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*Research Article*

# Solving Optimization Problems on Hermitian Matrix Functions with Applications

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We consider the extremal inertias and ranks of the matrix expressions $f(X,Y) = A_3 - B_3X - (B_3X)^* - C_3YD_3 - (C_3YD_3)^*$, where $A_3 = A_3^*$, $B_3$, $C_3$, and $D_3$ are known matrices and $Y$ and $X$ are the solutions to the matrix equations $A_1Y = C_1$, $YB_1 = D_1$, and $A_2X = C_2$, respectively. As applications, we present necessary and sufficient condition for the previous matrix function $f(X,Y)$ to be positive (negative), non-negative (positive) definite or nonsingular. We also characterize the relations between the Hermitian part of the solutions of the above-mentioned matrix equations. Furthermore, we establish necessary and sufficient conditions for the solvability of the system of matrix equations $A_1Y = C_1$, $YB_1 = D_1$, $A_2X = C_2$, and $B_3X + (B_3X)^* + C_3YD_3 + (C_3YD_3)^* = A_3$, and give an expression of the general solution to the above-mentioned system when it is solvable.

## 1. Introduction

Throughout, we denote the field of complex numbers by $\mathbb{C}$, the set of all $m \times n$ matrices over $\mathbb{C}$ by $\mathbb{C}^{m\times n}$, and the set of all $m \times m$ Hermitian matrices by $\mathbb{C}_h^{m\times m}$. The symbols $A^*$ and $\mathscr{R}(A)$ stand for the conjugate transpose, the column space of a complex matrix $A$ respectively. $I_n$ denotes the $n \times n$ identity matrix. The Moore-Penrose inverse [1] $A^\dagger$ of $A$, is the unique solution $X$ to the four matrix equations:

$$\begin{aligned} &\text{(i) } AXA = A,\\ &\text{(ii) } XAX = X,\\ &\text{(iii) } (AX)^* = AX,\\ &\text{(iv) } (XA)^* = XA. \end{aligned} \tag{1}$$

Moreover, $L_A$ and $R_A$ stand for the projectors $L_A = I - A^\dagger A$, $R_A = I - AA^\dagger$ induced by $A$. It is well known that the eigenvalues of a Hermitian matrix $A \in \mathbb{C}^{n\times n}$ are real, and the inertia of $A$ is defined to be the triplet

$$\mathbb{I}_n(A) = \{i_+(A), i_-(A), i_0(A)\}, \tag{2}$$

where $i_+(A)$, $i_-(A)$, and $i_0(A)$ stand for the numbers of positive, negative, and zero eigenvalues of $A$, respectively. The symbols $i_+(A)$ and $i_-(A)$ are called the positive index and the negative index of inertia, respectively. For two Hermitian matrices $A$ and $B$ of the same sizes, we say $A \geq B$ ($A \leq B$) in the Löwner partial ordering if $A - B$ is positive (negative) semidefinite. The Hermitian part of $X$ is defined as $H(X) = X + X^*$. We will say that $X$ is Re-nnd (Re-nonnegative semidefinite) if $H(X) \geq 0$, $X$ is Re-pd (Re-positive definite) if $H(X) > 0$, and $X$ is Re-ns if $H(X)$ is nonsingular.

It is well known that investigation on the solvability conditions and the general solution to linear matrix equations is very active (e.g., [2–9]). In 1999, Braden [10] gave the general solution to

$$BX + (BX)^* = A. \tag{3}$$

In 2007, Djordjević [11] considered the explicit solution to (3) for linear bounded operators on Hilbert spaces. Moreover, Cao [12] investigated the general explicit solution to

$$BXC + (BXC)^* = A. \tag{4}$$

Xu et al. [13] obtained the general expression of the solution of operator equation (4). In 2012, Wang and He [14] studied some necessary and sufficient conditions for the consistence of the matrix equation

$$A_1X + (A_1X)^* + B_1YC_1 + (B_1YC_1)^* = E_1 \tag{5}$$

and presented an expression of the general solution to (5).

Note that (5) is a special case of the following system:

$$\begin{gathered} A_1Y = C_1, \qquad YB_1 = D_1, \qquad A_2X = C_2, \\ B_3X + (B_3X)^* + C_3YD_3 + (C_3YD_3)^* = A_3. \end{gathered} \tag{6}$$

To our knowledge, there has been little information about (6). One goal of this paper is to give some necessary and sufficient conditions for the solvability of the system of matrix (6) and present an expression of the general solution to system (6) when it is solvable.

In order to find necessary and sufficient conditions for the solvability of the system of matrix equations (6), we need to consider the extremal ranks and inertias of (10) subject to (13) and (11).

There have been many papers to discuss the extremal ranks and inertias of the following Hermitian expressions:

$$p(X) = A_3 - B_3X - (B_3X)^*, \tag{7}$$

$$g(Y) = A - BYC - (BYC)^*, \tag{8}$$

$$h(X,Y) = A_1 - B_1XB_1^* - C_1YC_1^*, \tag{9}$$

$$f(X,Y) = A_3 - B_3X - (B_3X)^* - C_3YD_3 - (C_3YD_3)^*. \tag{10}$$

Tian has contributed much in this field. One of his works [15] considered the extremal ranks and inertias of (7). He and Wang [16] derived the extremal ranks and inertias of (7) subject to $A_1X = C_1$, $A_2XB_2 = C_2$. Liu and Tian [17] studied the extremal ranks and inertias of (8). Chu et al. [18] and Liu and Tian [19] derived the extremal ranks and inertias of (9). Zhang et al. [20] presented the extremal ranks and inertias of (9), where $X$ and $Y$ are Hermitian solutions of

$$A_2X = C_2, \tag{11}$$

$$YB_2 = D_2, \tag{12}$$

respectively. He and Wang [16] derived the extremal ranks and inertias of (10). We consider the extremal ranks and inertias of (10) subject to (11) and

$$A_1Y = C_1, \qquad YB_1 = D_1, \tag{13}$$

which is not only the generalization of the above matrix functions, but also can be used to investigate the solvability conditions for the existence of the general solution to the system (6). Moreover, it can be applied to characterize the relations between Hermitian part of the solutions of (11) and (13).

The remainder of this paper is organized as follows. In Section 2, we consider the extremal ranks and inertias of (10) subject to (11) and (13). In Section 3, we characterize the relations between the Hermitian part of the solution to (11) and (13). In Section 4, we establish the solvability conditions for the existence of a solution to (6) and obtain an expression of the general solution to (6).

## 2. Extremal Ranks and Inertias of Hermitian Matrix Function (10) with Some Restrictions

In this section, we consider formulas for the extremal ranks and inertias of (10) subject to (11) and (13). We begin with the following Lemmas.

**Lemma 1** (see [21]). *(a) Let $A_1$, $C_1$, $B_1$, and $D_1$ be given. Then the following statements are equivalent:*

(1) *system* (13) *is consistent,*

(2)

$$R_{A_1}C_1 = 0, \qquad D_1L_{B_1} = 0, \qquad A_1D_1 = C_1B_1. \tag{14}$$

(3)

$$\begin{gathered} r\begin{bmatrix} A_1 & C_1 \end{bmatrix} = r(A_1), \\ \begin{bmatrix} D_1 \\ B_1 \end{bmatrix} = r(B_1), \\ A_1D_1 = C_1B_1. \end{gathered} \tag{15}$$

*In this case, the general solution can be written as*

$$Y = A_1^{\dagger}C_1 + L_{A_1}D_1B_1^{\dagger} + L_{A_1}VR_{B_1}, \tag{16}$$

*where $V$ is arbitrary.*

*(b) Let $A_2$ and $C_2$ be given. Then the following statements are equivalent:*

(1) *equation* (11) *is consistent,*

(2)

$$R_{A_2}C_2 = 0, \tag{17}$$

(3)

$$r\begin{bmatrix} A_2 & C_2 \end{bmatrix} = r(A_2). \tag{18}$$

*In this case, the general solution can be written as*

$$X = A^{\dagger}C + L_AW, \tag{19}$$

*where $W$ is arbitrary.*

**Lemma 2** ([22, Lemma 1.5, Theorem 2.3]). *Let $A \in \mathbb{C}_h^{m\times m}$, $B \in \mathbb{C}^{m\times n}$, and $D \in \mathbb{C}_h^{n\times n}$, and denote that*

$$\begin{gathered} M = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \\ N = \begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix}, \\ L = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}, \\ G = \begin{bmatrix} P & ML_N \\ L_NM^* & 0 \end{bmatrix}. \end{gathered} \tag{20}$$

*Then one has the following*

(a) *the following equalities hold*

$$i_{\pm}(M) = r(B) + i_{\pm}(R_B A R_B), \tag{21}$$

$$i_{\pm}(N) = r(Q), \tag{22}$$

(b) *if* $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, *then* $i_{\pm}(L) = i_{\pm}(A) + i_{\pm}(D - B^* A^{\dagger} B)$). *Thus* $i_{\pm}(L) = i_{\pm}(A)$ *if and only if* $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ *and* $i_{\pm}(D - B^* A^{\dagger} B) = 0$,

(c)

$$i_{\pm}(G) = \begin{bmatrix} P & M & 0 \\ M^* & 0 & N^* \\ 0 & N & 0 \end{bmatrix} - r(N). \tag{23}$$

**Lemma 3** (see [23]). *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, *and* $C \in \mathbb{C}^{l\times n}$. *Then they satisfy the following rank equalities:*

(a) $r[A \quad B] = r(A) + r(E_A B) = r(B) + r(E_B A)$,

(b) $r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C)$,

(c) $r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_B A F_C)$,

(d) $r[B \quad AF_C] = r\begin{bmatrix} B & A \\ 0 & C \end{bmatrix} - r(C)$,

(e) $r\begin{bmatrix} C \\ E_B A \end{bmatrix} = r\begin{bmatrix} C & 0 \\ A & B \end{bmatrix} - r(B)$,

(f) $r\begin{bmatrix} A & BF_D \\ E_E C & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & E \\ 0 & D & 0 \end{bmatrix} - r(D) - r(E)$,

**Lemma 4** (see [15]). *Let* $A \in \mathbb{C}_h^{m\times m}$, $B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}_h^{n\times n}$, $Q \in \mathbb{C}^{m\times n}$, *and* $P \in \mathbb{C}^{p\times n}$ *be given, and* $T \in \mathbb{C}^{m\times m}$ *be nonsingular. Then one has the following*

(1) $i_{\pm}(TAT^*) = i_{\pm}(A)$,

(2) $i_{\pm}\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} = i_{\pm}(A) + i_{\pm}(C)$,

(3) $i_{\pm}\begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix} = r(Q)$,

(4) $i_{\pm}\begin{bmatrix} A & BL_P \\ L_P B^* & 0 \end{bmatrix} + r(P) = i_{\pm}\begin{bmatrix} A & B & 0 \\ B^* & 0 & P^* \\ 0 & P & 0 \end{bmatrix}$.

**Lemma 5** (see [22, Lemma 1.4]). *Let $S$ be a set consisting of (square) matrices over* $\mathbb{C}^{m\times m}$, *and let $H$ be a set consisting of (square) matrices over* $\mathbb{C}_h^{m\times m}$. *Then Then one has the following*

(a) *$S$ has a nonsingular matrix if and only if* $\max_{X\in S} r(X) = m$;

(b) *any* $X \in S$ *is nonsingular if and only if* $\min_{X\in S} r(X) = m$;

(c) $\{0\} \in S$ *if and only if* $\min_{X\in S} r(X) = 0$;

(d) $S = \{0\}$ *if and only if* $\max_{X\in S} r(X) = 0$;

(e) *$H$ has a matrix* $X > 0\,(X < 0)$ *if and only if* $\max_{X\in H} i_{+}(X) = m(\max_{X\in H} i_{-}(X) = m)$;

(f) *any* $X \in H$ *satisfies* $X > 0\,(X < 0)$ *if and only if* $\min_{X\in H} i_{+}(X) = m\ (\min_{X\in H} i_{-}(X) = m)$;

(g) *$H$ has a matrix* $X \geq 0\,(X \leq 0)$ *if and only if* $\min_{X\in H} i_{-}(X) = 0\ (\min_{X\in H} i_{+}(X) = 0)$;

(h) *any* $X \in H$ *satisfies* $X \geq 0\,(X \leq 0)$ *if and only if* $\max_{X\in H} i_{-}(X) = 0\ (\max_{X\in H} i_{+}(X) = 0)$.

**Lemma 6** (see [16]). *Let* $p(X,Y) = A - BX - (BX)^* - CYD - (CYD)^*$, *where A, B, C, and D are given with appropriate sizes, and denote that*

$$M_1 = \begin{bmatrix} A & B & C \\ B^* & 0 & 0 \\ C^* & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} A & B & D^* \\ B^* & 0 & 0 \\ D & 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} A & B & C & D^* \\ B^* & 0 & 0 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} A & B & C & D^* \\ B^* & 0 & 0 & 0 \\ C^* & 0 & 0 & 0 \end{bmatrix}, \quad M_5 = \begin{bmatrix} A & B & C & D^* \\ B^* & 0 & 0 & 0 \\ D & 0 & 0 & 0 \end{bmatrix}. \tag{24}$$

*Then one has the following:*

(1) *the maximal rank of* $p(X,Y)$ *is*

$$\max_{X\in\mathbb{C}^{n\times m},Y} r[p(X,Y)] = \min\{m, r(M_1), r(M_2), r(M_3)\}, \tag{25}$$

(2) *the minimal rank of* $p(X,Y)$ *is*

$$\min_{X\in\mathbb{C}^{n\times m},Y} r[p(X,Y)] = 2r(M_3) - 2r(B) + \max\{u_{+} + u_{-}, v_{+} + v_{-}, u_{+} + v_{-}, u_{-} + v_{+}\}, \tag{26}$$

(3) *the maximal inertia of* $p(X,Y)$ *is*

$$\max_{X\in\mathbb{C}^{n\times m},Y} i_{\pm}[p(X,Y)] = \min\{i_{\pm}(M_1), i_{\pm}(M_2)\}, \tag{27}$$

(4) *the minimal inertias of* $p(X,Y)$ *is*

$$\min_{X\in\mathbb{C}^{n\times m},Y} i_{\pm}[p(X,Y)] = r(M_3) - r(B) + \max\{i_{\pm}(M_1) - r(M_4), i_{\pm}(M_2) - r(M_5)\}, \tag{28}$$

*where*

$$u_{\pm} = i_{\pm}(M_1) - r(M_4), \qquad v_{\pm} = i_{\pm}(M_2) - r(M_5). \tag{29}$$

Now we present the main theorem of this section.

**Theorem 7.** *Let $A_1 \in \mathbb{C}^{m\times n}$, $C_1 \in \mathbb{C}^{m\times k}$, $B_1 \in \mathbb{C}^{k\times l}$, $D_1 \in \mathbb{C}^{n\times l}$, $A_2 \in \mathbb{C}^{t\times q}$, $C_2 \in \mathbb{C}^{t\times p}$, $A_3 \in \mathbb{C}_h^{p\times p}$, $B_3 \in \mathbb{C}^{p\times q}$, $C_3 \in \mathbb{C}^{p\times n}$, and $D_3 \in \mathbb{C}^{p\times n}$ be given, and suppose that the system of matrix equations* (13) *and* (11) *is consistent, respectively. Denote the set of all solutions to* (13) *by $S$ and* (11) *by $G$. Put*

$$E_1 = \begin{bmatrix} A_3 & C_3 & D_3^*C_1^* & B_3 & C_2^* \\ C_3^* & 0 & A_1^* & 0 & 0 \\ C_1D_3 & A_1 & 0 & 0 & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix},$$

$$E_2 = r\begin{bmatrix} A_3 & D_3^* & C_3D_1 & B_3 & C_2^* \\ D_3 & 0 & B_1 & 0 & 0 \\ D_1^*C_3^* & B_1^* & 0 & 0 & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} A_3 & B_3 & C_3^* & D_3^* & C_2^* \\ B_3^* & 0 & 0 & 0 & A_2^* \\ D_1^*C_3^* & 0 & 0 & B_1^* & 0 \\ C_1D_3 & 0 & A_1 & 0 & 0 \\ C_2 & A_2 & 0 & 0 & 0 \end{bmatrix}, \tag{30}$$

$$E_4 = \begin{bmatrix} A_3 & B_3 & C_3 & D_3^* & C_2^* & D_3^*C_1^* \\ B_3^* & 0 & 0 & 0 & A_2^* & 0 \\ C_3^* & 0 & 0 & 0 & 0 & A_1^* \\ 0 & 0 & 0 & B_1^* & 0 & 0 \\ C_1D_3 & 0 & A_1 & 0 & 0 & 0 \\ C_2 & A_2 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_5 = \begin{bmatrix} A_3 & B_3 & C_3 & D_3^* & C_2^* & C_3D_1 \\ B_3^* & 0 & 0 & 0 & A_2^* & 0 \\ D_3 & 0 & 0 & 0 & 0 & B_1 \\ D_1^*C_3^* & 0 & 0 & A_1^* & 0 & 0 \\ 0 & 0 & A_1 & 0 & 0 & 0 \\ C_2 & A_2 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

*Then one has the following:*

(a) *the maximal rank of* (10) *subject to* (13) *and* (11) *is*

$$\begin{aligned}\max_{X\in G, Y\in S} r[f(X,Y)] &= \min\{p, r(E_1) - 2r(A_1) - 2r(A_2), \\ &\quad r(E_2) - 2r(B_1) - 2r(A_2), \\ &\quad r(E_3) - 2r(A_2) - r(A_1) - r(B_1)\},\end{aligned} \tag{31}$$

(b) *the minimal rank of* (10) *subject to* (13) *and* (11) *is*

$$\begin{aligned}\min_{X\in G, Y\in S} r[f(X,Y)] &= 2r(E_3) - 2r\begin{bmatrix} B_3 \\ A_2 \end{bmatrix} \\ &\quad + \max\{r(E_1) - 2r(E_4), r(E_2) - 2r(E_5), \\ &\quad i_+(E_1) + i_-(E_2) - r(E_4) - r(E_5), \\ &\quad i_-(E_1) + i_+(E_2) - r(E_4) - r(E_5)\},\end{aligned} \tag{32}$$

(c) *the maximal inertia of* (10) *subject to* (13) *and* (11) *is*

$$\begin{aligned}\max_{X\in G, Y\in Sx} i_\pm[f(X,Y)] &= \min\{i_\pm(E_1) - r(A_1) - r(A_2), \\ &\quad i_\pm(E_2) - r(B_1) - r(A_2)\},\end{aligned} \tag{33}$$

(d) *the minimal inertia of* (10) *subject to* (13) *and* (11) *is*

$$\begin{aligned}\min_{X\in G, Y\in S} i_\pm[f(X,Y)] &= r(E_3) - r\begin{bmatrix} B_3 \\ A_2 \end{bmatrix} \\ &\quad + \max\{i_\pm(E_1) - r(E_4), \\ &\quad i_\pm(E_2) - r(E_5)\}.\end{aligned} \tag{34}$$

*Proof.* By Lemma 1, the general solutions to (13) and (11) can be written as

$$\begin{aligned} X &= A_2^\dagger C_2 + L_{A_2}W, \\ Y &= A_1^\dagger C_1 + L_{A_1}D_1B_1^\dagger + L_{A_1}ZR_{B_1}, \end{aligned} \tag{35}$$

where $W$ and $Z$ are arbitrary matrices with appropriate sizes. Put

$$\begin{aligned} Q &= B_3L_{A_2}, \qquad T = C_3L_{A_1}, \qquad J = R_{B_1}D_3, \\ P &= A_3 - B_3A_2^\dagger C_2 - \left(B_3A_2^\dagger C_2\right)^* \\ &\quad - C_3\left(A_1^\dagger C_1 + L_{A_1}D_1B_1^\dagger\right)D_3 \\ &\quad - \left(C_3\left(A_1^\dagger C_1 + L_{A_1}D_1B_1^\dagger\right)D_3\right)^*. \end{aligned} \tag{36}$$

Substituting (36) into (10) yields

$$f(X,Y) = P - QW - (QW)^* - TZJ - (TZJ)^*. \tag{37}$$

Clearly $P$ is Hermitian. It follows from Lemma 6 that

$$\begin{aligned}\max_{X\in G, Y\in S} r[f(X,Y)] &= \max_{W,Z} r(P - QW - (QW)^* - TZJ - (TZJ)^*) \\ &= \min\{m, r(N_1), r(N_2), r(N_3)\},\end{aligned} \tag{38}$$

$$\begin{aligned}\min_{X\in G, Y\in S} r[f(X,Y)] &= \max_{W,Z} r(P - QW - (QW)^* - TZJ - (TZJ)^*) \\ &= 2r(N_3) - 2r(Q) \\ &\quad + \max\{s_+ + s_-, t_+ + t_-, s_+ + t_-, s_- + t_+\},\end{aligned} \tag{39}$$

$$\max_{X\in G,Y\in S} i_{\pm}[f(X,Y)] = \max_{W,Z} r(P - QW - (QW)^* - TZJ - (TZJ)^*) = \min\{i_{\pm}(N_1), i_{\pm}(N_2)\}, \tag{40}$$

$$\min_{X\in G,Y\in S} i_{\pm}[f(X,Y)] = \max_{W,Z} r(P - QW - (QW)^* - TZJ - (TZJ)^*) = r(N_3) - r(Q) + \max\{s_{\pm}, t_{\pm}\}, \tag{41}$$

where

$$N_1 = \begin{bmatrix} P & Q & T \\ Q^* & 0 & 0 \\ T^* & 0 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} P & Q & J^* \\ Q^* & 0 & 0 \\ J & 0 & 0 \end{bmatrix}, \quad N_3 = \begin{bmatrix} P & Q & T & J^* \\ Q^* & 0 & 0 & 0 \end{bmatrix},$$
$$N_4 = \begin{bmatrix} P & Q & T & J^* \\ Q^* & 0 & 0 & 0 \\ T^* & 0 & 0 & 0 \end{bmatrix}, \quad N_5 = \begin{bmatrix} P & Q & T & J^* \\ Q^* & 0 & 0 & 0 \\ J & 0 & 0 & 0 \end{bmatrix},$$
$$s_{\pm} = i_{\pm}(N_1) - r(N_4), \quad t_{\pm} = i_{\pm}(N_2) - r(N_5). \tag{42}$$

Now, we simplify the ranks and inertias of block matrices in (38)–(41).

By Lemma 4, block Gaussian elimination, and noting that

$$L_S^* = (I - S^{\dagger}S)^* = I - S^*(S^*)^{\dagger} = R_{S^*}, \tag{43}$$

we have the following:

$$r(N_1) = r\begin{bmatrix} P & Q & T \\ Q^* & 0 & 0 \\ T^* & 0 & 0 \end{bmatrix} = r\begin{bmatrix} A_3 & C_3 & D_3^*C_1^* & B_3 & C_2^* \\ C_3^* & 0 & A_1^* & 0 & 0 \\ C_1D_3 & A_1 & 0 & 0 & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix} - 2r(A_1) - 2r(A_2). \tag{44}$$

By $C_1B_1 = A_1D_1$, we obtain

$$r(N_2) = r\begin{bmatrix} P & Q & J^* \\ Q^* & 0 & 0 \\ J & 0 & 0 \end{bmatrix} = r\begin{bmatrix} A_3 & D_3^* & C_3D_1 & B_3 & C_2^* \\ D_3 & 0 & B_1 & 0 & 0 \\ D_1^*C_3^* & B_1^* & 0 & 0 & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix} - 2r(B_1) - 2r(A_2),$$

$$r(N_3) = r\begin{bmatrix} P & Q & T & J^* \\ Q^* & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} A_3 & B_3 & C_3^* & D_3^* & C_2^* \\ B_3^* & 0 & 0 & 0 & A_2^* \\ D_1^*C_3^* & 0 & 0 & B_1^* & 0 \\ C_1D_3 & 0 & A_1 & 0 & 0 \\ C_2 & A_2 & 0 & 0 & 0 \end{bmatrix} - r(B_1) - 2r(A_2) - r(A_1),$$

$$r(N_4) = r\begin{bmatrix} P & Q & T & J^* \\ Q^* & 0 & 0 & 0 \\ T^* & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} A_3 & B_3 & C_3 & D_3^* & C_2^* & D_3^*C_1^* \\ B_3^* & 0 & 0 & 0 & A_2^* & 0 \\ C_3^* & 0 & 0 & 0 & 0 & A_1^* \\ 0 & 0 & 0 & B_1^* & 0 & 0 \\ C_1D_3 & 0 & A_1 & 0 & 0 & 0 \\ C_2 & A_2 & 0 & 0 & 0 & 0 \end{bmatrix} - r(B_1) - 2r(A_2) - 2r(A_1),$$

$$r(N_5) = r\begin{bmatrix} P & Q & T & J^* \\ Q^* & 0 & 0 & 0 \\ T^* & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} A_3 & B_3 & C_3 & D_3^* & C_2^* & C_3D_1 \\ B_3^* & 0 & 0 & 0 & A_2^* & 0 \\ D_3 & 0 & 0 & 0 & 0 & B_1 \\ D_1^*C_3^* & 0 & 0 & A_1^* & 0 & 0 \\ 0 & 0 & A_1 & 0 & 0 & 0 \\ C_2 & A_2 & 0 & 0 & 0 & 0 \end{bmatrix} - 2r(B_1) - 2r(A_2) - r(A_1). \tag{45}$$

By Lemma 2, we can get the following:

$$i_{\pm}(N_1) = i_{\pm}\begin{bmatrix} P & Q & T \\ Q^* & 0 & 0 \\ T^* & 0 & 0 \end{bmatrix}$$

$$= i_\pm \begin{bmatrix} A_3 & C_3 & D_3^* C_1^* & B_3 & C_2^* \\ C_3^* & 0 & A_1^* & 0 & 0 \\ C_1 D_3 & A_1 & 0 & 0 & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix} - r(A_1) - r(A_2), \tag{46}$$

$$i_\pm(N_2) = i_\pm \begin{bmatrix} P & Q & J^* \\ Q^* & 0 & 0 \\ J & 0 & 0 \end{bmatrix}$$
$$= i_\pm \begin{bmatrix} A_3 & D_3^* & C_3 D_1 & B_3 & C_2^* \\ D_3 & 0 & B_1 & 0 & 0 \\ D_1^* C_3^* & B_1^* & 0 & 0 & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix} - r(B_1) - r(A_2). \tag{47}$$

Substituting (44)-(47) into (38) and (41) yields (31)–(34), respectively. □

**Corollary 8.** *Let $A_1$, $C_1$, $B_1$, $D_1$, $A_2$, $C_2$, $A_3$, $B_3$, $C_3$, $D_3$, and $E_i$, ($i = 1, 2, \ldots, 5$) be as in Theorem 7, and suppose that the system of matrix equations* (13) *and* (11) *is consistent, respectively. Denote the set of all solutions to* (13) *by $S$ and* (11) *by $G$. Then, one has the following:*

(a) *there exist $X \in G$ and $Y \in S$ such that $A_3 - B_3X - (B_3X)^* - C_3YD_3 - (C_3YD_3)^* > 0$ if and only if*

$$i_+(E_1) - r(A_1) - r(A_2) \geq p,$$
$$i_+(E_2) - r(B_1) - r(A_2) \geq p. \tag{48}$$

(b) *there exist $X \in G$ and $Y \in S$ such that $A_3 - B_3X - (B_3X)^* - C_3YD_3 - (C_3YD_3)^* < 0$ if and only if*

$$i_-(E_1) - r(A_1) - r(A_2) \geq p,$$
$$i_-(E_2) - r(B_1) - r(A_2) \geq p, \tag{49}$$

(c) *there exist $X \in G$ and $Y \in S$ such that $A_3 - B_3X - (B_3X)^* - C_3YD_3 - (C_3YD_3)^* \geq 0$ if and only if*

$$r(E_3) - r\begin{bmatrix} B_3 \\ A_2 \end{bmatrix} + i_-(E_1) - r(E_4) \leq 0,$$
$$r(E_3) - r\begin{bmatrix} B_3 \\ A_2 \end{bmatrix} + i_-(E_2) - r(E_5) \leq 0. \tag{50}$$

(d) *there exist $X \in G$ and $Y \in S$ such that $A_3 - B_3X - (B_3X)^* - C_3YD_3 - (C_3YD_3)^* \leq 0$ if and only if*

$$r(E_3) - r\begin{bmatrix} B_3 \\ A_2 \end{bmatrix} + i_+(E_1) - r(E_4) \leq 0,$$
$$r(E_3) - r\begin{bmatrix} B_3 \\ A_2 \end{bmatrix} + i_+(E_2) - r(E_5) \leq 0, \tag{51}$$

(e) *$A_3 - B_3X - (B_3X)^* - C_3YD_3 - (C_3YD_3)^* > 0$ for all $X \in G$ and $Y \in S$ if and only if*

$$r(E_3) - r\begin{bmatrix} B_3 \\ A_2 \end{bmatrix} + i_+(E_1) - r(E_4) = p$$
$$\text{or} \quad r(E_3) - r\begin{bmatrix} B_3 \\ A_2 \end{bmatrix} + i_+(E_2) - r(E_5) = p, \tag{52}$$

(f) *$A_3 - B_3X - (B_3X)^* - C_3YD_3 - (C_3YD_3)^* < 0$ for all $X \in G$ and $Y \in S$ if and only if*

$$r(E_3) - r\begin{bmatrix} B_3 \\ A_2 \end{bmatrix} + i_-(E_1) - r(E_4) = p$$
$$\text{or} \quad r(E_3) - r\begin{bmatrix} B_3 \\ A_2 \end{bmatrix} + i_-(E_2) - r(E_5) = p, \tag{53}$$

(g) *$A_3 - B_3X - (B_3X)^* - C_3YD_3 - (C_3YD_3)^* \geq 0$ for all $X \in G$ and $Y \in S$ if and only if*

$$i_-(E_1) - r(A_1) - r(A_2) \leq 0$$
$$\text{or} \quad i_-(E_2) - r(B_1) - r(A_2) \leq 0, \tag{54}$$

(h) *$A_3 - B_3X - (B_3X)^* - C_3YD_3 - (C_3YD_3)^* \leq 0$ for all $X \in G$ and $Y \in S$ if and only if*

$$i_+(E_1) - r(A_1) - r(A_2) \leq 0$$
$$\text{or} \quad i_+(E_2) - r(B_1) - r(A_2) \leq 0, \tag{55}$$

(i) *there exist $X \in G$ and $Y \in S$ such that $A_3 - B_3X - (B_3X)^* - C_3YD_3 - (C_3YD_3)^*$ is nonsingular if and only if*

$$r(E_1) - 2r(A_1) - 2r(A_2) \geq p,$$
$$r(E_2) - 2r(B_1) - 2r(A_2) \geq p, \tag{56}$$
$$r(E_3) - 2r(A_2) - r(A_1) - r(B_1) \geq p.$$

## 3. Relations between the Hermitian Part of the Solutions to (13) and (11)

Now we consider the extremal ranks and inertias of the difference between the Hermitian part of the solutions to (13) and (11).

**Theorem 9.** *Let $A_1 \in \mathbb{C}^{m\times p}$, $C_1 \in \mathbb{C}^{m\times p}$, $B_1 \in \mathbb{C}^{p\times l}$, $D_1 \in \mathbb{C}^{p\times l}$, $A_2 \in \mathbb{C}^{t\times p}$, and $C_2 \in \mathbb{C}^{t\times p}$, be given. Suppose that the system of matrix equations* (13) *and* (11) *is consistent,* *respectively. Denote the set of all solutions to* (13) *by S and* (11) *by G. Put*

$$H_1 = \begin{bmatrix} 0 & I & C_1^* & -I & C_2^* \\ I & 0 & A_1^* & 0 & 0 \\ C_1 & A_1 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & A_2^* \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix},$$

$$H_2 = r\begin{bmatrix} 0 & I & D_1 & -I & C_2^* \\ I & 0 & B_1 & 0 & 0 \\ D_1^* & B_1^* & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & A_2^* \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix},$$

$$H_3 = \begin{bmatrix} 0 & -I & I & I & C_2^* \\ -I & 0 & 0 & 0 & A_2^* \\ D_1^* & 0 & 0 & B_1^* & 0 \\ C_1 & 0 & A_1 & 0 & 0 \\ C_2 & A_2 & 0 & 0 & 0 \end{bmatrix}, \quad (57)$$

$$H_4 = \begin{bmatrix} 0 & -I & I & I & C_2^* & C_1^* \\ -I & 0 & 0 & 0 & A_2^* & 0 \\ I & 0 & 0 & 0 & 0 & A_1^* \\ 0 & 0 & 0 & B_1^* & 0 & 0 \\ C_1 & 0 & A_1 & 0 & 0 & 0 \\ C_2 & A_2 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$H_5 = \begin{bmatrix} 0 & -I & I & I & C_2^* & D_1 \\ -I & 0 & 0 & 0 & A_2^* & 0 \\ I & 0 & 0 & 0 & 0 & B_1 \\ D_1^* & 0 & 0 & A_1^* & 0 & 0 \\ 0 & 0 & A_1 & 0 & 0 & 0 \\ C_2 & A_2 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

*Then one has the following:*

$$\max_{X\in G, Y\in S} r\left[(X+X^*)-(Y+Y^*)\right]$$
$$= \min\{p, r(H_1)-2r(A_1)-2r(A_2), r(H_2)-2r(B_1) -2r(A_2), r(H_3)-2r(A_2)-r(A_1)-r(B_1)\},$$

$$\min_{X\in G, Y\in S} r\left[(X+X^*)-(Y+Y^*)\right]$$
$$= 2r(H_3)-2p + \max\{r(H_1)-2r(H_4), r(H_2)-2r(H_5), i_+(H_1)+i_-(H_2)-r(H_4)-r(H_5), i_-(H_1)+i_+(H_2)-r(H_4)-r(H_5)\},$$

$$\max_{X\in G, Y\in S} i_\pm\left[(X+X^*)-(Y+Y^*)\right]$$
$$= \min\{i_\pm(H_1)-r(A_1)-r(A_2), i_\pm(H_2)-r(B_1)-r(A_2)\},$$

$$\min_{X\in G, Y\in S} i_\pm\left[(X+X^*)-(Y+Y^*)\right]$$
$$= r(E_3)-p+\max\{i_\pm(H_1)-r(H_4), i_\pm(H_2)-r(H_5)\}. \quad (58)$$

*Proof.* By letting $A_3 = 0$, $B_3 = -I$, $C_3 = I$, and $D_3 = I$ in Theorem 7, we can get the results. □

**Corollary 10.** *Let $A_1$, $C_1$, $B_1$, $D_1$, $A_2$, $C_2$, and $H_i$, $(i = 1, 2, \ldots, 5)$ be as in Theorem 9, and suppose that the system of matrix equations* (13) *and* (11) *is consistent, respectively. Denote the set of all solutions to* (13) *by S and* (11) *by G. Then, one has the following:*

(a) *there exist $X \in G$ and $Y \in S$ such that $(X+X^*) > (Y+Y^*)$ if and only if*

$$i_+(H_1)-r(A_1)-r(A_2) \geq p, \quad i_+(H_2)-r(B_1)-r(A_2) \geq p. \quad (59)$$

(b) *there exist $X \in G$ and $Y \in S$ such that $(X+X^*) < (Y+Y^*)$ if and only if*

$$i_-(H_1)-r(A_1)-r(A_2) \geq p, \quad i_-(H_2)-r(B_1)-r(A_2) \geq p, \quad (60)$$

(c) *there exist $X \in G$ and $Y \in S$ such that $(X+X^*) \geq (Y+Y^*)$ if and only if*

$$r(H_3)-p+i_-(H_1)-r(H_4) \leq 0, \quad r(H_3)-p+i_-(H_1)-r(H_4) \leq 0, \quad (61)$$

(d) *there exist $X \in G$ and $Y \in S$ such that $(X+X^*) \leq (Y+Y^*)$ if and only if*

$$r(H_3)-p+i_+(H_1)-r(H_4) \leq 0, \quad r(H_3)-p+i_+(H_2)-r(H_5) \leq 0, \quad (62)$$

(e) $(X+X^*) > (Y+Y^*)$ *for all $X \in G$ and $Y \in S$ if and only if*

$$r(H_3)-p+i_+(H_1)-r(H_4) = p \quad \text{or} \quad r(H_3)-p+i_+(H_2)-r(H_5) = p, \quad (63)$$

(f) $(X+X^*) < (Y+Y^*)$ *for all $X \in G$ and $Y \in S$ if and only if*

$$r(H_3)-p+i_-(H_1)-r(H_4) = p \quad \text{or} \quad r(H_3)-p+i_-(H_2)-r(H_5) = p, \quad (64)$$

(g) $(X+X^*) \geq (Y+Y^*)$ *for all $X \in G$ and $Y \in S$ if and only if*

$$i_-(H_1)-r(A_1)-r(A_2) \leq 0 \quad \text{or} \quad i_-(H_2)-r(B_1)-r(A_2) \leq 0, \quad (65)$$

(h) $(X+X^*) \leq (Y+Y^*)$ *for all* $X \in G$ *and* $Y \in S$ *if and only if*

$$i_+(H_1) - r(A_1) - r(A_2) \leq 0 \quad \text{or} \quad i_+(H_2) - r(B_1) - r(A_2) \leq 0, \tag{66}$$

(i) *there exist* $X \in G$ *and* $Y \in S$ *such that* $(X+X^*) - (Y+Y^*)$ *is nonsingular if and only if*

$$\begin{gathered} r(H_1) - 2r(A_1) - 2r(A_2) \geq p, \\ r(H_2) - 2r(B_1) - 2r(A_2) \geq p, \\ r(H_3) - 2r(A_2) - r(A_1) - r(B_1) \geq p. \end{gathered} \tag{67}$$

## 4. The Solvability Conditions and the General Solution to System (6)

We now turn our attention to (6). We in this section use Theorem 9 to give some necessary and sufficient conditions for the existence of a solution to (6) and present an expression of the general solution to (6). We begin with a lemma which is used in the latter part of this section.

**Lemma 11** (see [14]). *Let* $A_1 \in \mathbb{C}^{m\times n_1}$, $B_1 \in \mathbb{C}^{m\times n_2}$, $C_1 \in \mathbb{C}^{q\times m}$, *and* $E_1 \in \mathbb{C}_h^{m\times m}$ *be given. Let* $A = R_{A_1}B_1$, $B = C_1R_{A_1}$, $E = R_{A_1}E_1R_{A_1}$, $M = R_AB^*$, $N = A^*L_B$, *and* $S = B^*L_M$. *Then the following statements are equivalent:*

(1) *equation* (5) *is consistent,*

(2)

$$R_MR_AE = 0, \qquad R_AER_A = 0, \qquad L_BEL_B = 0, \tag{68}$$

(3)

$$\begin{gathered} r\begin{bmatrix} E_1 & B_1 & C_1^* & A_1 \\ A_1^* & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_1 & C_1^* & A_1 \end{bmatrix} + r(A_1), \\ r\begin{bmatrix} E_1 & B_1 & A_1 \\ A_1^* & 0 & 0 \\ B_1^* & 0 & 0 \end{bmatrix} = 2r\begin{bmatrix} B_1 & A_1 \end{bmatrix}, \\ r\begin{bmatrix} E_1 & C_1^* & A_1 \\ A_1^* & 0 & 0 \\ C_1 & 0 & 0 \end{bmatrix} = 2r\begin{bmatrix} C_1^* & A_1 \end{bmatrix}. \end{gathered} \tag{69}$$

*In this case, the general solution of* (5) *can be expressed as*

$$\begin{aligned} Y = {} & \frac{1}{2}\left[A^\dagger EB^\dagger - A^\dagger B^* M^\dagger EB^\dagger - A^\dagger S\left(B^\dagger\right)^* EN^\dagger A^* B^\dagger \right. \\ & \left. + A^\dagger E\left(M^\dagger\right)^* + \left(N^\dagger\right)^* EB^\dagger S^\dagger S\right] + L_AV_1 + V_2R_B \\ & + U_1L_SL_M + R_NU_2^*L_M - A^\dagger SU_2R_NA^*B^\dagger, \\ X = {} & A_1^\dagger\left[E_1 - B_1YC_1 - (B_1YC_1)^*\right] \\ & - \frac{1}{2}A_1^\dagger\left[E_1 - B_1YC_1 - (B_1YC_1)^*\right]A_1A_1^\dagger \\ & - A_1^\dagger W_1A_1^* + W_1^*A_1A_1^\dagger + L_{A_1}W_2, \end{aligned} \tag{70}$$

*where* $U_1$, $U_2$, $V_1$, $V_2$, $W_1$, *and* $W_2$ *are arbitrary matrices over* $\mathbb{C}$ *with appropriate sizes.*

Now we give the main theorem of this section.

**Theorem 12.** *Let* $A_i$, $C_i$, $(i = 1, 2, 3)$, $B_j$, *and* $D_j$, $(j = 1, 3)$ *be given. Set*

$$\begin{gathered} A = B_3L_{A_2}, \qquad B = C_3L_{A_1}, \\ C = R_{B_1}D_3, \qquad F = R_AB, \\ G = CR_A, \qquad M = R_FG^*, \\ N = F^*L_G, \qquad S = G^*L_M, \end{gathered} \tag{71}$$

$$\begin{aligned} D = {} & A_3 - B_3A_2^\dagger C_2 - \left(B_3A_2^\dagger C_2\right)^* \\ & - C_3\left(A_1^\dagger C_1 + L_{A_1}D_1B_1^\dagger\right)D_3 \\ & - D_3^*\left(A_1^\dagger C_1 + L_{A_1}D_1B_1^\dagger\right)^*C_3^*, \end{aligned} \tag{72}$$

$$E = R_ADR_A. \tag{73}$$

*Then the following statements are equivalent:*

(1) *system* (6) *is consistent,*

(2) *the equalities in* (14) *and* (17) *hold, and*

$$R_MR_FE = 0, \qquad R_FER_F = 0, \qquad L_GEL_G = 0, \tag{74}$$

(3) *the equalities in* (15) *and* (18) *hold, and*

$$\begin{gathered} r\begin{bmatrix} A_3 & C_3 & D_3^* & B_3 & C_2^* \\ B_3^* & 0 & 0 & 0 & A_2^* \\ C_1D_3 & A_1 & 0 & 0 & 0 \\ D_1^*C_3^* & 0 & B_1^* & 0 & 0 \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix} = r\begin{bmatrix} C_3 & D_3^* & B_3 \\ A_1 & 0 & 0 \\ 0 & B_1^* & 0 \\ 0 & 0 & A_2 \end{bmatrix} + r\begin{bmatrix} A_2 \\ B_3 \end{bmatrix}, \\ r\begin{bmatrix} A_3 & C_3 & B_3 & D_3^*C_1^* & C_2^* \\ C_3^* & 0 & 0 & A_1^* & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ C_1D_3 & A_1 & 0 & 0 & 0 \\ C_2 & 0 & A_2 & 0 & 0 \end{bmatrix} = 2r\begin{bmatrix} C_3 & B_3 \\ A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \\ r\begin{bmatrix} A_3 & D_3^* & B_3 & C_3D_1 & C_2^* \\ D_3 & 0 & 0 & B_1 & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ D_1^*C_3^* & B_1^* & 0 & 0 & 0 \\ C_2 & 0 & A_2 & 0 & 0 \end{bmatrix} = 2r\begin{bmatrix} D_3^* & B_3 \\ B_1^* & 0 \\ 0 & A_2 \end{bmatrix}. \end{gathered} \tag{75}$$

*In this case, the general solution of system* (6) *can be expressed as*

$$\begin{gathered} X = A_2^\dagger C_2 + L_{A_2}U, \\ Y = A_1^\dagger C_1 + L_{A_1}D_1B_1^\dagger + L_{A_1}VR_{B_1}, \end{gathered} \tag{76}$$

*where*

$$V = \frac{1}{2}\left[F^{\dagger}EG^{\dagger} - F^{\dagger}G^{*}M^{\dagger}EG^{\dagger} - F^{\dagger}S\left(G^{\dagger}\right)^{*}EN^{\dagger}F^{*}G^{\dagger} + F^{\dagger}E\left(M^{\dagger}\right)^{*} + \left(N^{\dagger}\right)^{*}EG^{\dagger}S^{\dagger}S\right] + L_FV_1 + V_2R_G + U_1L_SL_M + R_NU_2^{*}L_M - F^{\dagger}SU_2R_NF^{*}G^{\dagger},$$

$$U = A^{\dagger}\left[D - BVC - (BVC)^{*}\right] - \frac{1}{2}A^{\dagger}\left[D - BVC - (BVC)^{*}\right]AA^{\dagger} - A^{\dagger}W_1A^{*} + W_1^{*}AA^{\dagger} + L_AW_2, \tag{77}$$

*where* $U_1$, $U_2$, $V_1$, $V_2$, $W_1$, *and* $W_2$ *are arbitrary matrices over* $\mathbb{C}$ *with appropriate sizes.*

*Proof.* (2) $\Leftrightarrow$ (3): Applying Lemma 3 and Lemma 11 gives

$$\begin{aligned}
R_MR_FE = 0 &\Longleftrightarrow r\left(R_MR_FE\right) = 0 \Longleftrightarrow r\begin{bmatrix} D & B & C^{*} & A \\ A^{*} & 0 & 0 & 0 \end{bmatrix} = r\left[B \;\; C^{*} \;\; A\right] + r(A) \\
&\Longleftrightarrow r\begin{bmatrix} D & C_3 & D_3^{*} & B_3 & 0 \\ B_3^{*} & 0 & 0 & 0 & A_2^{*} \\ 0 & A_1 & 0 & 0 & 0 \\ 0 & 0 & B_1^{*} & 0 & 0 \\ 0 & 0 & 0 & A_2 & 0 \end{bmatrix} = r\begin{bmatrix} C_3 & D_3^{*} & B_3 \\ A_1 & 0 & 0 \\ 0 & B_1^{*} & 0 \\ 0 & 0 & A_2 \end{bmatrix} + r\begin{bmatrix} A_2 \\ B_3 \end{bmatrix} \\
&\Longleftrightarrow r\begin{bmatrix} A_3 & C_3 & D_3^{*} & B_3 & C_2^{*} \\ B_3^{*} & 0 & 0 & 0 & A_2^{*} \\ C_1D_3 & A_1 & 0 & 0 & 0 \\ D_1^{*}C_3^{*} & 0 & B_1^{*} & 0 & 0 \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix} = r\begin{bmatrix} C_3 & D_3^{*} & B_3 \\ A_1 & 0 & 0 \\ 0 & B_1^{*} & 0 \\ 0 & 0 & A_2 \end{bmatrix} + r\begin{bmatrix} A_2 \\ B_3 \end{bmatrix}.
\end{aligned} \tag{78}$$

By a similar approach, we can obtain that

$$\begin{aligned}
R_FER_F = 0 &\Longleftrightarrow r\begin{bmatrix} A_3 & C_3 & B_3 & D_3^{*}C_1^{*} & C_2^{*} \\ C_3^{*} & 0 & 0 & A_1^{*} & 0 \\ B_3^{*} & 0 & 0 & 0 & A_2^{*} \\ C_1D_3 & A_1 & 0 & 0 & 0 \\ C_2 & 0 & A_2 & 0 & 0 \end{bmatrix} = 2r\begin{bmatrix} C_3 & B_3 \\ A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \\
L_GEL_G = 0 &\Longleftrightarrow r\begin{bmatrix} A_3 & D_3^{*} & B_3 & C_3D_1 & C_2^{*} \\ D_3 & 0 & 0 & B_1 & 0 \\ B_3^{*} & 0 & 0 & 0 & A_2^{*} \\ D_1^{*}C_3^{*} & B_1^{*} & 0 & 0 & 0 \\ C_2 & 0 & A_2 & 0 & 0 \end{bmatrix} = 2r\begin{bmatrix} D_3^{*} & B_3 \\ B_1^{*} & 0 \\ 0 & A_2 \end{bmatrix}.
\end{aligned} \tag{79}$$

(1) $\Leftrightarrow$ (2): We separate the four equations in system (6) into three groups:

$$A_1Y = C_1, \qquad YB_1 = D_1, \tag{80}$$

$$A_2X = C_2, \tag{81}$$

$$B_3X + \left(B_3X\right)^{*} + C_3YD_3 + \left(C_3YD_3\right)^{*} = A_3. \tag{82}$$

By Lemma 1, we obtain that system (80) is solvable if and only if (14), (81) is consistent if and only if (17). The general solutions to system (80) and (81) can be expressed as (16) and (19), respectively. Substituting (16) and (19) into (82) yields

$$AU + (AU)^{*} + BVC + (BVC)^{*} = D. \tag{83}$$

Hence, the system (5) is consistent if and only if (80), (81), and (83) are consistent, respectively. It follows from Lemma 11 that (83) is solvable if and only if

$$R_MR_FE = 0, \qquad R_FER_F = 0, \qquad L_GEL_G = 0. \tag{84}$$

We know by Lemma 11 that the general solution of (83) can be expressed as (77). □

In Theorem 12, let $A_1$ and $D_1$ vanish. Then we can obtain the general solution to the following system:

$$\begin{gathered} A_2X = C_2, \qquad YB_1 = D_1, \\ B_3X + \left(B_3X\right)^{*} + C_3YD_3 + \left(C_3YD_3\right)^{*} = A_3. \end{gathered} \tag{85}$$

**Corollary 13.** *Let* $A_2$, $C_2$, $B_1$, $D_1$, $B_3$, $C_3$, $D_3$, *and* $A_3 = A_3^{*}$ *be given. Set*

$$\begin{gathered} A = B_3L_{A_2}, \qquad C = R_{B_1}D_3, \\ F = R_AC_3, \qquad G = CR_A, \\ M = R_FG^{*}, \qquad N = F^{*}L_G, \\ S = G^{*}L_M, \\ D = A_3 - B_3A_2^{\dagger}C_2 - \left(B_3A_2^{\dagger}C_2\right)^{*} - C_3D_1B_1^{\dagger}D_3 - \left(C_3D_1B_1^{\dagger}D_3\right)^{*}, \\ E = R_ADR_A. \end{gathered} \tag{86}$$

*Then the following statements are equivalent:*

(1) *system* (85) *is consistent*

(2)

$$R_{A_2}C_2 = 0, \qquad D_1 L_{B_1} = 0, \qquad R_M R_F E = 0, \tag{87}$$

$$R_F E R_F = 0, \qquad L_G E L_G = 0, \tag{88}$$

(3)

$$r\begin{bmatrix} A_2 & C_2 \end{bmatrix} = r(A_2), \qquad \begin{bmatrix} D_1 \\ B_1 \end{bmatrix} = r(B_1),$$

$$r\begin{bmatrix} A_3 & C_3 & D_3^* & B_3 & C_2^* \\ B_3^* & 0 & 0 & 0 & A_2^* \\ D_1^* C_3^* & 0 & B_1^* & 0 & 0 \\ C_2 & 0 & 0 & A_2 & 0 \end{bmatrix} = r\begin{bmatrix} C_3 & D_3^* & B_3 \\ 0 & B_1^* & 0 \\ 0 & 0 & A_2 \end{bmatrix} + r\begin{bmatrix} A_2 \\ B_3 \end{bmatrix},$$

$$r\begin{bmatrix} A_3 & C_3 & B_3 & C_2^* \\ C_3^* & 0 & 0 & 0 \\ B_3^* & 0 & 0 & A_2^* \\ C_2 & 0 & A_2 & 0 \end{bmatrix} = 2r\begin{bmatrix} C_3 & B_3 \\ 0 & A_2 \end{bmatrix},$$

$$r\begin{bmatrix} A_3 & D_3^* & B_3 & C_3 D_1 & C_2^* \\ D_3 & 0 & 0 & B_1 & 0 \\ B_3^* & 0 & 0 & 0 & A_2^* \\ D_1^* C_3^* & B_1^* & 0 & 0 & 0 \\ C_2 & 0 & A_2 & 0 & 0 \end{bmatrix} = 2r\begin{bmatrix} D_3^* & B_3 \\ B_1^* & 0 \\ 0 & A_2 \end{bmatrix}. \tag{89}$$

*In this case, the general solution of system* (6) *can be expressed as*

$$\begin{aligned} X &= A_2^{\dagger} C_2 + L_{A_2} U, \\ Y &= D_1 B_1^{\dagger} + V R_{B_1}, \end{aligned} \tag{90}$$

*where*

$$\begin{aligned} V = {} & \frac{1}{2}\left[ F^{\dagger} E G^{\dagger} - F^{\dagger} G^* M^{\dagger} E G^{\dagger} - F^{\dagger} S \left(G^{\dagger}\right)^* E N^{\dagger} F^* G^{\dagger} \right. \\ & \left. + F^{\dagger} E \left(M^{\dagger}\right)^* + \left(N^{\dagger}\right)^* E G^{\dagger} S^{\dagger} S \right] + L_F V_1 \\ & + V_2 R_G + U_1 L_S L_M + R_N U_2^* L_M - F^{\dagger} S U_2 R_N F^* G^{\dagger}, \\ U = {} & A^{\dagger}\left[D - C_3 V C - (C_3 V C)^*\right] \\ & - \frac{1}{2} A^{\dagger}\left[D - C_3 V C - (C_3 V C)^*\right] A A^{\dagger} \\ & - A^{\dagger} W_1 A^* + W_1^* A A^{\dagger} + L_A W_2, \end{aligned} \tag{91}$$

*where $U_1$, $U_2$, $V_1$, $V_2$, $W_1$, and $W_2$ are arbitrary matrices over $\mathbb{C}$ with appropriate sizes.*

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