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Research Article

On Partially Trace Distance Preserving Maps and Reversible Quantum Channels

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We give a characterization of trace-preserving and positive linear maps preserving trace distance partially, that is, preservers of trace distance of quantum states or pure states rather than all matrices. Applying such results, we give a characterization of quantum channels leaving Helstrom's measure of distinguishability of quantum states or pure states invariant and show that such quantum channels are fully reversible, which are unitary transformations.

1. Introduction

Linear preserver problems concern the characterization of linear maps on matrix spaces that leave certain functions, subsets, relations, and so forth, invariant, and have been an active research area in matrix theory. The earliest paper on linear preserver problems dates back to the year of 1897, and a great deal of effort has been devoted to the study of this type of questions (see [1–4] and their references). In 1975, Choi in [4] gave a characterization of completely positive linear maps on matrix algebras. In the theory of quantum information, a quantum channel just be a completely positive and trace preserving linear map. So Choi gave a mathematical characterization of quantum channels. Such a result is applied in quantum information extensively. In recent years, more and more researchers on linear preserver problems pay their attention to the theory of quantum information (see [5–9] and their references). In this paper, we will give a characterization of trace-preserving and positive linear maps preserving trace distance partially, that is, preservers of trace distance of quantum states or pure states rather than all matrices (see Theorems 6 and 10). Applying such results, we give a characterization of quantum channels leaving Helstrom's measure of distinguishability of quantum states or pure states invariant and show that such quantum channels are fully reversible, which are unitary transformations (see Theorems 7 and 11).

In the mathematical framework of quantum information, quantum states are positive operators with trace 1 on complex Hilbert space H, and we denote by $\mathcal{S}(H)$ the set of all quantum states on H, which is a convex subset of the space of trace-class operators $\mathcal{T}(H)$. Pure states are rank one projections. If $\dim H = n < \infty$, then $\mathcal{T}(H)$ is identical with $\mathcal{B}(H)$, that is, the $n \times n$ complex matrix algebra. In the case of $\dim H = n < \infty$, a quantum channel $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ has the following form:

$$\Phi(\rho) = \sum_{i=1}^{m} M_i \rho M_i^*, \quad (1)$$

where $M_i s \in \mathcal{B}(H)$ and $\sum_{i=1}^{m} M_i^* M_i = I$ with the identity I. For a quantum channel Φ, the aim of quantum error correction is to find another quantum channel Ψ such that

$$\Psi \circ \Phi(\rho) = \rho, \quad \text{for } \rho \in \mathcal{C} \subseteq \mathcal{S}(H). \quad (2)$$

Here, we call that Φ is reversible for the state ρ; if Φ is reversible for all states $\rho \in \mathcal{C}$, then Φ is reversible on $\mathcal{C}$. $\mathcal{C}$ is a subset of $\mathcal{S}(H)$ and is called error correction code. It is easy to check that if Φ is reversible for ρ, σ, then Φ is reversible for the arbitrary convex combination of ρ, σ. So, if $\mathcal{C}$ is an error correction code, we can assume that $\mathcal{C}$ is a convex set. The topic of quantum error correction or reversibility of quantum channels naturally arises in the analysis of questions in quantum information and plays an important role in the theory of quantum information (see [10–13] and their references). In the theory of reversibility of quantum channels, Helstrom's measure of distinguishability of quantum states plays an important role and is defined as follows.

Definition 1. Helstrom's measure of distinguishability of quantum states ρ, σ with respect to $\lambda \in [0,1]$ is defined by $P_H(\rho,\sigma,\lambda) = (1/2)(1 + \|\lambda\rho - (1-\lambda)\sigma\|_1)$.

Definition 2. A map $\Phi : \mathscr{B}(H) \rightarrow \mathscr{B}(H)$ preserves Helstrom's measure of distinguishability of quantum states ρ, σ if $P_H(\rho,\sigma,\lambda) = P_H(\Phi(\rho),\Phi(\sigma),\lambda)$; equivalently, $\|\lambda\Phi(\rho) - (1-\lambda)\Phi(\sigma)\|_1 = \|\lambda\rho - (1-\lambda)\sigma\|_1$ for arbitrary $\lambda \in [0,1]$.

In [10], Robin Blume-Kohout et al. discussed the trace norm $\|\cdot\|_1$ (defined by $\|A\|_1 = \mathrm{tr}(\sqrt{A^*A})$) and reversibility of quantum channels, where Helstrom's measure of distinguishability of quantum states is used as a criterion for reversibility of quantum channels. Robin Blume-Kohout et al., showed that for a quantum channel Φ, with the following additional condition:

$$\begin{array}{l}\text{the projection to the joint support of}\\ \text{all states in } \mathscr{C} \text{ is onto,}\end{array} \qquad (\dagger)$$

Φ preserves Helstrom's measure of distinguishability of all $\rho,\sigma \in \mathscr{C}$ if and only if Φ is reversible on $\mathscr{C}$. In this paper, we prove that, in the case of $\mathscr{C} = \mathscr{S}(H)$, that is, a quantum channel Φ preserves Helstrom's measure of distinguishability of all states $\rho,\sigma \in \mathscr{S}(H)$ if and only if Φ is reversible on $\mathscr{S}(H)$ (see Theorem 7). We call such a channel the fully reversible channel as follows.

Definition 3. A quantum channel $\Phi : \mathscr{B}(H) \rightarrow \mathscr{B}(H)$ is fully reversible if there exists a quantum channel Ψ such that $\Psi \circ \Phi(\rho) = \rho$ for all quantum states ρ.

In Definition 2, taking $\lambda = 1/2$, one can define the map preserving trace distance of quantum states as follows.

Definition 4. A map $\Phi : \mathscr{B}(H) \rightarrow \mathscr{B}(H)$ preserves trace distance of quantum states if for all $\rho,\sigma \in \mathscr{S}(H)$, $\|\Phi(\rho) - \Phi(\sigma)\|_1 = \|\rho - \sigma\|_1$.

The map preserving Helstrom's measure of distinguishability must preserve trace distance. Robin Blume-Kohout et al. in [10] showed that with the assumption ($\dagger$), the channel preserving Helstrom's measure of distinguishability is reversible for an error correction code $\mathscr{C}$, but the channel preserving trace distance of quantum states is not. In this paper, we will show that the channel preserving trace distance of all quantum states is also fully reversible (see Theorem 7). Indeed a fully reversible channel Φ is a unitary transformation; that is, there exists a unitary operator U such that $\Phi(\rho) = U\rho U^*$ for all quantum states ρ. Also many authors pay their attention to characterizing preservers of trace distance (see [14, 15] and their references). Let $\mathscr{P}_1(H)$ be the set of all pure states; furthermore, we introduce the following general partial preservers of trace distance of pure states.

Definition 5. A map $\Phi : \mathscr{B}(H) \rightarrow \mathscr{B}(H)$ preserves trace distance of pure states if for all $P,Q \in \mathscr{P}_1(H)$, $\|\Phi(P) - \Phi(Q)\|_1 = \|P - Q\|_1$.

The map preserving trace distance of quantum states must preserve trace distance of pure states. We also give a characterization of trace preserving and positive linear maps preserving trace distance of pure states and show that the channel preserving trace distance of all pure states is also fully reversible (see Theorems 10 and 11).

2. Partially Trace Distance Preservers and Fully Reversible Channels

In this section, we are first devoted to characterizing a class of positive and trace-preserving linear maps preserving trace distance of quantum states.

Theorem 6. *Let H be a finite dimensional complex Hilbert space with $\dim H = n$, $\Phi : \mathscr{B}(H) \rightarrow \mathscr{B}(H)$ being a positive and trace-preserving linear map; then the following statements are equivalent:*

(I) *Φ preserves trace distance of quantum states; that is, $\|\Phi(\rho) - \Phi(\sigma)\|_1 = \|\rho - \sigma\|_1$ for all $\rho,\sigma \in \mathscr{S}(H)$;*

(II) *there exists a unitary operator U on H such that $\Phi(\rho) = U\rho U^*$ for all states $\rho \in \mathscr{S}(H)$ or $\Phi(\rho) = U\rho^t U^*$ for all states $\rho \in \mathscr{S}(H)$, where ρ^t is the transpose of ρ with respect to an orthonormal basis.*

Applying Theorem 6, we will have the following main result.

Theorem 7. *Let H be a finite dimensional complex Hilbert space with $\dim H = n$, $\Phi : \mathscr{B}(H) \rightarrow \mathscr{B}(H)$ being a quantum channel, that is, a completely positive and trace-preserving linear map; then the following statements are equivalent:*

(I) *Φ is fully reversible;*

(II) *Φ preserves Helstrom's measure of distinguishability of quantum states, that is, $\|\lambda\Phi(\rho) - (1-\lambda)\Phi(\sigma)\|_1 = \|\lambda\rho - (1-\lambda)\sigma\|_1$ for all $\rho,\sigma \in \mathscr{S}(H)$ and arbitrary $\lambda \in [0,1]$;*

(III) *Φ preserves trace distance of quantum states; that is, $\|\Phi(\rho) - \Phi(\sigma)\|_1 = \|\rho - \sigma\|_1$ for all $\rho,\sigma \in \mathscr{S}(H)$;*

(IV) *Φ is a unitary transformation; that is, there exists a unitary operator U on H such that $\Phi(\rho) = U\rho U^*$ for all input states ρ.*

Remark 8. Theorem 7 shows that quantum channels leaving Helstrom's measure of distinguishability or trace distance of all quantum states invariant is fully reversible and vice versa. Also we prove that a quantum channel is fully reversible if and only if it is a unitary transformation. Indeed, such a result also can be induced by some other characterization of preservers in existence. Here, let us give a short proof; if a quantum channel is fully reversible, by [11], then the channel preserves the relative entropy of all quantum states. Such a map has been characterized in [16], where Molnár showed that the map preserving relative entropy of all quantum states has the following form: $A \mapsto UAU^*$, where U is a unitary or antiunitary operator. Applying the result in [16], it follows from linearity and complete positivity of quantum channels that fully reversible quantum channels are unitary transformations.

Before the proof of Theorems 6 and 7, we first give some primary observations. It is mentioned that a quantum state ρ is a pure state if and only if rank $\rho = 1$ if and only if $\rho^2 = \rho$; that is, pure states are rank one projections.

Lemma 9. *For a quantum state $\rho \in \mathcal{S}(H)$ with $\dim H = n < \infty$, ρ is a pure state if and only if there exist $n-1$ pairwise orthogonal states $\rho_1, \ldots, \rho_{n-1}$ such that $\rho\rho_i = 0$ for all $1 \le i \le n-1$.*

Proof of Lemma 9. If ρ is a pure state, then rank $\rho = 1$; one can find $n-1$ pairwise orthogonal pure states $\rho_1, \ldots, \rho_{n-1}$ such that $\rho\rho_i = 0$ for all $1 \le i \le n-1$ since $\dim H = n$.

Conversely, assume on the contrary that rank $\rho \ge 2$; then there exist $x, y \in \operatorname{Ran}\rho$ and $x \perp y$. If there exist $n-1$ pairwise orthogonal states $\rho_1, \ldots, \rho_{n-1}$ such that $\rho\rho_i = 0$ for all $1 \le i \le n-1$, then there exist $n-1$ vectors $e_i \in \operatorname{Ran}\rho_i$ for each ρ_i being pairwise orthogonal nonzero vectors since $\rho_1, \ldots, \rho_{n-1}$ are positive and pairwise orthogonal, but $e_i \perp [x, y]$ for all $1 \le i \le n-1$ since $\rho\rho_i = 0$ for all $1 \le i \le n-1$. It follows that $\dim H > n$. This is a contradiction to $\dim H = n$. The proof is completed. □

Now we prove Theorem 6.

Proof of Theorem 6. (II)⇒(I) is clear; we only need to check that (I)⇒(II).

The proof is divided into the following claims.

Claim 1. $\Phi(\mathcal{S}(H)) \subseteq \mathcal{S}(H)$, and Φ on $\mathcal{S}(H)$ is injective.

Since Φ is trace-preserving and positive, we have that $\Phi(\mathcal{S}(H)) \subseteq \mathcal{S}(H)$. Since Φ on $\mathcal{S}(H)$ is trace-distance-preserving, so we have that Φ on $\mathcal{S}(H)$ is injective.

Claim 2. Φ on $\mathcal{S}(H)$ preserves orthogonality; that is, for quantum states ρ, σ, $\rho\sigma = 0 \Rightarrow \Phi(\rho)\Phi(\sigma) = 0$.

A well-known proposition is that for arbitrary $A, B \in \mathcal{B}(H)$, $A^*B = B^*A = 0$ if and only if $\|A+B\|_1 + \|A-B\|_1 = 2(\|A\|_1 + \|B\|_1)$ (see [17]). For quantum states ρ, σ, one can check that

$$\|\rho\|_1 = \operatorname{tr}(\rho) = 1, \qquad \|\rho+\sigma\|_1 = \operatorname{tr}(\rho+\sigma) = 2. \tag{3}$$

Since quantum states are positive, so $\rho\sigma = 0$ if and only if $\|\rho-\sigma\|_1 = 2$. Since $\|\Phi(\rho)-\Phi(\sigma)\|_1 = \|\rho-\sigma\|_1$, so $\|\rho-\sigma\|_1 = 2 \Rightarrow \|\Phi(\rho)-\Phi(\sigma)\|_1 = 2$; that is, $\rho\sigma = 0 \Rightarrow \Phi(\rho)\Phi(\sigma) = 0$.

Claim 3. Φ maps pure states to pure states.

If the quantum state ρ is a pure state, that is, a rank one projection, by Lemma 9, then there exist $n-1$ pairwise orthogonal states $\rho_1, \ldots, \rho_{n-1}$ such that $\rho_i\rho = 0$ for all $1 \le i \le n-1$. It follows from Claims 1 and 2 that $\Phi(\rho_1), \ldots, \Phi(\rho_{n-1})$ are states, pairwise orthogonal, and $\Phi(\rho_i)\Phi(\rho) = 0$ for all $1 \le i \le n-1$. So by Lemma 9 again, $\Phi(\rho)$ is a rank one projection, that is, a pure state.

Claim 4. $\Phi(\rho) = U\rho U^*$ for arbitrary quantum state $\rho \in \mathcal{S}(H)$ or $\Phi(\rho) = U\rho^t U^*$ for arbitrary quantum state $\rho \in \mathcal{S}(H)$, where U is a unitary operator on H.

Now we have that Φ is linear and maps pure states to pure states; such a map had been characterized by Friedland et al. in [8] (see [8, Lemma 2.4]), where authors show that Φ has one of the following forms:

(I) there is a pure state R such that $\Phi(A) = \operatorname{tr}(A)R$ for all $A \in \mathcal{B}(H)$;

(II) there is a unitary operator U on H such that $\Phi(A) = UAU^*$ for all $A \in \mathcal{B}(H)$ or $\Phi(A) = UA^tU^*$ for all $A \in \mathcal{B}(H)$, where A^t is the transpose of A with respect to an orthonormal basis.

Since Φ preserves orthogonality of rank one projections, so the case (I) does not occur. Therefore, this claim holds true. The proof is completed. □

Next we will give the proof of Theorem 7.

Proof of Theorem 7. (II)⇒(III) and (IV)⇒(I) are clear; we only need to check that (I)⇒(II) and (III)⇒(IV).

First we check that (I)⇒(II). If Φ is reversible for all quantum states, assume that there exists a trace-preserving and completely positive linear map Ψ such that $\Psi \circ \Phi(\rho) = \rho$ for any state ρ. Since positive and trace-preserving linear maps are contractive under the trace norm (see [18]), that is, $\|\Delta(A)\|_1 \le \|A\|_1$ for trace-preserving and positive linear maps Δ and $A \in \mathcal{B}(H)$, so we have that for all $\rho, \sigma \in \mathcal{S}(H)$ and arbitrary $\lambda \in [0,1]$,

$$\begin{aligned} \|\lambda\rho - (1-\lambda)\sigma\|_1 &= \|\lambda\Psi\circ\Phi(\rho) - (1-\lambda)\Psi\circ\Phi(\sigma)\|_1 \\ &\le \|\Psi(\lambda\Phi(\rho) - (1-\lambda)\Phi(\sigma))\|_1 \\ &\le \|\lambda\Phi(\rho) - (1-\lambda)\Phi(\sigma)\|_1 \\ &\le \|\Phi(\lambda\rho - (1-\lambda)\sigma)\|_1 \\ &\le \|\lambda\rho - (1-\lambda)\sigma\|_1. \end{aligned} \tag{4}$$

So $\|\lambda\Phi(\rho) - (1-\lambda)\Phi(\sigma)\|_1 = \|\lambda\rho - (1-\lambda)\sigma\|_1$; (II) holds.

Next we show that (III) ⇒ (IV). Since the channel Φ satisfies the assumptions in Theorem 6, so there exists a unitary operator U on H such that (1) $\Phi(\rho) = U\rho U^*$ for all input states ρ or (2) $\Phi(\rho) = U\rho^tU^*$ for all input states ρ, where ρ^t is the transpose of ρ with respect to an orthonormal basis. One can note that the transpose map is positive but is not completely positive (see [9]), so for a quantum channel Φ, case (2) does not occur. So (IV) holds true. The proof is completed. □

3. Characterizing Channels Reversible for Pure States

In the section, we first give a characterization of positive and trace-preserving linear maps preserving trace distance of pure states.

Theorem 10. *Let H be a finite dimensional complex Hilbert space with $\dim H = n$, $\Phi : \mathscr{B}(H) \to \mathscr{B}(H)$ being a positive and trace-preserving linear map; then the following statements are equivalent:*

(I) Φ *preserves trace distance of pure states; that is, $\|\Phi(P) - \Phi(Q)\|_1 = \|P - Q\|_1$ for all $P, Q \in P_1(H)$;*

(II) *there exists a unitary operator U on H such that $\Phi(\rho) = U\rho U^*$ for all states $\rho \in \mathcal{S}(H)$ or $\Phi(\rho) = U\rho^t U^*$ for all states $\rho \in \mathcal{S}(H)$, where ρ^t is the transpose of ρ with respect to an orthonormal basis.*

Applying Theorem 10, we will have the following refined result.

Theorem 11. *Let H be a finite dimensional complex Hilbert space with $\dim H = n$, $\Phi : \mathscr{B}(H) \to \mathscr{B}(H)$ being a quantum channel, that is, a completely positive and trace-preserving linear map; then the following statements are equivalent:*

(I) Φ *is fully reversible;*

(II) Φ *is reversible on the set of pure states;*

(III) Φ *preserves Helstrom's measure of distinguishability of pure states; that is, $\|\lambda\Phi(P) - (1-\lambda)\Phi(Q)\|_1 = \|\lambda P - (1-\lambda)Q\|_1$ for all $P, Q \in \mathscr{P}_1(H)$ and arbitrary $\lambda \in [0, 1]$;*

(IV) Φ *preserves trace distance of pure states; that is, $\|\Phi(P) - \Phi(Q)\|_1 = \|P - Q\|_1$ for all $P, Q \in \mathscr{P}_1(H)$;*

(V) Φ *is a unitary transformation; that is, there exists a unitary operator U on H such that $\Phi(\rho) = U\rho U^*$ for all input states ρ.*

Now we prove Theorem 10.

Proof of Theorem 10. (II)⇒(I) is clear; we only need to check that (I)⇒(II).

Similar to the proof of Theorem 6, we can show that $\Phi(\mathcal{S}(H)) \subseteq \mathcal{S}(H)$ and Φ on $\mathscr{P}_1(H)$ is injective, so $\Phi(P) \neq 0$ for any pure state P. And Φ preserves orthogonality of pure states; that is, for pure states P, Q, $PQ = 0 \Rightarrow \Phi(P)\Phi(Q) = 0$.

Next we show that $\Phi(P)$ is a pure state for any pure state P. For any pure state P, by Lemma 9, then there exist $n-1$ pairwise orthogonal nonzero pure states $P_1, \ldots, P_{n-1}$ such that $P_i P = 0$ for all $1 \le i \le n-1$. Since Φ preserves orthogonality of pure states and is injective on pure states, it follows that $\Phi(P_1), \ldots, \Phi(P_{n-1})$ are states, pairwise orthogonal, and $\Phi(P_i)\Phi(P) = 0$ for all $1 \le i \le n-1$. So by Lemma 9 again, $\Phi(P)$ is a rank one projection.

Now we have that Φ is linear and maps pure states to pure states; similar to the proof of Theorem 6, one can complete the proof. □

Next we will give the proof of Theorem 11.

Proof of Theorem 11. (I)⇒(II), (III)⇒(IV), and (V)⇒(I) are clear; we only need to check that (II)⇒(III) and (IV)⇒(V). One can check that (II)⇒(III) holds true similar to the proof of (I)⇒(II) in Theorem 7.

Next we show that (IV)⇒(V). Since the channel Φ satisfies the assumptions in Theorem 10, so there exists a unitary operator U on H such that (1) $\Phi(\rho) = U\rho U^*$ for all input states ρ or (2) $\Phi(\rho) = U\rho^t U^*$ for all input states ρ, where ρ^t is the transpose of ρ with respect to an orthonormal basis. Since the transpose map is positive, but is not completely positive (see [9]), so for a quantum channel Φ, the case (2) does not occur. So (IV) holds true. The proof is completed. □

Remark 12. However, it is also mentioned that if the error correction code $\mathscr{C}$ is the proper convex subset of $\mathcal{S}(H)$, that is, $\mathcal{S}(H) \setminus \mathscr{C} \neq \emptyset$, the channel preserving trace distance of quantum states in $\mathscr{C}$ may not be reversible on $\mathscr{C}$ (see [10, Example 7]). Also a natural question is to give a characterization of the channel on a proper subset of $\mathcal{S}(H)$ preserving trace distance, and such a result may help us understand more deeply the theory of reversibility of quantum channels.

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