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Research Article

New Optimality Conditions for a Nondifferentiable Fractional Semipreinvex Programming Problem

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We study a nondifferentiable fractional programming problem as follows: $(P)\min_{x\in K} f(x)/g(x)$ subject to $x \in K \subseteq X$, $h_i(x) \leq 0$, $i = 1, 2, \ldots, m$, where K is a semiconnected subset in a locally convex topological vector space X, $f : K \to \mathbb{R}$, $g : K \to \mathbb{R}_+$ and $h_i : K \to \mathbb{R}$, $i = 1, 2, \ldots, m$. If f, $-g$, and h_i, $i = 1, 2, \ldots, m$, are arc-directionally differentiable, semipreinvex maps with respect to a continuous map $\gamma : [0, 1] \to K \subseteq X$ satisfying $\gamma(0) = 0$ and $\gamma'(0^+) \in K$, then the necessary and sufficient conditions for optimality of (P) are established.

1. Introduction

In recent years, there has been an increasing interest in studying the develpoment of optimality conditions for nondifferentiable multiobjective programming problems. Many authors established and employed some different Kuhn and Tucker type necessary conditions or other type necessary conditions to research optimal solutions; see [1–27] and references therein. In [7], Lai and Ho used the Pareto optimality condition to investigate multiobjective programming problems for semipreinvex functions. Lai [6] had obtained the necessary and sufficient conditions for optimality programming problems with semipreinvex assumptions. Some Pareto optimality conditions are established by Lai and Lin in [8]. Lai and Szilágyi [9] studied the programming with convex set functions and proved that the alternative theorem is valid for convex set functions defined on convex subfamily S of measurable subsets in X and showed that if the system

$$\begin{aligned} f(\Omega) &\ll \theta, \\ g(\Omega) &< \theta \end{aligned} \tag{1}$$

has on solution, where θ stands for zero vector in a topological vector space, then there exists a nonzero continuous linear function $(y^*, z^*) \in C^* \times D^*$ such that

$$\langle f(\Omega), y^* \rangle + \langle g(\Omega), z^* \rangle \geq 0 \quad \forall \Omega \in S. \tag{2}$$

In this paper, we study the following optimization problem:

$$\begin{aligned} &\min_{x\in K} \quad \frac{f(x)}{g(x)} \\ &\text{subject to} \quad x \in K \subseteq X, \quad h_i(x) \leq 0, \\ &\qquad\qquad\qquad\qquad i = 1, 2, \ldots, m, \end{aligned} \tag{P}$$

where K is a semiconnected subset in a locally convex topological vector space X, $f : K \to \mathbb{R}$, $g : K \to \mathbb{R}_+$ and $h_i : K \to (-\infty, 0]$, $i = 1, 2, \ldots, m$, are functions satisfying some suitable conditions. The purpose of this study is dealt with such constrained fractional semipreinvex programming problem. Finally, we established the Fritz John type necessary and sufficient conditions for the optimality of a fractional semipreinvex programming problem.

2. Preliminaries

Throughout this paper, we let X be a locally convex topological vector space over the real field $\mathbb{R}$. Denote $L^1(X)$ by the space of all linear operators from X into $\mathbb{R}$.

Let W be a nonempty convex subset of X. Let $f : W \to \mathbb{R}$ be differentiable at $x_0 \in K$. Then there is a linear operator $A = f'(x_0) \in L^1(X)$, such that

$$\lim_{\alpha \to 0} \frac{f((1-\alpha)x_0 + \alpha x) - f(x_0)}{\alpha} = f'(x_0)(x - x_0). \quad (3)$$

Recall that a function $f : W \to \mathbb{R}$ is called convex on W, if

$$f((1-\alpha)x_0 + \alpha x) \le (1-\alpha) f(x_0) + \alpha f(x) \quad (4)$$

or

$$\frac{f((1-\alpha)x_0 + \alpha x) - f(x_0)}{\alpha} \le f(x) - f(x_0). \quad (5)$$

If $f : W \to \mathbb{R}$ is convex and differentiable at $x_0 \in K$, then by (3) and (5), we have

$$f'(x_0)(x - x_0) \le f(x) - f(x_0). \quad (6)$$

In 1981, Hanson [13, 14] introduced a generalized convexity on X, so-called invexity; that is, $x - x_0$ is replaced by a vector $\tau(x_0, x) \in X$ in (6), or

$$f'(x_0)\tau(x_0, x) \le f(x) - f(x_0). \quad (7)$$

So an invex function is indeed a generalization of a convex differentiable function.

Definition 1 (see [6]). (1) A set $K \subseteq X$ is said to be semiconnected with respect to a given $\tau : X \times X \to \mathbb{R}$ if

$$x, y \in K,\ 0 \le \alpha \le 1 \Longrightarrow y + \alpha\tau(x, y, \alpha) \in K. \quad (8)$$

(2) A map $f : X \to \mathbb{R}$ is said to be semipreinvex on a semiconnected subset $K \subset X$ if each $(x, y, \alpha) \in K \times K \times [0,1]$ corresponds a vector $\tau(x, y, \alpha) \in X$ such that

$$\begin{gathered} f(x + \alpha\tau(x, y, \alpha)) \le (1-\alpha) f(x) + \alpha f(y), \\ \lim_{\alpha \downarrow 0} \alpha\tau(x, y, \alpha) = \theta, \end{gathered} \quad (9)$$

where θ stands for the zero vector of X.

The following is an example of a bounded semiconnected set in $\mathbb{R}$, which is semiconnected with respect to a nontrivial τ.

Example 2. Let $A := [4, 8]$, $B := [-8, -4]$ and $K := A \cup B$ be bounded sets. Let $\tau : K \times K \times [0,1] \to \mathbb{R}$ be defined by

$$\begin{aligned}
\tau(x, y, \alpha) &= \frac{x - y}{1-\alpha}, \quad \text{for } (x, y, \alpha) \in A \times A \times \left[0, \frac{1}{2}\right], \\
\tau(x, y, \alpha) &= \frac{x - y}{1-\alpha}, \quad \text{for } (x, y, \alpha) \in B \times B \times \left[0, \frac{1}{2}\right], \\
\tau(x, y, \alpha) &= \frac{-8 - y}{1-\alpha}, \quad \text{for } (x, y, \alpha) \in A \times B \times \left[0, \frac{1}{2}\right], \\
\tau(x, y, \alpha) &= \frac{4 - y}{1-\alpha}, \quad \text{for } (x, y, \alpha) \in B \times A \times \left[0, \frac{1}{2}\right], \\
\tau(x, y, \alpha) &= \frac{x - y}{\alpha}, \quad \text{for } (x, y, \alpha) \in A \times A \times \left[\frac{1}{2}, 1\right], \\
\tau(x, y, \alpha) &= \frac{x - y}{\alpha}, \quad \text{for } (x, y, \alpha) \in B \times B \times \left[\frac{1}{2}, 1\right], \\
\tau(x, y, \alpha) &= \frac{-8 - y}{\alpha}, \quad \text{for } (x, y, \alpha) \in A \times B \times \left[\frac{1}{2}, 1\right], \\
\tau(x, y, \alpha) &= \frac{4 - y}{\alpha}, \quad \text{for } (x, y, \alpha) \in B \times A \times \left[\frac{1}{2}, 1\right].
\end{aligned} \quad (10)$$

Then K is a bound semiconnected set with respect to τ.

Theorem 3 (see [6, Theorem 2.2]). *Let $K \subset X$ be a semiconnected subset and $f : K \to \mathbb{R}$ a semipreinvex map. Then any local minimum of f is also a global minimum of f over K.*

From the assumption in problem 9, there exists a positive number λ such that

$$\begin{gathered} \frac{f(y)}{g(y)} \ge \lambda \quad \forall y \in X, \\ f(y) - \lambda g(y) \ge 0. \end{gathered} \quad (11)$$

Consequently, we can reduce the problem 9 to an equivalent nonfractional parametric problem:

$$v(\lambda) := \min_{y \in X} (f(y) - \lambda g(y)) \ge 0, \quad (P_\lambda)$$

where $\lambda \in [0, \infty)$ is a parameter.

We will prove that the problem (P) is equivalent to the problem (P_{λ^*}) for the optimal value λ^*. The following result is our main technique to derive the necessary and sufficient optimality conditions for problem (P).

Theorem 4. *Problem (P) has an optimal solution y_0 with optimal value λ^* if and only if $v(\lambda^*) = 0$ and y_0 is an optimal solution of (P_{λ^*}).*

Proof. If y_0 is an optimal solution of (P) with optimal value λ^*, that is,

$$\lambda^* := \frac{f(y_0)}{g(y_0)} = \min_{z \in X} \frac{f(z)}{g(z)} \le \frac{f(z)}{g(z)} \quad \forall z \in X. \quad (12)$$

It follows from (12) that

$$f(z) - \lambda^* g(z) \geq 0 \quad \forall z \in X, \qquad f(y_0) - \lambda^* g(y_0) = 0. \tag{13}$$

Thus, we have

$$0 \leq \min_{z\in X} (f(z) - \lambda^* g(z)) \leq f(y_0) - \lambda^* g(y_0) = 0. \tag{14}$$

Then, by (14), we get

$$\nu(\lambda^*) = \min_{z\in X} (f(z) - \lambda^* g(z)) = f(y_0) - \lambda^* g(y_0) = 0. \tag{15}$$

Therefore, y_0 is an optimal solution of (P_{λ^*}) and $\nu(\lambda^*) = 0$.

Conversely, if y_0 is an optimal solution of (P_{λ^*}) with optimal value $\nu(\lambda^*) = 0$, then

$$f(y_0) - \lambda^* g(y_0) = \min_{z\in X} (f(z) - \lambda^* g(z)) = 0. \tag{16}$$

So

$$f(z) - \lambda^* g(z) \geq 0 = f(y_0) - \lambda^* g(y_0) \quad \forall z \in X. \tag{17}$$

It follows from (17) that

$$\frac{f(z)}{g(z)} \geq \lambda^* \quad \forall z \in X, \qquad \frac{f(y_0)}{g(y_0)} = \lambda^*, \tag{18}$$

and hence

$$\min_{z\in X} \frac{f(z)}{g(z)} \geq \lambda^*, \qquad \min_{z\in X} \frac{f(z)}{g(z)} \leq \frac{f(y_0)}{g(y_0)} = \lambda^*. \tag{19}$$

Therefore,

$$\min_{z\in X} \frac{f(z)}{g(z)} = \lambda^* = \frac{f(y_0)}{g(y_0)} \tag{20}$$

and we know y_0 is an optimal solution of (P) with optimal value λ^*. □

3. The Existence of the Necessary and Sufficient Conditions for Semipreinvex Functions

Definition 5 (see [6]). A mapping $f : K \subset X \to \mathbb{R}$ is said to be arcwise directionally (in short, arc-directionally) differentiable at $x_0 \in K$ with respect to a continuous arc $\beta : [0,1] \to K \subset X$ if $x_0 + \beta(t) \in K$ for $t \in [0,1]$ with

$$\beta(0) = \theta, \quad \beta'(0^+) = u \quad (\text{in } X), \tag{21}$$

that is, the continuous function β is differentiable from right at 0, and the limit

$$\lim_{t\downarrow 0} \frac{f(x_0 + \beta(t)) - f(x_0)}{t} \cong f'(x_0; u) \ \text{ exists.} \tag{22}$$

Note that the arc directional derivative $f'(x_0; \cdot)$ is a mapping from X into $\mathbb{R}$. Moreover, how can we make K to be a semiconnected set? Indeed, we can construct a function τ concerned with β defined as follows.

For any $x, y \in K$ and $t \in [0,1]$, we choose a vector

$$\tau(x, y, t) := \frac{\beta(t)}{t} = \frac{\beta(t) - \beta(0)}{t - 0}, \tag{23}$$

then

$$\lim_{t\downarrow 0} \tau(x, y, t) = \beta'(0^+) = u, \qquad \frac{d}{dt} [t\tau(x, y, t)] \Big|_{t=0^+} = \beta'(0^+) = u. \tag{24}$$

Let $f : X \to \mathbb{R}$, $-g : X \to \mathbb{R}_-$ and $h_i : X \to \mathbb{R}_-$, $i = 1, 2, \ldots, m$, be semipreinvex maps on a semiconnected subset K in X. Consider a constrained programming problem as (P).

The following Fritz John type theorem is essential in this section for programming problem (P).

Theorem 6 (Necessary Optimality Condition). *Suppose that $f, -g$ and h_i, $i = 1, 2, \ldots, m$ are arc-directionally differentiable at $x_0 \in K$ and semipreinvex on K with respect to a continuous arc β defined as in Definition 5. If x_0 minimizes locally for the semipreinvex programming problem (P), then there exist $\lambda^* \in (0, \infty)$ and $\{\gamma_i\}_{i=1}^m \subseteq [0, \infty)$ such that*

$$f'(x_0; u) - \lambda^* g'(x_0; u) + \sum_{i=1}^m \gamma_i h_i'(x_0; u) \geq 0, \tag{25}$$

where $u = \beta'(0^+)$ and

$$\sum_{i=1}^m \gamma_i h_i(x_0) = 0. \tag{26}$$

Proof. By Theorem 4, the minimum solution to (P) is also a minimum to (P_{λ^*}). Then x_0 is the local minimal solution to (P_{λ^*}). By Theorem 3, we have x_0 is the global minimal solution to (P_λ). It follows that the system

$$[f(x) - \lambda^* g(x)] - [f(x_0) - \lambda^* g(x_0)] < 0, \qquad h_i(x) \leq 0, \quad i = 1, 2, \ldots, m \tag{27}$$

has no solution in K, then we have

$$[f(x) - \lambda^* g(x)] - [f(x_0) - \lambda^* g(x_0)] + \sum_{i=1}^m \gamma_i h_i(x) < 0 \tag{28}$$

has no solution in K for any $\{\gamma_i\}_{i=1}^m \subseteq [0, \infty)$. Thus for any $x \in K$,

$$[f(x) - \lambda^* g(x)] - [f(x_0) - \lambda^* g(x_0)] + \sum_{i=1}^m \gamma_i h_i(x) \geq 0 \tag{29}$$

for some $\{\gamma_i\}_{i=1}^m \subseteq [0,\infty)$. Putting $x = x_0$ in (29), we get

$$\sum_{i=1}^m \gamma_i h_i(x_0) \geq 0. \tag{30}$$

Since $\gamma_i \geq 0$ and $h_i(x_0) \leq 0$, it follows that

$$\sum_{i=1}^m \gamma_i h_i(x_0) = 0. \tag{31}$$

So (26) is proved.

As K is a semiconnected set, for any $x \in K$ and $t \in [0,1]$, we have

$$x_0 + t\tau(x_0, x, t) \in K. \tag{32}$$

For $t \neq 0$, the point $\tilde{x} = x_0 + t\tau(x_0, x, t) \neq x_0$ does not solve the system (27). So substituting $\tilde{x}$ in (29) and using the result (26), we obtain

$$\begin{aligned} &[f(x_0 + t\tau(x_0, x, t)) - f(x_0)] \\ &\quad - \lambda^* [g(x_0 + t\tau(x_0, x, t)) - g(x_0)] \\ &\quad + \sum_{i=1}^m \gamma_i (h_i(x_0 + t\tau(x_0, x, t)) - h_i(x_0)) \geq 0. \end{aligned} \tag{33}$$

Since f and g are arc-directionally differentiable with respect to β, choose a vector $\tau(x_0, x, t)$ as (23), so that (24) hold. It follows that if we divide (33) by $t \neq 0$ and take the limit as $t \downarrow 0$, then we have

$$f'(x_0; u) - \lambda^* g'(x_0; u) + \sum_{i=1}^m \gamma_i h_i'(x_0; u) \geq 0, \tag{34}$$

which proves (25) and the proof of theorem is completed. □

Theorem 7 (Sufficient Optimality Condition). *Let f, $-g$ and h_i, $i = 1, 2, \ldots, m$ be arc-directionally differentiable at $x_0 \in K$ and semipreinvex on K with respect to a continuous arc β defined as in Definition 5. If there exist $\lambda \in (0,\infty)$ and $\{\gamma_i\}_{i=1}^m \subseteq [0,\infty)$ satisfying*

$$f'(x_0; u) - \lambda g'(x_0; u) + \sum_{i=1}^m \gamma_i h_i'(x_0; u) \geq 0, \tag{35}$$

with $u = \beta'(0^+)$ and

$$\sum_{i=1}^m \gamma_i h_i(x_0) = 0, \tag{36}$$

then x_0 is an optimal solution for problem (P).

Proof. Suppose to the contrary that x_0 is not optimal for problem (P) and $\lambda = f(x_0)/g(x_0)$. Then $f(x_0) - \lambda g(x_0) = 0$. Therefore,

$$0 \leq \min_{x \in X} (f(x) - \lambda g(x)) \leq f(x_0) - \lambda g(x_0) = 0, \tag{37}$$

thus $v(\lambda) = \min_{x \in X}(f(x) - \lambda g(x)) = 0$.

By Theorem 4, x_0 was not optimal for problem (P_λ). Then there is an $x \in X$ such that

$$\begin{gathered} f(x) - \lambda g(x) < f(x_0) - \lambda g(x_0), \\ h_i(x) \leq 0 \end{gathered} \tag{38}$$

for $i = 1, 2, \ldots, m$. Moreover, we have

$$[f(x) - \lambda g(x)] - [f(x_0) - \lambda g(x_0)] < 0, \tag{39}$$

$$\sum_{i=1}^m \gamma_i [h_i(x) - h_i(x_0)] \leq 0 \qquad \left(\text{since} \sum_{i=1}^m \gamma_i h_i(x_0) = 0\right) \tag{40}$$

for any $\{\gamma_i\}_{i=1}^m \subseteq [0,\infty)$. Thus

$$\begin{aligned} &[f(x) - \lambda g(x)] - [f(x_0) - \lambda g(x_0)] \\ &\quad + \sum_{i=1}^m \gamma_i [h_i(x) - h_i(x_0)] < 0. \end{aligned} \tag{41}$$

Since the semi-preinvex maps f, $-g$ and h_i, $i = 1, 2, \ldots, m$ are arc-directionally differentiable, it follows that for $(x, x_0, t) \in K \times K \times [0,1]$ there corresponds a vector $\tau(x, x_0, t) \in X$ such that

$$\begin{gathered} f(x_0 + t\tau(x, x_0, t)) \leq (1-t) f(x_0) + t f(x), \\ -g(x_0 + t\tau(x, x_0, t)) \leq (1-t)(-g)(x_0) + t(-g)(x), \\ h_i(x_0 + t\tau(x, x_0, t)) \leq (1-t) h_i(x_0) + t h_i(x), \end{gathered} \tag{42}$$

and so

$$\begin{gathered} \frac{f(x_0 + t\tau(x, x_0, t)) - f(x_0)}{t} \leq f(x) - f(x_0), \\ \frac{(-g)(x_0 + t\tau(x, x_0, t)) + g(x_0)}{t} \leq (-g)(x) + g(x_0), \\ \frac{h_i(x_0 + t\tau(x, x_0, t)) - h_i(x_0)}{t} \leq h_i(x) - h_i(x_0). \end{gathered} \tag{43}$$

Letting $t \downarrow 0$, we have $\lim_{t\downarrow 0} \tau(x, x_0, t) = \beta'(0^+) = u$ and the last inequalities imply

$$\begin{gathered} f'(x_0, u) \leq f(x) - f(x_0), \\ -g'(x_0, u) \leq -[g(x) - g(x_0)], \\ h_i'(x_0, u) \leq h_i(x) - h_i(x_0). \end{gathered} \tag{44}$$

Consequently, from (41) and (44), we obtain

$$f'(x_0; u) - \lambda g'(x_0; u) + \sum_{i=1}^m \gamma_i h_i'(x_0; u) < 0, \tag{45}$$

which contradicts the fact of (35). Therefore x_0 is an optimal solution of problem (P). □

Since any global minimal is a local minimal, applying Theorems 6 and 7, we can obtain the necessary and sufficient conditions for problem (P).

Theorem 8. *Suppose that $f, -g$ and h_i, $i = 1, 2, \ldots, m$ are arc-directionally differentiable at at $x_0 \in K$ and semi-preinvex on K with respect to a continuous arc β defined as in Definition 5. If x_0 minimizes globally for the semi-preinvex programming problem (P) if and only if there exists $(\lambda, \gamma_i) \in \mathbb{R}^+ \times (\mathbb{R}^+ \cup \{0\})$, $i = 1, 2, \ldots, m$, such that*

$$f'(x_0; u) - \lambda g'(x_0; u) + \sum_{i=1}^{m} \gamma_i h_i'(x_0; u) \geq 0, \tag{46}$$

where $u = \beta'(0^+)$ and

$$\sum_{i=1}^{m} \gamma_i h_i(x_0) = 0. \tag{47}$$

Remark 9. Our results also hold for preinvex functions.

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