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Research Article

Note on the Hahn-Banach Theorem in a Partially Ordered Vector Space

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Using a fixed point theorem in a partially ordered set, we give a new proof of the Hahn-Banach theorem in the case where the range space is a partially ordered vector space.

1. Introduction

The Hahn-Banach theorem is one of the most fundamental theorems in the functional analysis theory. This theorem is well known in the case where the range space is the real number system as follows.

Let p be a sublinear mapping from a vector space X into the real number system R, Y a subspace of X, and q a linear mapping from Y into R such that $q \leq p$ on Y. Then there exists a linear mapping g from X into R such that $g = q$ on Y and $g \leq p$ on X.

It is known that this theorem is established in the case where the range space is a Dedekind complete Riesz space as follows [1–3].

Let p be a sublinear mapping from a vector space X into a Dedekind complete Riesz space E, Y a subspace of X and q a linear mapping from Y into E such that $q \leq p$ on Y. Then there exists a linear mapping g from X into E such that $g = q$ on Y and $g \leq p$ on X.

On the other hand, Hirano et al. [4] showed the Hahn-Banach theorem by using the Markov-Kakutani fixed point theorem [5] in the case where the range space is the real number system.

In this paper, motivated by Hirano et al. [4], we give a proof of the Hahn-Banach theorem using a fixed point theorem. We show the Hahn-Banach theorem in the case where the range space is a Dedekind complete partially ordered vector space (Theorem 10). Moreover, we show the Mazur-Orlicz theorem in a Dedekind complete partially ordered vector space (Theorem 11).

2. Preliminaries

Let $(E, \leq)$ be a partially ordered set and F a subset of E. The set F is called a *chain* if any two elements are comparable; that is, $x \leq y$ or $y \leq x$ for any $x, y \in F$. An element $x \in E$ is called a *lower bound* of F if $x \leq y$ for any $y \in F$. An element $x \in E$ is called the *minimum* of F if x is a lower bound of F and $x \in F$. If there exists a lower bound of F, then F is said to be *bounded from below*. An element $x \in E$ is called an *upper bound* of F if $y \leq x$ for any $y \in F$. An element $x \in E$ is called the *maximum* of F if x is an upper bound and $x \in F$. If there exists an upper bound of F, then F is said to be *bounded from above*. If the set of all lower bounds of F has the maximum, then the maximum is called an *infimum* of F and denoted by $\inf F$. If the set of all upper bounds of F has the minimum, then the minimum is called a *supremum* of F and denoted by $\sup F$. An element $x \in F$ is called a *minimal* of F if $y \leq x$ and $y \in F$ implies $y = x$. A partially ordered set E is said to be *complete* if every nonempty chain of E has an infimum; E is said to be *chain complete* if every nonempty chain of E

which is bounded from below has an infimum; E is said to be *Dedekind complete* if every nonempty subset of E which is bounded from below has an infimum. A mapping f from E into E is said to be *decreasing* if $f(x) \leq x$ for any $x \in E$. For further information of a partially ordered set, see [1, 2, 6–10].

In a complete partially ordered set, the following theorem is obtained; see [11–14].

Theorem 1 (Bourbaki-Kneser). *Let E be a complete partially ordered set. Let f be a decreasing mapping from E into E. Then f has a fixed point.*

A partially ordered set E is called a partially ordered vector space if E is a vector space and $x+z \leq y+z$ and $\alpha x \leq \alpha y$ hold whenever $x, y, z \in E$, $x \leq y$ and α is a nonnegative real number. If a partially ordered vector space E is a lattice, that is, any two elements in E have a supremum and an infimum, then E is called a *Riesz space*.

Let X be a vector space and E a partially ordered vector space. A mapping f from X into E is said to be *concave* if $f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$ for any $x, y \in X$ and $t \in [0, 1]$. A mapping p from X into E is said to be *sublinear* if the following conditions are satisfied.

(S1) For any $x, y \in X$, $p(x+y) \leq p(x) + p(y)$.

(S2) For any $x \in X$ and nonnegative real number α, $p(\alpha x) = \alpha p(x)$.

Let E^X be the set of mappings from X into E. Throughout this paper, E^X is ordered as follows. For $f, g \in E^X$, let $f \leq g$ mean that $f(x) \leq g(x)$ for any $x \in X$. It is easy to check that E^X is also a partially ordered vector space.

The following lemmas are useful for the proof of our main results.

Lemma 2. *Let X be a vector space, E a chain complete partially ordered vector space, and Z a nonempty chain of E^X which is bounded from below. Then there exists* $\inf\{h(x) \mid h \in Z\}$ *for any $x \in X$. Moreover, if $p \in E^X$ is defined by $p(x) = \inf\{h(x) \mid h \in Z\}$ for any $x \in X$, then $p = \inf Z$; that is, E^X is chain complete.*

Proof. Let $x \in X$ be fixed. Since Z is a nonempty chain, so is $\{h(x) \mid h \in Z\}$. Let f be a lower bound of Z. Since $f(x) \leq h(x)$ for any $h \in Z$, $\{h(x) \mid h \in Z\}$ is bounded from below. Therefore, since E is chain complete, there exists $\inf\{h(x) \mid h \in Z\}$.

Define $p \in E^X$ by $p(x) = \inf\{h(x) \mid h \in Z\}$ for any $x \in X$. Then it is clear that $p \leq h$ for any $h \in Z$; that is, p is a lower bound of Z. Let q be a lower bound of Z. Since $q(x) \leq h(x)$ for any $x \in X$ and $h \in Z$, $q(x)$ is a lower bound of $\{h(x) \mid h \in Z\}$ for any $x \in X$. Therefore, $q(x) \leq \inf\{h(x) \mid h \in Z\} = p(x)$ for any $x \in X$ and thus $p = \inf Z$. □

Lemma 3. *Let X be a vector space, E a Dedekind complete partially ordered vector space, and Z a nonempty subset in E^X which is bounded from below. Then there exists* $\inf\{h(x) \mid h \in Z\}$ *for any $x \in X$. Moreover, if $p \in E^X$ is defined by $p(x) = \inf\{h(x) \mid h \in Z\}$ for any $x \in X$, then $p = \inf Z$; that is, E^X is Dedekind complete.*

Proof. The proof is similar to that of Lemma 2. □

Lemma 4. *Let X, E, E^X, Z, and p be the same as in Lemma 2. Suppose that*

(1) *for any $h \in Z$, $x \in X$ and $\alpha > 0$, there exists $h' \in Z$ such that $h(\alpha x) = \alpha h'(x)$;*

(2) *$p(0) = 0$;*

(3) *for any $h_1, h_2 \in Z$ and $x, y \in X$, there exists $h \in Z$ such that $h(x+y) \leq h_1(x) + h_2(y)$.*

Then p is sublinear.

Proof. Let $x \in X$ and $\alpha > 0$ be fixed. It is clear from (1) that $\{h(\alpha x) \mid h \in Z\} \subset \{\alpha h'(x) \mid h' \in Z\}$. Since $\alpha x \in X$ and $1/\alpha > 0$, by (1), for any $h' \in Z$ there exists $h \in Z$ such that

$$\alpha h'(x) = \alpha h'\left(\frac{1}{\alpha}\alpha x\right) = \alpha \frac{1}{\alpha} h(\alpha x) = h(\alpha x) \tag{1}$$

and hence $\{\alpha h'(x) \mid h' \in Z\} \subset \{h(\alpha x) \mid h \in Z\}$. Therefore, we conclude that $\{h(\alpha x) \mid h \in Z\} = \{\alpha h'(x) \mid h' \in Z\}$. Thus we obtain that

$$\begin{aligned} p(\alpha x) &= \inf\{h(\alpha x) \mid h \in Z\} = \inf\left\{\alpha h'(x) \mid h' \in Z\right\} \\ &= \alpha \inf\left\{h'(x) \mid h' \in Z\right\} = \alpha p(x). \end{aligned} \tag{2}$$

Moreover, (2) shows that $p(0x) = p(0) = 0 = 0p(x)$. Therefore, (S2) holds.

Let $x, y \in X$ be fixed. By (3), for any $h_1, h_2 \in Z$, there exists $h \in Z$ such that $h(x+y) \leq h_1(x) + h_2(y)$. Thus we have

$$p(x+y) \leq h_1(x) + h_2(y) \tag{3}$$

for any $h_1, h_2 \in Z$. This shows that $p(x+y) - h_2(y)$ is a lower bound of $\{h(x) \mid h \in Z\}$ for any $h_2 \in Z$ and hence we have

$$p(x+y) - h_2(y) \leq p(x) \tag{4}$$

for any $h_2 \in Z$. This shows that $p(x+y) - p(x)$ is a lower bound of $\{h(y) \mid h \in Z\}$ and hence we have $p(x+y) - p(x) \leq p(y)$. Therefore, (S1) holds. This completes the proof. □

Lemma 5. *Let X, E, E^X, Z, and p be the same as in Lemma 3. Suppose that*

(1) *for any $h \in Z$, $x \in X$ and $\alpha > 0$, there exists $h' \in Z$ such that $h(\alpha x) = \alpha h'(x)$;*

(2) *$p(0) = 0$;*

(3) *for any $h_1, h_2 \in Z$ and $x, y \in X$, there exists $h \in Z$ such that $h(x+y) \leq h_1(x) + h_2(y)$.*

Then p is sublinear.

Proof. The proof is similar to that of Lemma 4. □

3. Main Results

To obtain our main results, we need the following.

Lemma 6. *Let g be a sublinear mapping from a vector space X into a chain complete partially ordered vector space E and $y \in X$. Let ϕ be a mapping from X into E defined by*

$$\phi(x) = \inf\{g(x+ty) - g(ty) \mid t \geq 0\} \tag{5}$$

for any $x \in X$. Then ϕ is sublinear and $g^* \leq \phi \leq g$ on X, where g^* is a mapping from X into E defined by $g^*(x) = -g(-x)$ for $x \in X$.

Proof. For any $x \in X$ and $t \geq 0$, put $\tau_t(x) = g(x+ty) - g(ty)$. Then $Z = \{\tau_t \mid t \geq 0\}$ is a nonempty chain and bounded from below in E^X. Indeed, since $g = \tau_0 \in Z$, Z is nonempty. If $s \leq t$, then

$$\begin{aligned}\tau_s(x) - \tau_t(x) &= g(x+sy) - g(sy) - (g(x+ty) - g(ty)) \\ &= g(x+sy) + (g(ty) - g(sy)) - g(x+ty) \\ &= g(x+sy) + (t-s)g(y) - g(x+ty) \\ &= g(x+sy) + g((t-s)y) - g(x+ty) \\ &\geq g(x+sy+(t-s)y) - g(x+ty) = 0\end{aligned} \tag{6}$$

for any $x \in X$. Thus Z is a chain in E^X. Since

$$\begin{aligned}\tau_t(x) = g(x+ty) - g(ty) \geq g(ty) \\ - g(-x) - g(ty) = -g(-x) = g^*(x)\end{aligned} \tag{7}$$

for any $x \in X$ and $t \geq 0$, g^* is a lower bound of Z. Hence Z is bounded from below in E^X. Lemma 2 shows that $\phi(x) = \inf Z$ is well defined.

We next check (1), (2), and (3) in Lemma 4. Let $t \geq 0$, $x \in X$, and $\alpha > 0$. We have

$$\begin{aligned}\tau_t(\alpha x) &= g(\alpha x + ty) - g(ty) \\ &= \alpha\left(g\left(x + \frac{t}{\alpha}y\right) - g\left(\frac{t}{\alpha}y\right)\right) \\ &= \alpha\tau_{t/\alpha}(x).\end{aligned} \tag{8}$$

Clearly, $\tau_{t/\alpha} \in Z$ and hence (1) in Lemma 4 holds. Since $\phi(0) = \inf\{0 \mid t \geq 0\} = 0$, (2) in Lemma 4 holds. Let $t_1, t_2 \geq 0$ and $x_1, x_2 \in X$. Since we have

$$\begin{aligned}\tau_{t_1+t_2}(x_1+x_2) &= g(x_1+x_2+(t_1+t_2)y) \\ &\quad - g((t_1+t_2)y) \leq g(x_1+t_1y) \\ &\quad + g(x_2+t_2y) - (t_1+t_2)g(y) \\ &= g(x_1+t_1y) - g(t_1y) \\ &\quad + g(x_2+t_2y) - g(t_2y) \\ &= \tau_{t_1}(x_1) + \tau_{t_2}(x_2),\end{aligned} \tag{9}$$

(3) in Lemma 4 holds. Therefore, Lemma 4 implies that ϕ is sublinear.

Finally, it is clear that $\phi \leq g$. This inequality and (7) imply that $g^* \leq \phi \leq g$ on X. □

By Theorem 1 and Lemma 6, we obtain the following. For the case that E is a Dedekind complete Riesz space, see [2].

Theorem 7. *Let f be a sublinear mapping from a vector space X into a chain complete partially ordered vector space E. Then there exists a linear mapping g from X into E such that $g \leq f$ on X.*

Proof. Let Y be a subset of E^X defined by

$$Y = \left\{h \in E^X \mid h \quad \text{is sublinear}, \quad f^* \leq h \leq f\right\}, \tag{10}$$

where f^* is defined by $f^*(x) = -f(-x)$ for any $x \in X$. Then it is clear that $f \in Y$ and hence Y is nonempty. Moreover Y is complete. In fact, let $Z \subset Y$ be a nonempty chain. Since for any $h \in Z$, $f^* \leq h$, Z is bounded from below. It follows from Lemma 2 that there exists $\inf Z \in E^X$. By Lemma 4, $\inf Z$ is sublinear. Since $f^* \leq h \leq f$ for any $h \in Z$, we have $f^* \leq \inf Z \leq f$. Thus $\inf Z \in Y$ and hence Y is complete. Furthermore Y has a minimal. In fact, we suppose that Y does not have a minimal element. Then, for any $h \in Y$, there exists $\widehat{h} \in Y$ such that $\widehat{h} \leq h$ and $\widehat{h} \neq h$. We define a mapping T from Y into Y by $Th = \widehat{h}$. Since the mapping T is decreasing, there exists $h_0 \in Y$ satisfying $Th_0 = h_0$ by Theorem 1. This is a contradiction.

Let g be a minimal in Y. Let $x \in X$. Let ϕ be a mapping from X into E defined by

$$\phi(z) = \inf\{g(z+tx) - g(tx) \mid t \geq 0\} \tag{11}$$

for any $z \in X$, then ϕ is sublinear and $g^* \leq \phi \leq g$ on X by Lemma 6. Moreover $\phi \in Y$. In fact, since $g \leq f$ and $f^* \leq g^*$, we have $f^* \leq g^* \leq \phi \leq g \leq f$ for any $f \in Z$. This shows that $\phi \in Y$. Since g is minimal, $\phi = g$. Then we have

$$\begin{aligned}g(-x) &= \phi(-x) \\ &= \inf\{g(-x+tx) - g(tx) \mid t \geq 0\} \\ &\leq g(-x+x) - g(x) \\ &= g(0) - g(x) = -g(x).\end{aligned} \tag{12}$$

Since g is sublinear and $0 = g(0) \leq g(x+z) + g(-x-z)$, we have

$$\begin{aligned}-g(x+z) &\leq g(-x-z) \\ &\leq g(-x) + g(-z) \\ &\leq -g(x) - g(z).\end{aligned} \tag{13}$$

Thus $g(x)+g(z) \leq g(x+z)$. Since g is sublinear, we also have $g(x+z) \leq g(x) + g(z)$ for any $x, z \in X$. Then we obtain that for any $x, z \in X$, $g(x+z) = g(x) + g(z)$. Let $x \in X$ and $\alpha > 0$. Since

$$0 = g(\alpha x - \alpha x) = \alpha g(x) + g(-\alpha x), \tag{14}$$

we have $g(-\alpha x) = -\alpha g(x)$. Then for any real number α, we have $g(\alpha x) = \alpha g(x)$. Thus g is linear. Therefore, g is a linear mapping from X into E such that $g \leq f$ on X. □

Since Dedekind completeness implies chain completeness, we obtain the following.

Corollary 8. *Let f be a sublinear mapping from a vector space X into a Dedekind complete partially ordered vector space E. Then there exists a linear mapping g from X into E such that $g \leq f$ on X.*

To give the Hahn-Banach Theorem in the case where the range space is a Dedekind complete partially ordered vector space, we need the following.

Lemma 9. *Let p be a sublinear mapping from a vector space X into a Dedekind complete partially ordered vector space E, K a nonempty convex subset of X, and q a concave mapping from K into E such that $q \leq p$ on K. For any $x \in X$, let*

$$\phi(x) = \inf\{p(x+ty) - tq(y)t \geq 0,\ y \in K\}. \tag{15}$$

Then ϕ is a sublinear mapping such that $\phi \leq p$ on X. Moreover, if g is a linear mapping from X into E, then $g \leq \phi$ on X is equivalent to $g \leq p$ on X and $q \leq g$ on K.

Proof. First, we show that ϕ is well defined and $\phi(x) \geq -p(-x)$ for any $x \in X$. Let $Z = \{\tau_{t,y} \mid t \geq 0 \text{ and } y \in K\}$, where

$$\tau_{t,y}(x) = p(x+ty) - tq(y) \tag{16}$$

for any $x \in X$ and $t \geq 0$. For any $\tau_{t,y} \in Z$ and $x \in X$,

$$\begin{aligned}\tau_{t,y}(x) &= p(x+ty) - tq(y)\\ &\geq p(ty) - p(-x) - tq(y) \geq -p(-x),\end{aligned} \tag{17}$$

and thus $\phi(x) \geq -p(-x)$ and Z is bounded from below in E^X. Since E is Dedekind complete, ϕ is well defined by Lemma 3.

We next check (1), (2), and (3) in Lemma 5.

(1) Let $\tau_{t,y} \in Z$. For any $x \in X$ and $\alpha > 0$, we have

$$\begin{aligned}\tau_{t,y}(\alpha x) &= p(\alpha x + ty) - tq(y)\\ &= \alpha\left(p\left(x + \frac{t}{\alpha}y\right) - \frac{t}{\alpha}q(y)\right)\\ &= \alpha\tau_{t/\alpha,y}(x).\end{aligned} \tag{18}$$

(2) By the definition of ϕ, $\phi(x) \leq p(x)$ for any $x \in X$. Therefore $\phi(0) \leq p(0) = 0$. Since $p \geq q$ on K, we have

$$\begin{aligned}\phi(0) &= \inf\{p(ty) - tq(y)\ \ t \geq 0,\ y \in K\}\\ &= \inf\{tp(y) - tq(y)t \geq 0,\ y \in K \geq 0\}.\end{aligned} \tag{19}$$

Hence we have $\phi(0) = 0$.

(3) Let $\tau_{t_1,y_1}, \tau_{t_2,y_2} \in Z$ satisfying $t_1+t_2 \neq 0$. Let $x_1, x_2 \in X$. Since K is convex and q is concave, we have

$$\begin{aligned}&\tau_{t_1,y_1}(x_1) + \tau_{t_2,y_2}(x_2)\\ &\quad = p(x_1 + t_1y_1) - t_1q(y_1) + p(x_2 + t_2y_2) - t_2q(y_2)\\ &\quad \geq p(x_1 + x_2 + (t_1+t_2)w) - (t_1+t_2)q(w)\\ &\quad = \tau_{t_1+t_2,w}(x_1+x_2),\end{aligned} \tag{20}$$

where $w = (1/(t_1+t_2))(t_1y_1 + t_2y_2) \in K$. Since p is sublinear, we have

$$\begin{aligned}\tau_{0,w}(x_1+x_2) &= p(x_1+x_2) \leq p(x_1) + p(x_2)\\ &= \tau_{0,y_1}(x_1) + \tau_{0,y_2}(x_2).\end{aligned} \tag{21}$$

Therefore, for any $x_1, x_2 \in X$ and $t_1, t_2 \geq 0$, we have $\tau_{t_1,y_1}(x_1) + \tau_{t_2,y_2}(x_2) \geq \tau_{t_1+t_2,w}(x_1+x_2)$.

Thus by Lemma 5, ϕ is sublinear. Moreover, by the definition of ϕ, we have $\phi \leq p$ on X.

Let g be a linear mapping from X into E. Suppose that $g \leq \phi$ on X. Since $\phi \leq p$ on X, we have $g \leq p$ on X. Moreover, since for any $y \in K$,

$$\begin{aligned}-g(y) = g(-y) &\leq \phi(-y)\\ &\leq p(-y+y) - q(y) = -q(y),\end{aligned} \tag{22}$$

we have $g \geq q$ on K. To prove the converse, suppose that $g \leq p$ on X and $q \leq g$ on K. For any $x \in X$, $y \in K$ and $t \geq 0$, we have

$$g(x) = g(x+ty) - tg(y) \leq p(x+ty) - tq(y). \tag{23}$$

This implies that $g \leq \phi$ on X. □

By Corollary 8 and Lemma 9, we have the Hahn-Banach theorem in the case where the range space is a Dedekind complete partially ordered vector space. For the case that E is a Dedekind complete Riesz space, see [2].

Theorem 10. *Let p be a sublinear mapping from a vector space X into a Dedekind complete partially ordered vector space E, Y a subspace of X, and q a linear mapping from Y into E such that $q \leq p$ on Y. Then there exists a linear mapping g from X into Y such that $g = q$ on Y and $g \leq p$ on X.*

Proof. Let ϕ be a mapping from X into E defined by

$$\phi(x) = \inf\{p(x+ty) - tq(y)t \geq 0,\ y \in K\} \tag{24}$$

for any $x \in X$. By Lemma 9, ϕ is a sublinear mapping such that $\phi \leq p$ on X. By Corollary 8, there exists a linear mapping g such that $g \leq \phi$ on X. Then putting $K = Y$ in Lemma 9, we have $g \leq p$ on X and $q \leq g$ on Y. Since Y is a subspace, for any $y \in Y$, we have $-y \in Y$. Then $q(-y) \leq g(-y)$. Since q and g are linear, we have $-q(y) \leq -g(y)$. Then $g \leq q$ on Y. Thus $g = q$ on Y. □

Moreover, by Corollary 8, we obtain the Mazur-Orlicz theorem in a Dedekind complete partially ordered vector space. For the case that E is a Dedekind complete Riesz space, see [1, 15].

Theorem 11. *Let p be a sublinear mapping from a vector space X into a Dedekind complete partially ordered vector space E. Let $\{x_j \mid j \in J\}$ be a family of elements of X and $\{y_j \mid j \in J\}$ a family of elements of E. Then the following (1) and (2) are equivalent.*

(1) *There exists a linear mapping g from X into E such that $g \leq p$ on X and $y_j \leq g(x_j)$ for any $j \in J$.*

(2) *For any natural number n, nonnegative real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ and $j_1, j_2, \ldots, j_n \in J$, one has*

$$\sum_{i=1}^{n} \alpha_i y_{j_i} \leq p\left(\sum_{i=1}^{n} \alpha_i x_{j_i}\right). \tag{25}$$

Proof. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ and $j_1, j_2, \ldots, j_n \in J$ for a natural number n. By (1), we have

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i y_{j_i} &\leq \sum_{i=1}^{n} \alpha_i g\left(x_{j_i}\right) \\ &= g\left(\sum_{i=1}^{n} \alpha_i x_{j_i}\right) \leq p\left(\sum_{i=1}^{n} \alpha_i x_{j_i}\right). \end{aligned} \tag{26}$$

Thus (2) is established.

Next by (2), for any $x \in X$, we have

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i y_{j_i} \leq p\left(\sum_{i=1}^{n} \alpha_i x_{j_i}\right) &= p\left(x + \sum_{i=1}^{n} \alpha_i x_{j_i} - x\right) \\ &\leq p\left(x + \sum_{i=1}^{n} \alpha_i x_{j_i}\right) + p(-x), \\ -p(-x) \leq p\left(x + \sum_{i=1}^{n} \alpha_i x_{j_i}\right) &- \sum_{i=1}^{n} \alpha_i y_{j_i} \end{aligned} \tag{27}$$

for any natural number n, $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ and $j_1, j_2, \ldots, j_n \in J$. Put

$$\begin{aligned} &p_0(x) \\ &= \inf\left\{p\left(x + \sum_{i=1}^{n} \alpha_i x_{j_i}\right) - \sum_{i=1}^{n} \alpha_i y_{j_i} \mid n \in N,\right. \\ &\qquad\qquad \left. \alpha_i \geq 0, j_i \in J,\ i = 1, 2, \ldots, n\right\} \end{aligned} \tag{28}$$

for $x \in X$, where N is the set of all natural numbers. By Lemma 3, p_0 is well defined. Since p is sublinear, p_0 is also sublinear. Thus by Corollary 8, there exists a linear mapping g from X into E such that $g \leq p_0$ on X. Since $p_0(-x_j) \leq p(-x_j + x_j) - y_j = -y_j$, we have

$$y_j \leq -p_0\left(-x_j\right) \leq -g\left(-x_j\right) = g\left(x_j\right). \tag{29}$$

Since $p_0(x) \leq p(x)$ for any $x \in X$, we have $g(x) \leq p(x)$ for any $x \in X$. Thus (1) is established. □

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