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Research Article

Exact Number of Positive Solutions for a Class of Two-Point Boundary Value Problems

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This paper considers the existence of positive solutions of the boundary value problems $v'' + \lambda(v^r + v^p - v^q) = 0$ and $v(-1) = v(1) = 0$, where $p > r > q > -1$ and λ is a positive parameter. Using a time-map approach, we obtain the exact number of positive solutions in different cases.

1. Introduction and Main Results

The study of multiplicity results to boundary value problem

$$\begin{gathered} v'' + \lambda f(v) = 0, \quad t \in (a,b), \\ v(a) = v(b) = 0, \end{gathered} \tag{1}$$

where $\lambda > 0$ is a positive parameter, is very interesting because of its applications. As we know, when $f(v) = v^p(x) + v^q(x)$, the boundary value problem

$$\begin{gathered} -v''(x) = \mu v^p(x) + v^q(x), \quad a \le x \le b, \\ v(x) > 0, \quad x \in (a,b), \\ v(a) = v(b) = 0, \end{gathered} \tag{2}$$

where $0 < q < 1 < p$ and $k \ge 0$ are fixed given numbers and $\mu > 0$ is a parameter, comes from the elliptic problem

$$\begin{gathered} -\Delta u = \lambda u^q + u^p, \quad x \in \Omega, \\ u > 0, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{3}$$

with $0 < q < 1 < p$, which was raised by Ambrosetti et al. in [1].

Under different assumptions on f, there are many results for the above problems and elliptic equations (see [2–8]).

In [9, 10], Liu considered the case of $f(v) = v^p + (1/\lambda)v^q$ and $f(v) = v^p + v^q + kv$ and gave the exact number of solutions and many interesting properties of the solutions.

Cheng [11] investigated the following two-point boundary value problem:

$$\begin{gathered} -y'' = \lambda(y^p - y^q), \quad t \in (-1,1), \\ y(-1) = y(1) = 0, \end{gathered} \tag{4}$$

where $\lambda > 0$ is a positive parameter and $p > q > -1$ and got the exact number of positive solutions.

Now, in this paper we consider the more general case

$$\begin{gathered} -v'' = \lambda(v^r + v^p - v^q), \quad t \in (-1,1), \\ v(-1) = v(1) = 0, \end{gathered} \tag{5}$$

where $\lambda > 0$ is a positive parameter, $p > r > q > -1$, and $(1/(r+1)) + (1/(p+1)) - (1/(q+1)) < 0$.

Define β, where β satisfies

$$\frac{\beta^r}{r+1} + \frac{\beta^p}{p+1} - \frac{\beta^q}{q+1} = 0. \tag{6}$$

For $p > r > q$ and $-1 < q < 1$, let λ_1 be given by

$$\begin{aligned} \lambda_1 &= \frac{r+1}{2}\beta^{1-r} \\ &\quad \times \left(\int_0^1 \frac{dt}{\sqrt{t^{q+1}\left[\left(1 - t^{r-q}\right) + \left((r+1)/(p+1)\right)\beta^{p-r}\left(1 - t^{p-q}\right)\right]}} \right)^2. \end{aligned} \tag{7}$$

The main results of this paper are as follows.

Theorem 1. *If $p > r > q \geq 1$, (5) has exactly one positive solution for any $\lambda > 0$.*

Theorem 2. *If $p > r \geq 1 > q > -1$, (5) has exactly one positive solution for $\lambda \in (0, \lambda_1]$ and none for $\lambda > \lambda_1$.*

Theorem 3. *If $p > 1 > r > q \geq 0$, (5) has exactly one positive solution for $\lambda \in (0, \lambda_1]$ and none for $\lambda > \lambda_1$.*

Theorem 4. *If $(1/3) \geq p > r > q \geq 0$, (5) has exactly one positive solution for $\lambda \in (\lambda_1, +\infty)$ and none for $\lambda < \lambda_1$.*

Theorem 5. *Assume that $1 > p > r > 0 > q > -1$. Define*

$$\begin{aligned}\theta(r,p,q) &= \int_0^1 \frac{1-r}{\left[t^{q+1}\left(1-m_1\beta^{r-q}t^{r-q}-m_2\beta^{p-q}t^{p-q}\right)\right]^{1/2}}dt \\ &\quad -(r-q)\int_0^1 \frac{1-t^{q+1}}{\left[t^{q+1}\left(1-m_1\beta^{r-q}t^{r-q}-m_2\beta^{p-q}t^{p-q}\right)\right]^{3/2}}dt \\ &\quad -\frac{(p-r)(q+1)}{p+1}\beta^{p-q} \\ &\quad \times \int_0^1 \frac{1-t^{p+1}}{\left[t^{q+1}\left(1-m_1\beta^{r-q}t^{r-q}-m_2\beta^{p-q}t^{p-q}\right)\right]^{3/2}}dt,\end{aligned} \tag{8}$$

where $m_1 = (q+1)/(r+1)$ and $m_2 = (q+1)/(p+1)$. One has the following.

(1) *If $\theta(r,p,q) \geq 0$, (5) has exactly one positive solution for $\lambda \in (\lambda_1, +\infty)$ and none for $\lambda \in (0, \lambda_1)$.*

(2) *If $\theta(r,p,q) < 0$, there exists $\lambda_0 \in (0, \lambda_1)$ such that (5) has exactly two positive solutions for $\lambda \in (\lambda_0, \lambda_1]$, exactly one for $\lambda \in (\lambda_1, +\infty)$ or $\lambda = \lambda_0$, and none for $\lambda \in (0, \lambda_0)$.*

2. The Proofs of Theorems 1–4

We assume throughout this section that $p > r > q > -1$ and

$$\frac{1}{r+1} + \frac{1}{p+1} - \frac{1}{q+1} < 0. \tag{9}$$

Denote that $E = (\beta, +\infty)$, where β is given by (6). For $a \geq \beta$ and $t \in (0,1)$, let

$$P(a,t) = \int_{at}^{a} \left(z^r + z^p\right) dz, \qquad Q(a,t) = \int_{at}^{a} z^q dz. \tag{10}$$

Define a function $F : E \to (0, +\infty)$ as

$$F(a) = \int_0^a \left[2\int_v^a \left(z^r + z^p - z^q\right) dz\right]^{-1/2} dv \quad \text{for } a \in E. \tag{11}$$

It is clear that $\int_0^\beta (z^r + z^p - z^q)dz = 0$. Now we claim that $a^r + a^p - a^q > 0$ for $a \geq \beta$ and $\int_0^a [2\int_v^a (z^r + z^p - z^q)dz]^{-1/2} dv < \infty$ if and only if $a \in E$.

Let $g_1(z) = z^p$ and $g_2(z) = z^r - z^q$; then $f(z) = g_1(z) + g_2(z)$. Immediately, we get that $g_2'(x) = rz^{r-1} - qz^{q-1} = z^{q-1}(rz^{r-q} - q)$. In order to judge the sign of $g_2'(x)$, we just judge the sign of $T(z) = rz^{r-q} - q$. Since $T(0) = -q$, $T(1) = r - q$, and $T'(z) = r(r-q)z^{r-q-1}$, we can get the following two results.

(1) For $r > q > 0$, the function $g_2(z)$ has a stable point $z_0 = (q/r)^{1/(r-q)}$. When $z \in (0, z_0)$, we have $g_2'(z) < 0$, and when $z \in (z_0, +\infty)$, we have $g_2'(z) > 0$.

(2) For $r > 0 > q$, we have $g_2'(z) > 0$ on $z \in (0, +\infty)$. Combining $g_1'(x) = pz^{p-1}$ and $g_1''(x) = p(p-1)z^{p-2}$ with the above two results, we obtain the monotony of $f(z)$.

(3) For $p \geq 1 > r > q > 0$, the function $f(z)$ has a stable point z_* on $(0,1)$. When $z \in (0, z_*)$, we have $f'(z) < 0$ and when $z \in (0, z_*)$, we have $f'(z) > 0$.

(4) For $1 > p > r > q > 0$, we have that $f'(z) > 0$ on $(0, +\infty)$.

(5) For $p > r > 0 > q$ or $p > 0 > r > q$, we have that $f'(z) > 0$ on $(0, +\infty)$.

Then from $\int_0^\beta (z^r + z^p - z^q)dz = 0$, we infer that $a^r + a^p - a^q > 0$ for $a \geq \beta$.

We consider the integration $\int_0^a [2\int_v^a (z^r + z^p - z^q)dz]^{-1/2} dv$. It is clear that $v = a$ is a flaw point. Since $\int_0^a [2\int_v^a (z^r + z^p - z^q)dz]^{-1/2} dv < \int_0^a [2\int_v^a (z^p - z^q)dz]^{-1/2} dv$ on E, we consider the integration $\int_0^a [2\int_v^a (z^p - z^q)dz]^{-1/2} dv$.

Using Lagrange theorem, we obtain that

$$\begin{aligned}&\int_0^a \left[2\int_v^a \left(z^p - z^q\right) dz\right]^{-1/2} dv \\ &\quad = \int_0^a \left[2\left(\frac{a^{p+1}}{p+1} - \frac{v^{p+1}}{p+1}\right) - 2\left(\frac{a^{q+1}}{q+1} - \frac{v^{q+1}}{q+1}\right)\right]^{-1/2} dv \\ &\quad = \int_0^a \left\{2\left[v + \theta_1(a-v)\right]^p (a-v) - 2\left[v + \theta_2(a-v)\right]^q (a-v)\right\}^{-1/2} dv \\ &\quad = \int_0^a \left[2(a-v)\right]^{-1/2} \left\{\left[v + \theta_1(a-v)\right]^p - \left[v + \theta_2(a-v)\right]^q\right\}^{-1/2} dv \\ &\quad < \infty,\end{aligned} \tag{12}$$

where $\theta_1, \theta_2 \in (0,1)$ are constants.

The following Lemma 6 is listed to show that to study the number of positive solutions of (5) is equivalent to study the shape of the map $F(a)$ on E. Lemmas 7–9 show some properties of $F(a)$ on E.

Lemma 6. *Let $u(a,t)$ be the unique solution of the problem*

$$0 \le u(t) \le a,$$

$$\int_{u(t)}^{a} \left[2\int_{v}^{a} \left(z^r + z^p - z^q\right) dz\right]^{-1/2} dv = F(a)\, t, \quad t \in [0,1], \tag{13}$$

where $a \in E$. One has the following.

(1) *If $\lambda > 0$ and v is a positive solution of* (5), *$v(0) \in E$, $F(v(0)) = \sqrt{\lambda}$, and $v(t) = u(v(0), |t|)$ for $t \in [-1,1]$.*

(2) *If $a \in E$ and $F(a) = \sqrt{\lambda}$, $v(t) = u(v(0), |t|)$ is a positive solution of* (5) *with $v(0) = a$.*

Proof. (1) Assume that $\lambda > 0$ and v is a positive solution of (5). Let $\tau \in [-1,1]$ satisfy $v(\tau) = \max_{t\in[-1,1]} v(t) = a$. It follows from $v'(\tau) = 0$ that

$$\begin{aligned}\left[v'(s)\right]^2 &= \int_{\tau}^{s} 2v''(\eta)\, v'(\eta)\, d\eta \\ &= 2\lambda \int_{v(s)}^{a} \left(z^r + z^p - z^q\right) dz, \quad s \in (-1,1).\end{aligned} \tag{14}$$

This implies that $a \ge \beta$ and $v'(s) \ne 0$ if $v(s) < a$. And combine $v''(\tau) = -\lambda(a^r + a^p - a^q) < 0$ to obtain

$$\begin{aligned} v'(s) &> 0 \quad \text{for } s \in (-1,\tau), \\ v'(s) &< 0 \quad \text{for } s \in (\tau,1). \end{aligned} \tag{15}$$

Then, we have that

$$\begin{aligned} v'(s)\left[2\int_{v(s)}^{a} \left(z^r + z^p - z^q\right) dz\right]^{-1/2} &= \sqrt{\lambda}, \quad s \in (-1,\tau), \\ -v'(s)\left[2\int_{v(s)}^{a} \left(z^r + z^p - z^q\right) dz\right]^{-1/2} &= \sqrt{\lambda}, \quad s \in (\tau,1). \end{aligned} \tag{16}$$

It follows that

$$\int_{v(t)}^{a} \left[2\int_{v}^{a} \left(z^r + z^p - z^q\right) dz\right]^{-1/2} dv = (\tau - t)\sqrt{\lambda}, \quad t \in [-1,\tau], \tag{17}$$

$$\int_{v(t)}^{a} \left[2\int_{v}^{a} \left(z^r + z^p - z^q\right) dz\right]^{-1/2} dv = (t - \tau)\sqrt{\lambda}, \quad t \in [\tau,1], \tag{18}$$

$$\begin{aligned}(1-\tau)\sqrt{\lambda} &= \int_{0}^{a} \left[2\int_{v}^{a} \left(z^r + z^p - z^q\right) dz\right]^{-1/2} dv \\ &= (\tau+1)\sqrt{\lambda}.\end{aligned} \tag{19}$$

From (19) and $a \ge \beta$ we have that $\tau = 0$. With (17) and (18) we obtain the result (1) of this theorem.

(2) Since $u(a,0) = a > 0$ and $u(a,t)$ is a positive solution of the boundary value problem

$$\begin{aligned} u'' + [F(a)]^2\left(u^r + u^p - u^q\right) &= 0, \quad t \in (0,1), \\ u'(0) = 0, \qquad u(1) &= 0, \end{aligned} \tag{20}$$

we have that $v(t) = u(a, |t|)$ is a positive solution of (5). □

Lemma 7. *F is differentiable on (β, ∞), and*

$$F(a) = \frac{1}{4\sqrt{2}} \int_{0}^{1} H_0(a,t)\, dt, \quad a > \beta, \tag{21}$$

$$F'(a) = \frac{1}{4\sqrt{2}} \int_{0}^{1} H_1(a,t)\, dt, \quad a > \beta, \tag{22}$$

where

$$\begin{aligned} H_0(a,t) &= 4a[P(a,t) - Q(a,t)]^{-1/2}, \\ H_1(a,t) &= 2[P(a,t) - Q(a,t)]^{-3/2} \\ &\quad \times \left[\frac{1-r}{r+1} a^{r+1}\left(1 - t^{r+1}\right) + \frac{1-p}{p+1} a^{p+1}\left(1 - t^{p+1}\right) \right. \\ &\quad \left. - \frac{1-q}{q+1} a^{q+1}\left(1 - t^{q+1}\right)\right]. \end{aligned} \tag{23}$$

Proof. Equation (21) can be obtained by (11), immediately. From

$$\begin{aligned} \frac{\partial P(a,t)}{\partial a} &= \frac{\partial}{\partial a} \\ &\quad \times \left[\frac{1-r}{r+1} a^{r+1}\left(1 - t^{r+1}\right) + \frac{1-p}{p+1} a^{p+1}\left(1 - t^{p+1}\right)\right] \\ &= a^r\left(1 - t^{r+1}\right) + a^p\left(1 - t^{p+1}\right), \\ \frac{\partial Q(a,t)}{\partial a} &= \frac{\partial}{\partial a}\left[\frac{1}{q+1} a^{q+1}\left(1 - t^{q+1}\right)\right] = a^q\left(1 - t^{q+1}\right), \end{aligned} \tag{24}$$

we have that

$$\begin{aligned} \frac{\partial H_0(a,t)}{\partial a} &= 2[P(a,t) - Q(a,t)]^{-3/2} \\ &\quad \times \left[\frac{1-r}{r+1} a^{r+1}\left(1 - t^{r+1}\right) + \frac{1-p}{p+1} a^{p+1}\left(1 - t^{p+1}\right) \right. \\ &\quad \left. - \frac{1-q}{q+1} a^{q+1}\left(1 - t^{q+1}\right)\right] \\ &= H_1(a,t). \end{aligned} \tag{25}$$

It follows from (21) that F is differentiable on (β, ∞) and (22) is true. □

Lemma 8. *Consider the following:*

$$\lim_{a\to\beta} F(a) = \begin{cases} +\infty, & q \geq 1, \\ \sqrt{\lambda_1}, & q < 1, \end{cases} \tag{26}$$

where λ_1 is given by (7).

Proof. From

$$\lim_{a\to\beta} \int_\beta^a \frac{dv}{\sqrt{2\int_v^a (z^r + z^p - z^q)\,dz}} = 0 \tag{27}$$

and the Lebesgue theorem, we have that

$$\begin{aligned} \lim_{a\to\beta} F(a) &= \lim_{a\to\beta} \int_0^\beta \frac{dv}{\sqrt{2\int_v^a (z^r + z^p - z^q)\,dz}} \\ &= \int_0^\beta \frac{dv}{\sqrt{2\int_v^\beta (z^r + z^p - z^q)\,dz}}. \end{aligned} \tag{28}$$

On the other hand, from $\int_0^\beta (z^r + z^p - z^q)dz = 0$, we can obtain that

$$\begin{aligned} &\int_0^\beta \frac{dv}{\sqrt{2\int_v^\beta (z^r + z^p - z^q)\,dz}} \\ &= \int_0^\beta \frac{dv}{\sqrt{2\int_0^v (z^q - z^r - z^p)\,dz}} \\ &= \int_0^1 (\beta dt) \\ &\quad \times \left(\sqrt{2\left[\frac{(\beta t)^{q+1}}{q+1} - \frac{(\beta t)^{r+1}}{r+1} - \frac{(\beta t)^{p+1}}{p+1}\right]}\right)^{-1} \\ &= \int_0^1 (\beta dt) \\ &\quad \times \left(2\left[\frac{\beta^{r+1}}{r+1} t^{q+1} + \frac{\beta^{p+1}}{p+1} t^{q+1} \right.\right. \\ &\qquad \left.\left. - \frac{(\beta t)^{r+1}}{r+1} - \frac{(\beta t)^{p+1}}{p+1}\right]\right)^{-1/2} \\ &= \sqrt{\frac{r+1}{2}}\beta^{(1-r)/2} \\ &\quad \times \int_0^1 \frac{dt}{\sqrt{t^{q+1}\left[(1-t^{r-q}) + \left((r+1)/(p+1)\right)\beta^{p-r}(1-t^{p-q})\right]}}. \end{aligned} \tag{29}$$

This completes the proof of Lemma 8. □

Lemma 9. *Consider the following:*

$$\lim_{a\to+\infty} F(a) = \begin{cases} 0, & p > 1,\ r \geq 1 \text{ or } p > 1,\ r < 1, \\ \dfrac{\pi}{2\sqrt{2}}, & p = 1,\ r = 1, \\ +\infty, & p < 1,\ r < 1. \end{cases} \tag{30}$$

Proof. From

$$\begin{aligned} &\lim_{a\to+\infty} \frac{Q(a,t)}{P(a,t)} \\ &= \lim_{a\to+\infty} \frac{a^{q+1}\left(1-t^{q+1}\right)/(q+1)}{a^{r+1}\left(1-t^{r+1}\right)/(r+1) + a^{p+1}\left(1-t^{p+1}\right)/(p+1)} \\ &= \lim_{a\to+\infty} \left(\left(1-t^{q+1}\right)(r+1)(p+1)\right) \\ &\qquad \times \left((q+1)(p+1)a^{r-q}\left(1-t^{r+1}\right)\right. \\ &\qquad \left. + (q+1)(r+1)a^{p-q}(1-t^{p+1})\right)^{-1} \\ &= 0, \end{aligned} \tag{31}$$

we have that

$$\begin{aligned} &\lim_{a\to+\infty} \frac{a}{\sqrt{P(a,t) - Q(a,t)}} \\ &= \lim_{a\to+\infty} \frac{a}{\sqrt{P(a,t)}} \\ &= \lim_{a\to+\infty} \frac{1}{\sqrt{(1/(r+1))a^{r-1}\left(1-t^{r+1}\right) + (1/(p+1))a^{p-1}\left(1-t^{p+1}\right)}}. \end{aligned} \tag{32}$$

From Lemma 7 and the Lebesgue theorem, we can obtain the results of this lemma. □

In the following section, we give the proofs of Theorems 1–5. For convenience, we denote that

$$\begin{aligned} I(a,t) &= \frac{1-r}{r+1}a^{r+1}\left(1-t^{r+1}\right) + \frac{1-p}{p+1}a^{p+1}\left(1-t^{p+1}\right) \\ &\quad - \frac{1-q}{q+1}a^{q+1}\left(1-t^{q+1}\right). \end{aligned} \tag{33}$$

Hence from Lemma 7, we have that

$$F'(a) = \frac{1}{4\sqrt{2}}\int_0^1 2(P-Q)^{-3/2} I(a,t)\,dt, \quad \text{for } a > \beta. \tag{34}$$

Proof of Theorem 1. From $p > r > q \geq 1$, we obtain that

$$\begin{aligned} I(a,t) &< \frac{1-r}{r+1}a^{r+1}\left(1-t^{r+1}\right) + \frac{1-p}{p+1}a^{p+1}\left(1-t^{p+1}\right) \\ &\quad - \frac{1-q}{q+1}a^{p+1}\left(1-t^{p+1}\right) \\ &= \frac{1-r}{r+1}a^{r+1}\left(1-t^{r+1}\right) + \frac{2(q-p)}{(p+1)(q+1)}a^{p+1} \\ &\quad \times \left(1-t^{p+1}\right) < 0, \end{aligned} \tag{35}$$

for $a > \beta$, $t \in (0,1)$. Thus $F'(a) < 0$ for $a > \beta$. By Lemmas 6, 8, and 9, we have the results of this theorem. □

Proof of Theorem 2. From $p > r \geq 1 > q > -1$, (33), and (34), we have that $F'(a) < 0$ for $a > \beta$. It follows from Lemmas 6, 8, and 9 that the results of Theorem 2 hold. □

Proof of Theorem 3. Conditions $p > 1 > r > q \geq 0$ and $(1/(r+1))+(1/(p+1))-(1/(q+1)) < 0$ imply that $rp-1 > q(r+p+2) > 0$. With (33), we obtain that

$$\begin{aligned} I(a,t) &< \frac{1-r}{r+1}a^{p+1}\left(1-t^{p+1}\right)+\frac{1-p}{p+1}a^{p+1}\left(1-t^{p+1}\right) \\ &\quad -\frac{1-q}{q+1}a^{q+1}\left(1-t^{q+1}\right) \\ &< \frac{2(1-rp)}{(p+1)(r+1)}a^{p+1}\left(1-t^{p+1}\right)-\frac{1-q}{q+1}a^{q+1} \\ &\quad \times\left(1-t^{q+1}\right)<0, \end{aligned} \tag{36}$$

for $a > \beta$, $t \in (0,1)$, which means that $F'(a) < 0$ for $a > \beta$. Thus by Lemmas 6, 8, and 9, we have the results of this theorem. □

Proof of Theorem 4. From $(1/3) \geq p > r > q \geq 0$ and (33), we have that

$$\begin{aligned} I(a,t) &> \frac{1-r}{r+1}a^{q+1}\left(1-t^{q+1}\right)+\frac{1-p}{p+1}a^{q+1}\left(1-t^{q+1}\right) \\ &\quad -\frac{1-q}{q+1}a^{q+1}\left(1-t^{q+1}\right) \\ &> \left(2\frac{1-p}{p+1}-\frac{1-q}{q+1}\right)a^{q+1}\left(1-t^{q+1}\right) \\ &= \frac{q(3-p)+(1-3p)}{(p+1)(q+1)}a^{q+1}\left(1-t^{q+1}\right)>0, \end{aligned} \tag{37}$$

for $a > \beta$, $t \in (0,1)$. Hence $F'(a) > 0$ for $a > \beta$. By Lemmas 6, 8, and 9, we have the results of this theorem. □

3. The Proof of Theorem 5

In this section we always assume that $1 > p > r > 0 > q > -1$. Denote that

$$S(a,t)=\frac{P(a,t)}{Q(a,t)},\quad a>\beta,\ t\in(0,1), \tag{38}$$

$$h_1(s)=(s-1)(s-n),\quad s\in(-\infty,\infty), \tag{39}$$

$$h_2(s)=-3(s-m)(s-n)+2(s-1)(s-l),\quad s\in(-\infty,\infty), \tag{40}$$

where $P(a,t)$, $Q(a,t)$, and β are given by (10), (6), and

$$\begin{aligned} m=m(t)&=\frac{q+1}{r+1}-\frac{(p-r)(q+1)}{(p+1)(r+1)}a^{p-q}\frac{1-t^{p+1}}{1-t^{q+1}}, \\ &\qquad a>\beta,\quad t\in(0,1), \\ n=n(t)&=\frac{1-q}{1-r}+\frac{(p-r)(q+1)}{(p+1)(1-r)}a^{p-q}\frac{1-t^{p+1}}{1-t^{q+1}}, \\ &\qquad a>\beta,\quad t\in(0,1), \\ l=l(t)&=\frac{1-q^2}{1-r^2}+\frac{\left(p^2-r^2\right)(q+1)}{(p+1)(1-r^2)}a^{p-q}\frac{1-t^{p+1}}{1-t^{q+1}}, \\ &\qquad a>\beta,\quad t\in(0,1). \end{aligned} \tag{41}$$

Remark 10. From $1 > p > r > 0 \geq q > -1$, (6), and (41), it is obvious that $m < 1$, $n > 1$, and $l < n$.

Lemma 11. *For $a > \beta$ and $t \in (0,1)$,*

$$\frac{q+1}{r+1}a^{r-q}+\frac{q+1}{p+1}a^{p-q}<\frac{P(a,t)}{Q(a,t)}<a^{r-q}+a^{p-q}. \tag{42}$$

Proof. Let

$$\begin{aligned} K(t)&=\frac{a^r}{r+1}\left[(r-q)t^{r+q+1}+(q+1)t^q-(r+1)t^r\right] \\ &\quad+\frac{a^p}{p+1}\left[(p-q)t^{p+q+1}+(q+1)t^q-(p+1)t^p\right]. \end{aligned} \tag{43}$$

Condition $1 > p > r > 0 > q > -1$ implies that

$$\begin{aligned} K'(t)&=\frac{a^r}{r+1}\left[(r-q)(r+q+1)t^{r+q}\right. \\ &\qquad\left.+(q+1)qt^{q-1}-(r+1)rt^{r-1}\right] \\ &\quad+\frac{a^p}{p+1}\left[(p-q)(p+q+1)t^{p+q}\right. \\ &\qquad\left.+(q+1)qt^{q-1}-(p+1)pt^{p-1}\right] \\ &<\frac{a^r}{r+1}\left[(r-q)(r+q+1)\right. \\ &\qquad\left.+(q+1)q-(r+1)r\right]t^{r+q} \\ &\quad+\frac{a^p}{p+1}\left[(p-q)(p+q+1)+(q+1)q\right. \\ &\qquad\left.-(p+1)p\right]t^{p+q}=0,\quad t\in(0,1). \end{aligned} \tag{44}$$

With $K(1) = 0$ and (44), we have that $K(t) > 0$ for $t \in (0,1)$.

It follows that

$$\frac{\partial}{\partial t}\frac{P(a,t)}{Q(a,t)}=\frac{q+1}{\left(1-t^{q+1}\right)^2}a^{-q}K(t)>0,\quad t\in(0,1). \tag{45}$$

Combining

$$\begin{aligned} \lim_{t\to 0}\frac{P(a,t)}{Q(a,t)}&=\frac{q+1}{r+1}a^{r-q}+\frac{q+1}{p+1}a^{p-q}, \\ \lim_{t\to 1}\frac{P(a,t)}{Q(a,t)}&=a^{r-q}+a^{p-q}, \end{aligned} \tag{46}$$

we have the results of this lemma. □

Lemma 12. *For $1 > p > r > 0 > q > -1$,*

$$\lim_{a\to\beta}F'(a)=\frac{1}{2\sqrt{2}}\left(\frac{q+1}{\beta^{q+1}}\right)^{1/2}\theta(r,p,q). \tag{47}$$

Proof. From Lemma 7 we have that

$$\begin{aligned}
&2\sqrt{2}F'(a)\\
&\quad=\int_0^1\left(\left(\frac{1-r}{r+1}a^{r+1}\left(1-t^{r+1}\right)+\frac{1-p}{p+1}a^{p+1}\left(1-t^{p+1}\right)\right.\right.\\
&\qquad\left.-\frac{1-q}{q+1}a^{q+1}\left(1-t^{q+1}\right)\right)\\
&\qquad\left.\times\left([P(a,t)-Q(a,t)]^{3/2}\right)^{-1}\right)dt\\
&\quad=\int_0^1\frac{1-r}{[P(a,t)-Q(a,t)]^{1/2}}dt\\
&\qquad-\frac{r-q}{q+1}a^{q+1}\int_0^1\frac{1-t^{q+1}}{[P(a,t)-Q(a,t)]^{3/2}}dt\\
&\qquad-\frac{p-r}{p+1}a^{p+1}\int_0^1\frac{1-t^{p+1}}{[P(a,t)-Q(a,t)]^{3/2}}dt.
\end{aligned}\tag{48}$$

It follows from Lebesgue theorem that

$$\begin{aligned}
&\lim_{a\to\beta}\int_0^1\frac{1-r}{[P(a,t)-Q(a,t)]^{1/2}}dt\\
&\quad=\int_0^1\frac{1-r}{[P(\beta,t)-Q(\beta,t)]^{1/2}}dt,\\
&\lim_{a\to\beta}\frac{r-q}{q+1}a^{q+1}\int_0^1\frac{1-t^{q+1}}{[P(a,t)-Q(a,t)]^{3/2}}dt\\
&\qquad+\lim_{a\to\beta}\frac{p-r}{p+1}a^{p+1}\int_0^1\frac{1-t^{p+1}}{[P(a,t)-Q(a,t)]^{3/2}}dt\\
&\quad=\frac{r-q}{q+1}\beta^{q+1}\int_0^1\frac{1-t^{q+1}}{[P(\beta,t)-Q(\beta,t)]^{3/2}}dt\\
&\qquad+\frac{p-r}{p+1}\beta^{p+1}\int_0^1\frac{1-t^{p+1}}{[P(\beta,t)-Q(\beta,t)]^{3/2}}dt.
\end{aligned}\tag{49}$$

Finally, $\int_0^\beta(z^r+z^p-z^q)dz=0$ implies that

$$\begin{aligned}
&P(\beta,t)-Q(\beta,t)\\
&\quad=\int_0^{\beta t}\left(z^q-z^r-z^p\right)dz\\
&\quad=\frac{(\beta t)^{q+1}}{q+1}\left(1-m_1\beta^{r-q}t^{r-q}-m_2\beta^{p-q}t^{p-q}\right).
\end{aligned}\tag{50}$$

Combing (49) and (50), we complete the proof of this lemma. □

Lemma 13. *$F(a)$ has continuous derivatives up to second order on (β,∞) as follows:*

$$F'(a)=\frac{1-r}{2\sqrt{2}}\int_0^1 G(a,t)\,h_1(S(a,t))\,dt,\quad a>\beta,\tag{51}$$

$$F''(a)=\frac{1-r^2}{4a\sqrt{2}}\int_0^1 G(a,t)\,h_2(S(a,t))\,dt,\quad a>\beta,\tag{52}$$

where $G(a,t)=[S(a,t)-1]^{-5/2}[Q(a,t)]^{-1/2}$.

Proof. Equation (51) can be obtained by

$$\begin{aligned}
H_1(a,t)&=2[P(a,t)-Q(a,t)]^{-3/2}\\
&\quad\times\left[(1-r)P(a,t)-(1-q)Q(a,t)\right.\\
&\qquad\left.-\frac{p-r}{p+1}a^{p+1}\left(1-t^{p+1}\right)\right]\\
&=2\left[\frac{P(a,t)}{Q(a,t)}-1\right]^{-5/2}[Q(a,t)]^{-1/2}\\
&\quad\times(1-r)\left[\frac{P(a,t)}{Q(a,t)}-\frac{1-q}{1-r}-\frac{(p-r)(q+1)}{(p+1)(1-r)}\right.\\
&\qquad\left.\times a^{p-q}\frac{1-t^{p+1}}{1-t^{q+1}}\right]\left[\frac{P(a,t)}{Q(a,t)}-1\right]\\
&=2(1-r)G(a,t)\,h_1(S(a,t))
\end{aligned}\tag{53}$$

and Lemma 7, immediately. From (24) we have that

$$\begin{aligned}
a\frac{\partial H_1(a,t)}{\partial a}&=-3(P-Q)^{-5/2}\\
&\quad\times\left[(r+1)P-(q+1)Q\right.\\
&\qquad\left.+\frac{p-r}{p+1}a^{p+1}\left(1-t^{p+1}\right)\right]\\
&\quad\times\left[(1-r)P-(1-q)Q-\frac{p-r}{p+1}a^{p+1}\right.\\
&\qquad\left.\times\left(1-t^{p+1}\right)\right]+2(P-Q)^{-3/2}\\
&\quad\times\left[\left(1-r^2\right)P-\left(1-q^2\right)Q\right.\\
&\qquad\left.-\frac{p^2-r^2}{p+1}a^{p+1}\left(1-t^{p+1}\right)\right]\\
&=\left(\frac{P}{Q-1}\right)^{-3/2}\left(1-r^2\right)\\
&\quad\times\left\{-3\left[P-\frac{q+1}{r+1}Q+\frac{(p-r)}{(p+1)(r+1)}\right.\right.\\
&\qquad\left.\times a^{p+1}\left(1-t^{p+1}\right)\right]\\
&\qquad\times\left[P-\frac{1-q}{1-r}Q-\frac{(p-r)}{(p+1)(1-r)}a^{p+1}\right.\\
&\qquad\left.\times\left(1-t^{p+1}\right)\right]+2(P-Q)
\end{aligned}$$

$$\times \left[P - \frac{1-q^2}{1-r^2} Q - \frac{\left(p^2 - r^2\right)}{(p+1)(1-r^2)} \times a^{p+1}\left(1 - t^{p+1}\right) \right] \Bigg\}$$
$$= \left(1 - r^2\right) G(a,t)\, h_2(S(a,t)), \tag{54}$$

and so (52) holds. □

Lemma 14. *There exist $\delta(r,p,q) > 0$ and $\eta(r,p,q) > 0$ such that*

$$F''(a) + \delta(r,p,q) F'(a) \frac{1+r}{2a} > \eta(r,p,q)\left(a^{r-q} + a^{p-q}\right)^{-5/2} a^{-(3+q)/2} \quad \text{for } a > \beta. \tag{55}$$

Proof. From (41) and $1 > p > r > 0 > q > -1$, we have that

$$\begin{aligned} \min_{s\in[1,n]} h_2(s) &= \min\{h_2(1), h_2(n)\} \\ &= (n-1)\min\{3(1-m), 2(n-l)\} \\ &\geq 2(n-1)\min\{(1-m),(n-l)\} \\ &> 2(n-1)\min\left\{\frac{r-q}{r+1}, \frac{(1-q)(r-q)}{1-r^2}\right\} \\ &= 2(n-1)\frac{r-q}{r+1}. \end{aligned} \tag{56}$$

And since $h_1((1+n)/2) = -((n-1)^2/4) < 0$, let $\delta(r,p,q) > 0$ satisfy

$$\min_{s\in[1,n]} \left[h_2(s) + \delta(r,p,q) h_1(s)\right] = 2(n-1)\frac{r-q}{r+1}. \tag{57}$$

Set

$$h(s) = h_2(s) + \delta(r,p,q) h_1(s), \quad s \in (-\infty, \infty). \tag{58}$$

From $h(s) = (\delta-1)s^2 + (3m+3n-2l-2-\delta-\delta n)s - 3mn + 2l + \delta n$, we have that $\delta(r,p,q) - 1 > 0$ and

$$h(s) \geq \min_{s\in[1,n]} h(s) = 2(n-1)\frac{r-q}{r+1}, \quad s \in (-\infty, \infty). \tag{59}$$

On the other hand, from (10), (38), and Lemma 11, we have that

$$\begin{aligned} 0 < Q(a,t) &< \frac{1}{q+1} a^{q+1}, \\ 0 < \frac{q+1}{a^{q+1}} \int_0^a \left(z^r + z^p - z^q\right) dz &= \frac{q+1}{r+1} a^{r-q} + \frac{q+1}{p+1} a^{p-q} - 1 \\ &< S(a,t) - 1 < S(a,t) < a^{r-q} + a^{p-q}, \end{aligned} \tag{60}$$

for $a > \beta$, $t \in (0,1)$.

This means that

$$\begin{aligned} G(a,t) &= [S(a,t) - 1]^{-5/2} [Q(a,t)]^{-1/2} \\ &> \left(a^{r-q} + a^{p-q}\right)^{-5/2} a^{-(1+q)/2} \sqrt{1+q}, \end{aligned} \tag{61}$$

for $a > \beta$, $t \in (0,1)$. It follows from Lemma 13 and (58) that

$$\begin{aligned} F''(a) + \delta(r,p,q) F'(a) \frac{1+r}{2a} &= \frac{1-r^2}{4a\sqrt{2}} \int_0^1 G(a,t)\, h(S(a,t))\, dt. \end{aligned} \tag{62}$$

Now, from (59)–(62) we have (55), where

$$\eta(r,p,q) = \frac{(1-r)(r-q)\sqrt{1+q}}{2\sqrt{2}} \int_0^1 [n(t) - 1]\, dt. \tag{63}$$

□

Lemma 15. *If $\theta(r,p,q) \geq 0$, then $F'(a) > 0$ for $a > \beta$. If $\theta(r,p,q) < 0$, then there exists $a^* > \beta$, such that $F'(a) < 0$ for $a \in (\beta, a^*)$ and $F'(a) > 0$ for $a > a^*$.*

Proof. It follows from Lemma 14 that if

$$F'(a) = 0, \quad \text{then } F''(a) > 0, \tag{64}$$

and then

$$F'(a) \text{ has at most one zero in } (\beta, \infty). \tag{65}$$

By (40), (42), and (51) we can obtain that

$$F'(a) > 0 \quad \text{for } \frac{P(a,t)}{Q(a,t)} > n(t). \tag{66}$$

With (10) and (41), (66) implies that

$$F'(a) > 0 \quad \text{for } \frac{q+1}{r+1} a^r + \left(\frac{q+1}{p+1} - \frac{p-r}{1-r}\right) a^p > \frac{1-q}{1-r} a^q. \tag{67}$$

If $\theta(r,p,q) \neq 0$, then from Lemma 12 and (64)–(67) we have the results of this lemma, immediately. If $\theta(r,p,q) = 0$, then by Lemmas 12 and 14 we have that $F''(a) > 0$ for a near β, and so $F'(a) > 0$ for a near β. Thus, it follows from (64)–(67) that $F'(a) > 0$ for $a > \beta$. □

Proof of Theorem 5. From Lemmas 6, 8, 9, and 15, we can obtain the results of Theorem 5. □

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