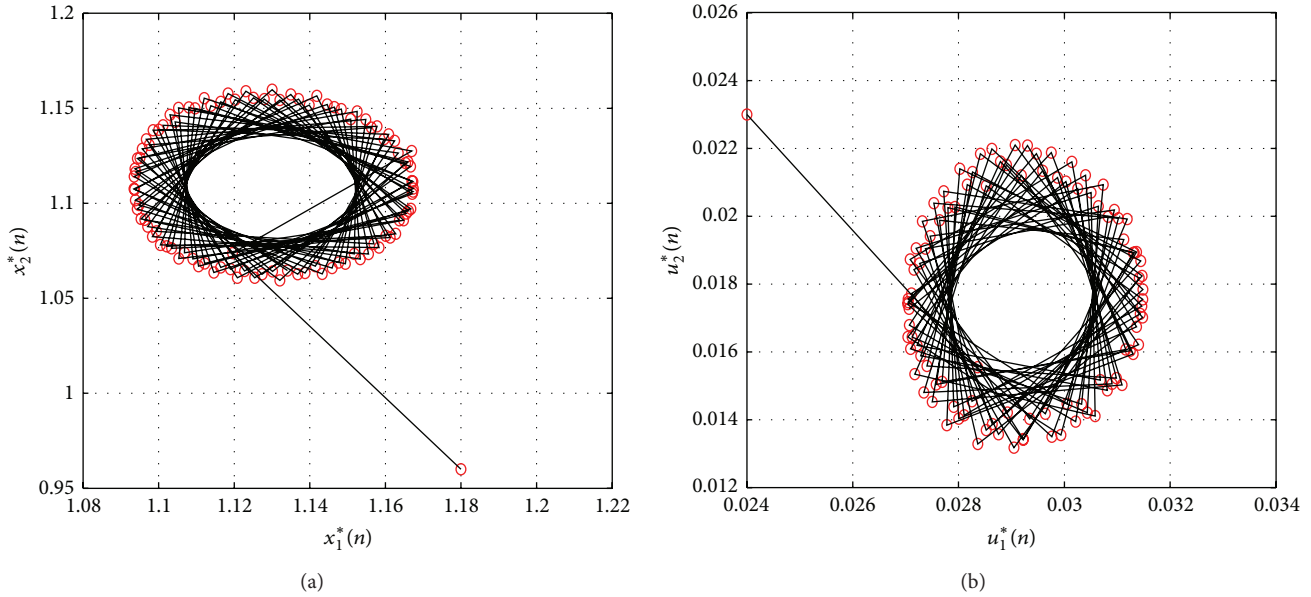


FIGURE 2: 2-dimensional phase portrait. (a) Phase portrait of  $x_1^*(n)$  and  $x_2^*(n)$  with initial values  $x_1^*(0) = 1.18$ ,  $x_2^*(0) = 0.96$  for  $n \in [0, 100]$ , respectively. (b) Phase portrait of  $u_1^*(n)$  and  $u_2^*(n)$  with initial values  $u_1^*(0) = 0.024$ ,  $u_2^*(0) = 0.023$  for  $n \in [0, 100]$ , respectively.

$$\begin{aligned}
 & \times (y_i(n) - z_i(n)) (y_j(n) - z_j(n)) \Big| \\
 & + 2 \left| (a_i(n) e_i(n) \theta_i(n) + d_i(n) (1 - b_i(n)) \theta_i(n) \right. \\
 & \quad \left. - e_i(n) (y_i(n) - z_i(n)) (u_i(n) - v_i(n)) \right| \\
 & + 2 \left| \frac{e_i(n) c_j(n) \sigma_j(n)}{(1 + \sigma_j(n))^2} (u_i(n) - v_i(n)) \right. \\
 & \quad \left. \times (y_j(n) - z_j(n)) \right| + \frac{c_j^2(n) \sigma_j^2(n)}{(1 + \sigma_j(n))^4} \\
 & \times (y_j(n) - z_j(n))^2 + (e_i^2(n) + b_i^2(n) - 2b_i(n)) \\
 & \times (u_i(n) - v_i(n))^2 \Big\}.
 \end{aligned} \tag{58}$$

Now, we set that

$$\begin{aligned}
 \Delta V_{(48)}(n, Y, Z) &= \sum_{i=1}^2 \{V_{1i} + V_{2ij} + V_{3i} + V_{4ij} + V_{5j} + V_{6i}\}, \\
 i, j &= 1, 2, \quad i \neq j,
 \end{aligned} \tag{59}$$

where

$$\begin{aligned}
 V_{1i} &= (-2a_i(n) \theta_i(n) + a_i^2(n) \theta_i^2(n) + d_i^2(n) \theta_i^2(n)) \\
 & \times (y_i(n) - z_i(n))^2
 \end{aligned}$$

$$\begin{aligned}
 & \leq (-2a_i^L x_{i*} + (a_i^U)^2 (x_i^*)^2 + (d_i^U)^2 (x_i^*)^2) \\
 & \times (y_i(n) - z_i(n))^2, \\
 V_{2ij} &= 2 \left| \left( \frac{a_i(n) c_j(n) \theta_i(n) \sigma_j(n)}{(1 + \sigma_j(n))^2} - \frac{c_j(n) \sigma_j(n)}{(1 + \sigma_j(n))^2} \right) \right. \\
 & \quad \left. \times (y_i(n) - z_i(n)) (y_j(n) - z_j(n)) \right| \\
 & \leq \left( \frac{a_i^U c_j^U x_i^* x_j^*}{(1 + x_{j*})^2} + \frac{c_j^U x_j^*}{(1 + x_{j*})^2} \right) \\
 & \times \{(y_i(n) - z_i(n))^2 + (y_j(n) - z_j(n))^2\}, \\
 V_{3i} &= 2 \left| (a_i(n) e_i(n) \theta_i(n) + d_i(n) (1 - b_i(n)) \theta_i(n) - e_i(n) \right. \\
 & \quad \left. \times (y_i(n) - z_i(n)) (u_i(n) - v_i(n)) \right| \\
 & \leq (a_i^U e_i^U x_i^* + d_i^U (1 - b_i^L) x_i^* + e_i^U) \\
 & \times \{(y_i(n) - z_i(n))^2 + (u_i(n) - v_i(n))^2\}, \\
 V_{4ij} &= 2 \left| \frac{e_i(n) c_j(n) \sigma_j(n)}{(1 + \sigma_j(n))^2} (u_i(n) - v_i(n)) (y_j(n) - z_j(n)) \right| \\
 & \leq \frac{e_i^U c_j^U x_j^*}{(1 + x_{j*})^2} \{(u_i(n) - v_i(n))^2 + (y_j(n) - z_j(n))^2\}, \\
 V_{5j} &= \frac{c_j^2(n) \sigma_j^2(n)}{(1 + \sigma_j(n))^4} (y_j(n) - z_j(n))^2
 \end{aligned}$$

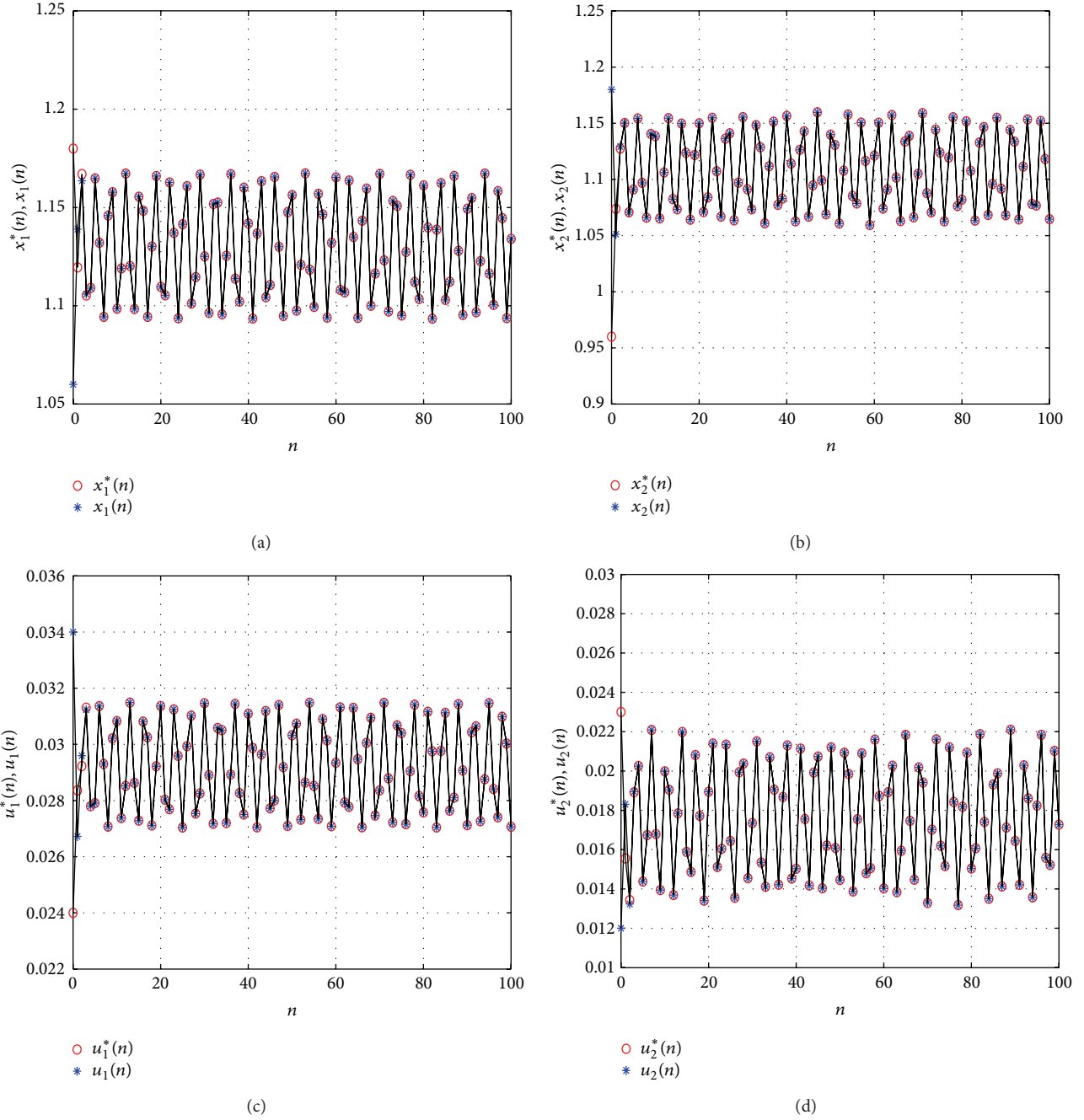


FIGURE 3: Uniformly asymptotic stability. (a)–(d) Time-series of  $x_1^*(n)$ ,  $x_2^*(n)$ ,  $u_1^*(n)$  and  $u_2^*(n)$  with initial values  $x_1^*(0) = 1.18$ ,  $x_2^*(0) = 0.96$ ,  $u_1^*(0) = 0.024$ , and  $u_2^*(0) = 0.023$  and  $x_1(n)$ ,  $x_2(n)$ ,  $u_1(n)$ , and  $u_2(n)$  with initial values  $x_1(0) = 1.06$ ,  $x_2(0) = 1.18$ ,  $u_1(0) = 0.034$ , and  $u_2(0) = 0.012$  for  $n \in [0, 100]$ , respectively.

$$\begin{aligned}
 &\leq \frac{(c_j^U)^2 (x_j^*)^2}{(1 + x_{j*})^4} (y_j(n) - z_j(n))^2, \\
 V_{6i} &= (e_i^2(n) + b_i^2(n) - 2b_i(n)) (u_i(n) - v_i(n))^2 \\
 &\leq ((e_i^U)^2 + (b_i^U)^2 - 2b_i^L) (u_i(n) - v_i(n))^2,
 \end{aligned} \tag{60}$$

which, together with (58), yields that

$$\begin{aligned}
 &\Delta V_{(48)}(n, Y, Z) \\
 &\leq \sum_{i=1}^2 \left\{ \left( -2a_i^L x_{i*} + (a_i^U)^2 (x_i^*)^2 + (d_i^U)^2 (x_i^*)^2 \right. \right. \\
 &\quad \left. \left. + \frac{a_i^U c_j^U x_i^* x_j^*}{(1 + x_{j*})^2} + \frac{c_j^U x_j^*}{(1 + x_{j*})^2} + a_i^U e_i^U x_i^* \right) \right\}
 \end{aligned}$$

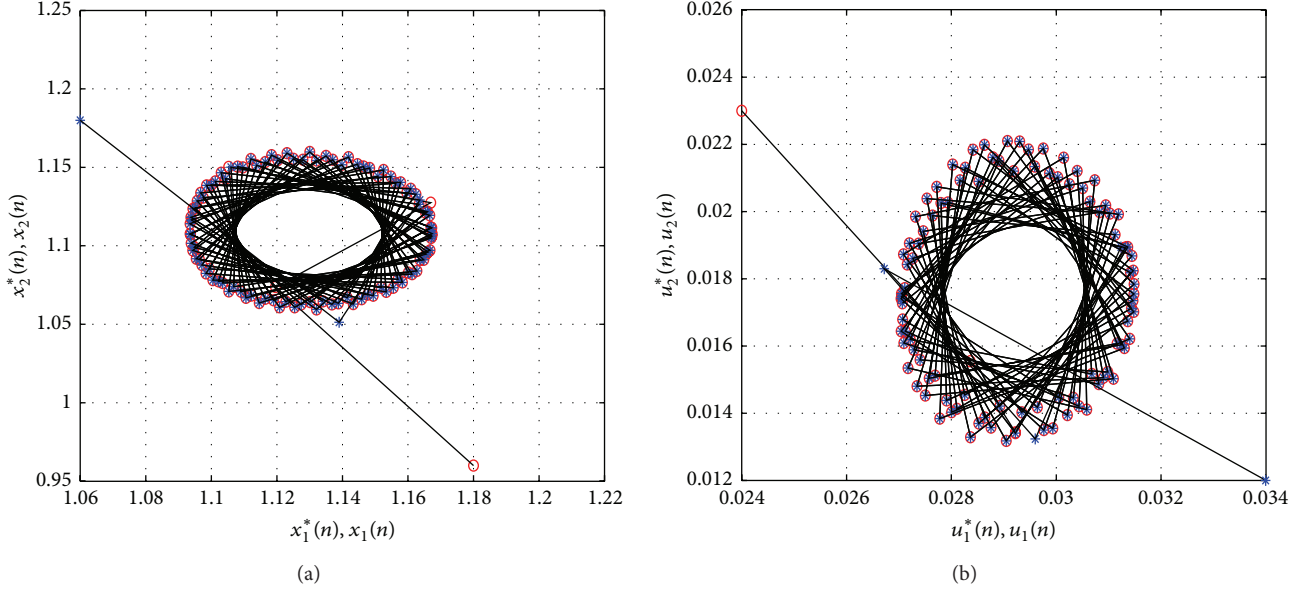


FIGURE 4: 2-dimensional phase portrait. (a) Phase portrait of  $x_1^*(n)$  and  $x_2^*(n)$  with initial values  $x_1^*(0) = 1.18$ ,  $x_2^*(0) = 0.96$  and  $x_1(n)$  and  $x_2(n)$  with initial values  $x_1(0) = 1.06$ ,  $x_2(0) = 1.18$  for  $n \in [0, 100]$ , respectively. (b) Phase portrait of  $u_1^*(n)$  and  $u_2^*(n)$  with initial values  $u_1^*(0) = 0.024$ ,  $u_2^*(0) = 0.023$  and  $u_1(n)$  and  $u_2(n)$  with initial values  $u_1(0) = 0.034$ ,  $u_2(0) = 0.012$  for  $n \in [0, 100]$ , respectively.

$$\begin{aligned}
& + d_i^U (1 - b_i^L) x_i^* + e_i^U \Big) (y_i(n) - z_i(n))^2 \\
& + \left( \frac{a_i^U c_j^U x_i^* x_j^*}{(1 + x_{j*})^2} + \frac{c_j^U x_j^*}{(1 + x_{j*})^2} + \frac{e_i^U c_j^U x_j^*}{(1 + x_{j*})^2} \right. \\
& \quad \left. + \frac{(c_j^U)^2 (x_j^*)^2}{(1 + x_{j*})^4} \right) (y_j(n) - z_j(n))^2 \\
& + \left( a_i^U e_i^U x_i^* + d_i^U (1 - b_i^L) x_i^* + e_i^U + \frac{e_i^U c_j^U x_j^*}{(1 + x_{j*})^2} \right. \\
& \quad \left. + (e_i^U)^2 + (b_i^U)^2 - 2b_i^L \right) (u_i(n) - v_i(n))^2 \Big\} \\
& = \sum_{i=1}^2 \left\{ \left( -2a_i^L x_{i*} + (a_i^U)^2 (x_i^*)^2 + (d_i^U)^2 (x_i^*)^2 \right. \right. \\
& \quad + \frac{a_i^U c_j^U x_i^* x_j^*}{(1 + x_{j*})^2} + \frac{c_j^U x_j^*}{(1 + x_{j*})^2} + a_i^U e_i^U x_i^* \\
& \quad \left. + d_i^U (1 - b_i^L) x_i^* + e_i^U \right) (y_i(n) - z_i(n))^2 \\
& \quad + \left( \frac{a_j^U c_i^U x_i^* x_j^*}{(1 + x_{i*})^2} + \frac{c_i^U x_i^*}{(1 + x_{i*})^2} + \frac{e_j^U c_i^U x_i^*}{(1 + x_{i*})^2} \right. \\
& \quad \left. + \frac{(c_i^U)^2 (x_i^*)^2}{(1 + x_{i*})^4} \right) (y_i(n) - z_i(n))^2 \\
& \quad + \left( 2a_i^L x_{i*} - (a_i^U)^2 (x_i^*)^2 - (d_i^U)^2 (x_i^*)^2 \right. \\
& \quad - \frac{a_i^U c_j^U x_i^* x_j^*}{(1 + x_{j*})^2} - \frac{c_j^U x_j^*}{(1 + x_{j*})^2} - a_i^U e_i^U x_i^* \\
& \quad - d_i^U (1 - b_i^L) x_i^* - e_i^U - \frac{a_j^U c_i^U x_i^* x_j^*}{(1 + x_{i*})^2} \\
& \quad - \frac{c_i^U x_i^*}{(1 + x_{i*})^2} - \frac{e_j^U c_i^U x_i^*}{(1 + x_{i*})^2} - \frac{(c_i^U)^2 (x_i^*)^2}{(1 + x_{i*})^4} \Big) \\
& \quad \times (y_i(n) - z_i(n))^2 \\
& \quad + \left( 2b_i^L - a_i^U e_i^U x_i^* - d_i^U (1 - b_i^L) x_i^* - e_i^U \right. \\
& \quad \left. - \frac{e_i^U c_j^U x_j^*}{(1 + x_{j*})^2} - (e_i^U)^2 - (b_i^U)^2 \right) \\
& \quad \times (u_i(n) - v_i(n))^2 \Big\}
\end{aligned}$$

$$\begin{aligned}
&= -\sum_{i=1}^2 \{s_{ij}(y_i(n) - z_i(n))^2 + s_{ij}^*(u_i(n) - v_i(n))^2\} \\
&\leq -\gamma \sum_{i=1}^2 \{(y_i(n) - z_i(n))^2 + (u_i(n) - v_i(n))^2\} \\
&= -\gamma V(n),
\end{aligned} \tag{61}$$

where  $\gamma = \min\{s_{ij}, s_{ij}^*\}$ ,  $i, j = 1, 2, i \neq j$ . It follows from the conditions of Theorem 8 that we have  $0 < \gamma < 1$ , then the condition (iii) of Lemma 4 is satisfied. Therefore, we have from Lemma 4 that there exists a unique uniformly asymptotically stable almost periodic solution  $(y_1^*(n), y_2^*(n), u_1^*(n), u_2^*(n))$  of system (46) which is bounded by  $\mathbb{S}$  for all  $n \in \mathbb{Z}^+$ , which implies that there exists a unique uniformly asymptotically stable positive almost periodic solution  $(x_1^*(n), x_2^*(n), u_1^*(n), u_2^*(n))$  of system (2) which is bounded by  $\Omega$  for all  $n \in \mathbb{Z}^+$ . This completed the proof.  $\square$

**Remark 9.** If we neglect the role of feedback controls, that is,  $b_i(n) = 0$ ,  $d_i(n) = 0$  and  $e_i(n) = 0$ ,  $i = 1, 2$ , then system (2) can be reduced to system (1). Propositions 5 and 6 and Theorem 8 can come down to our corresponding main results (see [1]). The fact shows that the feedback controls have influence on the existence and uniformly asymptotic stability of unique positive almost solution of system (2).

#### 4. Numerical Simulations

In this section we give a numerical example in support to our analytical findings.

**Example 10.** Consider the following discrete system with feedback controls:

$$\begin{aligned}
x_1(n+1) &= x_1(n) \exp \left[ \begin{aligned} &1.20 - 0.02 \sin(\sqrt{2}n\pi) \\ &- (1.05 + 0.01 \sin(\sqrt{2}n\pi)) x_1(n) \\ &- \frac{(0.025 + 0.002 \cos(\sqrt{2}n\pi)) x_2(n)}{1 + x_2(n)} \\ &- (0.015 + 0.001 \sin(\sqrt{3}n\pi)) u_1(n) \end{aligned} \right], \\
x_2(n+1) &= x_2(n) \exp \left[ \begin{aligned} &1.15 - 0.02 \cos(\sqrt{2}n\pi) \\ &- (1.02 + 0.02 \cos(\sqrt{2}n\pi)) x_2(n) \\ &- \frac{(0.035 + 0.005 \sin(\sqrt{3}n\pi)) x_1(n)}{1 + x_1(n)} \\ &- (0.025 + 0.003 \cos(\sqrt{3}n\pi)) u_2(n) \end{aligned} \right],
\end{aligned}$$

$$\begin{aligned}
\Delta u_1(n) &= - (0.93 - 0.03 \cos(\sqrt{2}n\pi)) u_1(n) \\
&\quad + (0.024 - 0.002 \cos(\sqrt{2}n\pi)) x_1(n), \\
\Delta u_2(n) &= - (0.95 - 0.03 \sin(\sqrt{3}n\pi)) u_2(n) \\
&\quad + (0.015 + 0.003 \sin(\sqrt{2}n\pi)) x_2(n).
\end{aligned} \tag{62}$$

A simple computation shows that

$$\begin{aligned}
x_1^* &\approx 1.1982, & x_2^* &\approx 1.1853, \\
u_1^* &\approx 0.0346, & u_2^* &\approx 0.0232, \\
x_{1*} &\approx 0.9665, & x_{2*} &\approx 0.9076, \\
u_{1*} &\approx 0.0221, & u_{2*} &\approx 0.0111, \\
r_1^L - e_1^U u_1^* - c_2^U &\approx 1.1524 > 0, \\
r_2^L - e_2^U u_2^* - c_1^U &\approx 1.0894 > 0.
\end{aligned} \tag{63}$$

Obviously, the assumptions in (19) are satisfied, and moreover, one has

$$\begin{aligned}
s_{12} &\approx 0.3086, & s_{21} &\approx 0.1831, \\
s_{12}^* &\approx 0.8386, & s_{21}^* &\approx 0.8142,
\end{aligned} \tag{64}$$

that is,  $0 < \gamma = \min\{s_{12}, s_{21}, s_{12}^*, s_{21}^*\} \approx 0.1831 < 1$ , so the assumptions of Theorem 8 are satisfied. Thus, there exists a unique uniformly asymptotically stable positive almost periodic solution of system (62). From Figure 1, we can easily see that system (62) exists a positive almost periodic solution  $(x_1^*(t), x_2^*(t), u_1^*(n), u_2^*(n))$ , and the 2-dimensional phase portraits of almost periodic system (62) are displayed in Figure 2, respectively. Figure 3 shows that any positive solution  $(x_1(n), x_2(n), u_1(n), u_2(n))$  tends to the pervious almost periodic solution  $(x_1^*(n), x_2^*(n), u_1^*(n), u_2^*(n))$ ; furthermore, the 2-dimensional phase portraits reflect the fact in Figure 4.

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