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Research Article

Some Identities between the Extended q-Bernstein Polynomials with Weight α and q-Bernoulli Polynomials with Weight (α,β)

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Using bosonic p-adic q-integral on $\mathbb{Z}_p$, we give some interesting relationships between q-Bernoulli numbers with weight (α,β) and q-Bernstein polynomials with weight α. Also, using q-Bernstein polynomials with two variables, we derive some interesting properties associated with q-Bernoulli numbers with weight (α,β).

1. Introduction

In many areas the Bernstein polynomials have many applications: in statistics, in computer applications, in approximations of functions, in numerical analysis, in p-adic analysis, and in the solution of differential equations. In particular, recognition of human speech has long been a hot issue among artificial intelligence and signal processing researchers, and q-Bernstein polynomials, which are mathematical operators, have been applied for pattern recognition, and a new method has thus been developed. For q-Bernoulli numbers and polynomials, several results have been studied by Carlitz (see [1, 2]), Kim (see [3–7]), Ozden et al. (see [8]), and M.-S. Kim et al. (see [9]). Bernoulli numbers and polynomials possess many interesting properties and arise in many areas of mathematics, mathematical physics, and statistical physics. Recently, many mathematicians have studied in the area of Bernoulli numbers and polynomials. First, we introduce the q-Bernoulli numbers and polynomials with weight (α,β) as extended measure. And we investigate some identities due to the q-Bernoulli numbers and polynomials with weight (α,β). Second, we consider the p-adic analogue of the extended q-Bernstein polynomials on $\mathbb{Z}_p$ and investigate some properties of several extended q-Bernstein polynomials by using the bosonic p-adic q-integral. Finally, we investigate the relation of Bernstein polynomials and Bernoulli numbers and polynomials using p-adic q-integral on $\mathbb{Z}_p$.

Throughout this paper, we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{Z}_+ = \mathbb{N}\cup\{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x\log q)$ for $|x|_p \le 1$. Throughout this paper we use the following notation:

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q} \tag{1}$$

(see [1–22]). The q-numbers $[x]_q$ satisfy many simple relations, all easily verified, as below:

$$[x+y]_q = [x] + q^x[y]_q,$$

$$[x]_{q^{-1}} = q^{1-x}[x]_q,$$

$$[-x]_q = -q^{-x}[x]_q,$$
$$[-x]_{q^{-1}} = -q[x]_q,$$
$$[1-x]_q = 1-[x]_{q^{-1}}. \tag{2}$$

For a fixed positive integer f with $(f,p)=1$, let

$$X = X_f = \lim_{\overleftarrow{N}}\left(\frac{Z}{fp^N\mathbb{Z}}\right), \qquad X_1 = \mathbb{Z}_p,$$
$$X^* = \bigcup_{\substack{0<a<fp^N\\(a,p)=1}}\left(a+fp^N\mathbb{Z}_p\right), \tag{3}$$
$$a+fp^N\mathbb{Z}_p = \left\{x\in X \mid x\equiv a\left(\operatorname{mod} fp^N\right)\right\},$$

where $a\in\mathbb{Z}$ and $0\le a<fp^N$ (see [3]). We say that f is a uniformly differential function at a point $a\in\mathbb{Z}_p$ and denotes this property by $f\in \mathrm{UD}(\mathbb{Z}_p)$ if the difference quotients

$$F_f(x,y) = \frac{f(x)-f(y)}{x-y} \tag{4}$$

have a $\lim_{(x,y)\to(a,a)}F_f(x,y) = f'(a)$. For $f\in\mathrm{UD}(\mathbb{Z}_p)$, let us begin with the expression

$$\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1} f(x)q^x = \sum_{0\le x<p^N} f(x)\mu_q\left(x+p^N\mathbb{Z}_p\right) \tag{5}$$

representing a q-analogue of the Riemann sum for f. The integral of f on $\mathbb{Z}_p$ is defined as the limit ($N\to\infty$) of the sums (if exists). The p-adic q-integral (or q-Volkenborn integrals of $f\in\mathrm{UD}(\mathbb{Z}_p)$) is defined by T. Kim (see [4]) as follows

$$\begin{aligned} I_q(f) &= \int_X f(x)\,d\mu_q(x)\\ &= \int_{Z_p} f(x)\,d\mu_q(x)\\ &= \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{0\le x<p^N} f(x)q^x \end{aligned} \tag{6}$$

(see [4]). If we take $f_1(x) = f(x+1)$ in (6), then we easily see that

$$\begin{aligned} q^\beta I_{q^\beta}(f_1) &= I_{q^\beta}(f) + \left(q^\beta-1\right)f(0)\\ &\quad + \frac{q^\beta-1}{\log q^\beta}f'(0). \end{aligned} \tag{7}$$

From (6), we obtain

$$\begin{aligned} q^{\beta n}I_{q^\beta}(f_n) &= I_{q^\beta}(f) + \left(q^\beta-1\right)\sum_{l=0}^{n-1} q^{\beta l}f(l)\\ &\quad + \frac{q^\beta-1}{\log q^\beta}\sum_{l=0}^{n-1} f'(l), \end{aligned} \tag{8}$$

where $f_n(x) = f(x+n)$ (cf. [1–19]).

Carlitz's q-Bernoulli numbers $\beta_{k,q}$ can be defined recursively by $\beta_{0,q}=1$ and by the value that

$$q\left(q\beta_q+1\right)^k - \beta_{k,q} = \begin{cases}1, & \text{if } k=1\\ 0, & \text{if } n>1,\end{cases} \tag{9}$$

with the usual convention of replacing $\beta_q^i = \beta_{i,q}$.

Carliz's q-Bernoulli polynomials are also defined by

$$\beta_{n,q}(x) = \left(q^x\beta+[x]_q\right)^k = \sum_{i=0}^{k}\binom{k}{i}q^{ix}\beta_{i,q}[x]_q^{k-i}. \tag{10}$$

It is well known that

$$\begin{aligned} \beta_{n,q} &= \int_{\mathbb{Z}_p}[x]_q^n d\mu_q(x),\\ \beta_{n,q}(x) &= \int_{\mathbb{Z}_p}[x+y]_q^n d\mu_q(y), \quad n\in Z_+, \end{aligned} \tag{11}$$

where $\beta_{n,q}(x)$ are called the nth Carlitz's q-Bernoulli polynomials.

In this paper, we define the q-Bernoulli numbers and polynomials with weight (α,β) as below.

For $\alpha,\beta\in\mathbb{Z}$ and $q\in\mathbb{C}_p$ with $|1-q|_p<1$, the q-Bernoulli numbers $\beta_{n,q}^{(\alpha,\beta)}$ with weight (α,β) are defined by

$$\beta_{n,q}^{(\alpha,\beta)} = \int_{\mathbb{Z}_p}[x]_{q^\alpha}^n d\mu_{q^\beta}(x). \tag{12}$$

For $\alpha,\beta\in\mathbb{Z}$ and $q\in\mathbb{C}_p$ with $|1-q|_p<1$, the q-Bernoulli polynomials with weight (α,β) are also defined by

$$\beta_{n,q}^{(\alpha,\beta)}(x) = \int_{\mathbb{Z}_p}[x+y]_{q^\alpha}^n d\mu_{q^\beta}(y). \tag{13}$$

2. The Properties of the q-Bernoulli Numbers and Polynomials with Weight (α,β)

Our primary goal of this section is to find some properties of the q-Bernoulli numbers with weight (α,β), respectively, $\beta_{n,q}^{(\alpha,\beta)}$, $\beta_{n,q}^{(\alpha,\beta)}(x)$. This is a very important process to find the relationship between Bernoulli numbers and polynomials and Bernstein polynomials. In this section, by using the bosonic p-adic q-integral on $\mathbb{Z}_p$, we obtain some properties.

For $\alpha\in\mathbb{Z}$ and $q\in\mathbb{C}_p$ with $|1-q|_p<1$, q-Bernoulli numbers with weight (α,β) are represented as follows by simple calculus:

$$\begin{aligned} \int_{\mathbb{Z}_p}[x]_{q^\alpha}^n d\mu_{q^\beta}(x) &= \lim_{N\to\infty}\frac{1}{[p^N]_{q^\beta}}\sum_{x=0}^{p^N-1}[x]_{q^\alpha}^n\left(q^\beta\right)^x\\ &= \left(\frac{1}{1-q^\alpha}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1+(\alpha/\beta)l}{[1+(\alpha/\beta)l]_{q^\beta}}. \end{aligned} \tag{14}$$

We set

$$F_q^{(\alpha,\beta)}(t) = \sum_{n=0}^{\infty}\beta_{n,q}^{(\alpha,\beta)}\frac{t^n}{n!}. \tag{15}$$

By (14) and (15), we have

$$\begin{aligned} F_q^{(\alpha,\beta)}(t) &= \sum_{n=0}^{\infty} \beta_{n,q}^{(\alpha,\beta)} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{1-q^\alpha}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1+(\alpha/\beta)l}{[1+(\alpha/\beta)l]_{q^\beta}} \frac{t^n}{n!} \\ &= \left(1-q^\beta\right) \left\{ \sum_{m=0}^{\infty} q^{\beta m} e^{[m]_{q^\alpha} t} - \frac{\alpha}{\beta} t \sum_{m=0}^{\infty} \frac{q^{(\alpha+\beta)m}}{1-q^\alpha} e^{[m]_{q^\alpha} t} \right\}. \end{aligned} \tag{16}$$

Note that for $\alpha = \beta = 1$,

$$\beta_{n,q}^{(1,1)}(x) = B_{n,q}(x), \tag{17}$$

where $B_{n,q}(x)$ are the nth Carliz's q-Bernoulli polynomials. In particular, $x = 0$, $\beta_{n,q}^{(1,1)}(0) = \beta_{n,q}$ are the nth Carliz's q-Bernoulli numbers.

For $\alpha \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, since $[x+y]_{q^\alpha} = [x]_{q^\alpha} + q^{\alpha x}[y]_{q^\alpha}$, we get the formula as below:

$$\begin{aligned} \beta_{n,q}^{(\alpha,\beta)}(x) &= \int_{\mathbb{Z}_p} [y+x]_{q^\alpha}^n d\mu_{q^\beta}(y) \\ &= \left([x]_{q^\alpha} + q^{\alpha x} \beta_q^{(\alpha,\beta)}\right)^n, \end{aligned} \tag{18}$$

with usual convention about replacing $(\beta_q^{(\alpha,\beta)})^n$ by $(\beta_{n,q}^{(\alpha,\beta)})$. Also, by (6) and the bosonic p-adic q-integral on $\mathbb{Z}_p$, we get the finite summation formula as below:

$$\begin{aligned} \beta_{n,q}^{(\alpha,\beta)}(x) &= \int_{\mathbb{Z}_p} [y+x]_{q^\alpha}^n d\mu_{q^\beta}(y) \\ &= \left(\frac{1}{1-q^\alpha}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha x l} \frac{1+(\alpha/\beta)l}{[1+(\alpha/\beta)l]_{q^\beta}}. \end{aligned} \tag{19}$$

Therefore, by (18) and (19), we have the following theorem.

Theorem 1. *For $\alpha, \beta \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, one has*

$$\begin{aligned} \beta_{n,q}^{(\alpha,\beta)}(x) &= \left([x]_{q^\alpha} + q^{\alpha x} \beta_q^{(\alpha,\beta)}\right)^n \\ &= \left(\frac{1}{1-q^\alpha}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha x l} \frac{1+(\alpha/\beta)l}{[1+(\alpha/\beta)l]_{q^\beta}}, \end{aligned} \tag{20}$$

and if $x = 0$, $\beta_{n,q}^{(\alpha,\beta)}(0) = \beta_{n,q}^{(\alpha,\beta)}$.

We also get from Theorem 1 that for $n \in \mathbb{Z}_+$,

$$\beta_{n,q^{-1}}^{(\alpha,\beta)}(1-x) = (-1)^n q^{\alpha n} \beta_{n,q}^{(\alpha,\beta)}(x). \tag{21}$$

From (7) we get the following formula:

$$\sum_{n=0}^{\infty} q^\beta \beta_{n,q}^{(\alpha,\beta)}(1) - \beta_{n,q}^{(\alpha,\beta)} \frac{t^n}{n!} = q^\beta - 1 + t \tag{22}$$

and by comparison of coefficients we obtain as below:

$$q^\beta \beta_{n,q}^{(\alpha,\beta)}(1) - \beta_{n,q}^{(\alpha,\beta)} = \begin{cases} q^\beta - 1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n \geq 2. \end{cases} \tag{23}$$

From (8), we get the following formula:

$$\begin{aligned} q^{\beta n} \beta_{k,q}^{(\alpha,\beta)}(n) &= \beta_{k,q}^{(\alpha,\beta)} + \left(q^\beta - 1\right) \sum_{l=0}^{n-1} q^{\beta l} [l]_{q^\alpha}^k \\ &\quad + \frac{\alpha}{\beta} \left[\frac{\beta}{\alpha}\right]_{q^\alpha} \sum_{l=0}^{n-1} k q^{(\alpha+\beta)l} [l]_{q^\alpha}^{k-1}. \end{aligned} \tag{24}$$

Specially, we consider when $n = 1$ and $n = 2$, then we obtain the following Theorem.

Theorem 2. *For $\alpha, \beta \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, one has*

$$\begin{aligned} &q^\beta \beta_{k,q}^{(\alpha,\beta)}(1) = \beta_{k,q}^{(\alpha,\beta)}, \quad \text{if } n = 1, \\ &q^{2\beta} \beta_{k,q}^{(\alpha,\beta)}(2) \\ &\quad = \beta_{k,q}^{(\alpha,\beta)} + \left(q^\beta - 1\right)\left(q^\beta\right) + k \frac{\alpha}{\beta} \left[\frac{\alpha}{\beta}\right]_{q^\alpha} q^{(\alpha+\beta)}, \quad \text{if } n = 2. \end{aligned} \tag{25}$$

By the bosonic p-adic q-integral on $\mathbb{Z}_p$ and (21), for $n \in \mathbb{N}$

$$\begin{aligned} \int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^n d\mu_{q^\beta}(x) &= (-1)^n q^{\alpha n} \int_{\mathbb{Z}_p} [x-1]_{q^\alpha}^n d\mu_{q^\beta}(x) \\ &= (-1)^n q^{\alpha n} \beta_{n,q}^{(\alpha,\beta)}(-1) \\ &= \beta_{n,q^{-1}}^{(\alpha,\beta)}(2). \end{aligned} \tag{26}$$

Therefore, from Theorem 2 and (26), we obtain the following theorem.

Theorem 3. *For $\alpha, \beta \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, one has*

$$\begin{aligned} &\int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^n d\mu_{q^\beta}(x) \\ &\quad = q^{2\beta} \beta_{n,q^{-1}}^{(\alpha,\beta)} + 1 - q^\beta + n q^{(\beta-\alpha)} \frac{\alpha}{\beta} \left[\frac{\beta}{\alpha}\right]_{q^{-\alpha}}. \end{aligned} \tag{27}$$

3. Properties of q-Bernstein Polynomials

In this section, we introduce the q-Bernstein polynomials and we get some properties to use. Let $C[0,1]$ be the set of continuous functions on $[0,1]$. Then, the classical Bernstein polynomials of degree n for $f \in C[0,1]$ are defined by

$$\mathbb{B}_n(f) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x), \quad 0 \leq x \leq 1, \tag{28}$$

where $\mathbb{B}_n(f)$ is called the Bernstein operator and

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k} \tag{29}$$

are called the Bernstein basis polynomials (or the Bernstein polynomials of degree n). In recent years, Gupta et al. have studied the generation function for q-Bernstein polynomials (see [19]). Their generating function for $\widetilde{B}_{k,n}(x \mid q) = \binom{n}{k}[x]_q^k[1-x]_q^{n-k}$ is given by

$$F_{k,q}(t,x) = \frac{t^k e^{[1-x]_q t}[x]_q^k}{k!} = \sum_{n=k}^{\infty} \widetilde{B}_{k,n}(x \mid q)\frac{t^n}{n!}, \tag{30}$$

where $[x]_q = (1-q^x)/(1-q)$. Observe that

$$\lim_{q\to 1} \widetilde{B}_{k,n}(x \mid q) = B_{k,n}(x). \tag{31}$$

Also, q-Bernstein polynomials with weight α, $\widetilde{B}_{k,n}^{(\alpha)}(x \mid q)$ are studied by T. Kim (see [7]), and the formula is as below:

$$\widetilde{B}_{k,n}^{(\alpha)}(x \mid q) = \binom{n}{k}[x]_{q^\alpha}^k[1-x]_{q^\alpha}^{n-k}. \tag{32}$$

Let f be continuous functions on $[0,1]$. q-Bernstein operator is defined as below:

$$\begin{aligned}\widetilde{\mathbb{B}}_{n,q}(f \mid x) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\widetilde{B}_{k,n}(x,q)\\ &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}\sum_{s=k}^{n}(-1)^{s-k}[x]_q^s.\end{aligned} \tag{33}$$

From now on, we introduce some properties of q-Bernstein polynomials with weight α and q-Bernstein operator. It is easy to show that these properties are true by some calculus are

(1) $\widetilde{B}_{n-k,n}^{(\alpha)}(1-x \mid q^{-1}) = \widetilde{B}_{k,n}^{(\alpha)}(x \mid q)$,

(2) $[1-x]_{q^{-\alpha}}\widetilde{B}_{k,n-1}^{(\alpha)}(x \mid q) + [x]_{q^\alpha}\widetilde{B}_{k-1,n-1}^{(\alpha)}(x \mid q) = \widetilde{B}_{k,n}^{(\alpha)}(x \mid q)$,

(3) $(d/dx)\widetilde{B}_{k,n}^{(\alpha)}(x \mid q) = (-n\alpha q^{\alpha x}\log q/(1-q^\alpha))(\widetilde{B}_{k,n}^{(\alpha)}(x \mid q) - \widetilde{B}_{k-1,n-1}^{(\alpha)}(x \mid q))$,

(4) $\widetilde{\mathbb{B}}_{n,q}(f \mid x) = \sum_{k=0}^{n} f(k/n)\widetilde{B}_{k,n}^{(\alpha)}(x \mid q)$,

In the above property 4, we consider special cases $f = 1$, $f = t$, $f = t^2$ as below.

If $f = 1$,

$$\widetilde{\mathbb{B}}_{n,q}(1 \mid x) = 1. \tag{34}$$

If $f = t$,

$$\begin{aligned}\widetilde{\mathbb{B}}_{n,q}(t \mid x) &= [x]_{q^\alpha}\sum_{s=0}^{n-1}\binom{n-1}{s}[x]_{q^\alpha}^s\left(1-[x]_{q^\alpha}\right)^{n-1-s}\\ &= [x]_{q^\alpha}.\end{aligned} \tag{35}$$

If $f = t^2$,

$$\begin{aligned}\widetilde{\mathbb{B}}_{n,q}(t^2 \mid x) &= \sum_{k=0}^{n}\frac{k^2}{n^2}\binom{n}{k}[x]_{q^\alpha}^k[1-x]_{q^{-\alpha}}^{n-k}\\ &= \sum_{s=0}^{n-1}\frac{s+1}{n}\binom{n-1}{s}[x]_{q^\alpha}^{s+1}[1-x]_{q^{-\alpha}}^{n-s-1}\\ &= \frac{n-1}{n}[x]_{q^\alpha}^2 + \frac{[x]_{q^\alpha}}{n},\end{aligned} \tag{36}$$

then one has that $\lim_{n\to\infty}\widetilde{\mathbb{B}}_{n,q}(t^2 \mid x) = [x]_{q^\alpha}^2$.

From definition of $\widetilde{B}_{k,n}^{(\alpha)}(x \mid q)$ with $n \in \mathbb{N}$, $k \in \mathbb{Z}_+$ and $x \in [0,1]$, for $0 \le k \le n$, we have that

$$\frac{n-k}{n}\widetilde{B}_{k,n}^{(\alpha)}(x \mid q) + \frac{k+1}{n}\widetilde{B}_{k+1,n}^{(\alpha)}(x \mid q) = \widetilde{B}_{k,n-1}^{(\alpha)}(x \mid q). \tag{37}$$

It is possible to write $[x]_{q^\alpha}^k$ as a linear combination of $\widetilde{B}_{k,n}(x \mid q)$ by using the degree evaluation formula and mathematical induction:

$$\sum_{k=1}^{n}\frac{\binom{k}{1}}{\binom{n}{1}}\widetilde{B}_{k,n}^{(\alpha)}(x \mid q) = [x]_{q^\alpha}. \tag{38}$$

By the same method, we get

$$\sum_{k=1}^{n}\frac{\binom{k}{2}}{\binom{n}{2}}\widetilde{B}_{k,n}^{(\alpha)}(x \mid q) = [x]_{q^\alpha}^2. \tag{39}$$

Continuing this process, we have the following:

$$\sum_{k=j}^{n}\frac{\binom{k}{j}}{\binom{n}{j}}\widetilde{B}_{n,k}^{(\alpha)}(x \mid q) = [x]_{q^{(\alpha)}}^j. \tag{40}$$

For $x \in [0,1]$, $n \in \mathbb{N}$, $k \in \mathbb{Z}_+$,

$$\widetilde{B}_{n,k}^{(\alpha)} = \frac{[x]_{q^\alpha}^k}{1+[x]_{q^\alpha}}\binom{n}{k}\left(1-\left(-[x]_{q^\alpha}\right)^{n-k+1}\right). \tag{41}$$

By some calculus, we obtain the following:

$$\widetilde{B}_{n,k}^{(\alpha)} = \sum_{l=0}^{k}\binom{n}{k}\binom{k}{l}(-1)^l\frac{1-\left(-[x]_{q^\alpha}\right)^{n+l-k+1}}{1+[x]_{q^\alpha}}. \tag{42}$$

Hence, from (41) and (42) we have the following theorem.

Theorem 4. *For $\alpha \in \mathbb{Z}$, $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $n, k \in \mathbb{N}$, one has*

$$\begin{aligned}&[x]_{q^\alpha}^k\left(1-\left(-[x]_{q^\alpha}\right)^{n-k+1}\right)\\ &\quad= \sum_{l=0}^{k}\binom{k}{l}(-1)^l\left(1-\left(-[x]_{q^\alpha}\right)^{n+l-k+1}\right).\end{aligned} \tag{43}$$

4. Relations of q-Bernoulli Numbers and Polynomials with Weight (α,β) and q-Bernstein Polynomials

In this section, assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. With bosonic p-adic q-integral on $\mathbb{Z}_p$, we get from Theorem 2 and (26) that

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \widetilde{B}_{k,n}^{(\alpha)}(x \mid q)\, d\mu_{q^\beta}(x) \\
&= \int_{\mathbb{Z}_p} \binom{n}{k} [x]_{q^\alpha}^{k} [1-x]_{q^{-\alpha}}^{n-k} d\mu_{q^\beta}(x) \\
&= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^l \int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^{n-l} d\mu_{q^\beta}(x) \\
&= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^l \beta_{n-l,q^{-1}}^{(\alpha,\beta)}(2) \\
&= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^l \\
&\quad \times \left\{ q^{2\beta} \beta_{n-l,q^{-1}}^{(\alpha,\beta)} + 1 - q^\beta + (n-l) q^{\beta-\alpha} \frac{\alpha}{\beta} \left[\frac{\beta}{\alpha}\right]_{q^{-\alpha}} \right\}.
\end{aligned} \tag{44}$$

Therefore, we have the following theorem.

Theorem 5. *For $\alpha, \beta \in \mathbb{Z}$, $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $n, k \in \mathbb{N}$, one has*

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \widetilde{B}_{k,n}^{(\alpha)}(x \mid q)\, d\mu_{q^\beta}(x) \\
&= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^l \beta_{n-l,q^{-1}}^{(\alpha,\beta)}(2) \\
&= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^l \\
&\quad \times \left\{ q^{2\beta} \beta_{n-l,q^{-1}}^{(\alpha,\beta)} + 1 - q^\beta + (n-l) q^{\beta-\alpha} \frac{\alpha}{\beta} \left[\frac{\beta}{\alpha}\right]_{q^{-\alpha}} \right\}.
\end{aligned} \tag{45}$$

Using $[1-x]_{q^{-\alpha}} = (1-[x]_{q^\alpha})$, we get the other identities as below:

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \widetilde{B}_{k,n}^{(\alpha)}(x \mid q)\, d\mu_{q^\beta}(x) \\
&= \sum_{l=0}^{n-k} \binom{n}{k} \binom{n-k}{l} (-1)^l \beta_{k+l,q}^{(\alpha,\beta)}.
\end{aligned} \tag{46}$$

From Theorem 5 and (46), we obtain the following:

$$\begin{aligned}
&\sum_{l=0}^{k} \binom{k}{l} (-1)^l \beta_{n-l,q^{-1}}^{(\alpha,\beta)}(2) \\
&= \sum_{l=0}^{n-k} \binom{n}{k} \binom{n-k}{l} (-1)^l \beta_{k+l,q}^{(\alpha,\beta)}.
\end{aligned} \tag{47}$$

Specially, if $k = n$, then

$$\sum_{l=0}^{n} \binom{n}{l} (-1)^l \beta_{n-l,q^{-1}}^{(\alpha,\beta)}(2) = \beta_{n,q}^{(\alpha,\beta)}. \tag{48}$$

If $n = k+1$, then

$$\begin{aligned}
&\sum_{l=0}^{k} \binom{k}{l} (-1)^l \beta_{k+1-l,q^{-1}}^{(\alpha,\beta)}(2) \\
&= \beta_{k,q}^{(\alpha,\beta)} - \beta_{k+1,q}^{(\alpha,\beta)}.
\end{aligned} \tag{49}$$

Note that for $m, n, k \in \mathbb{Z}_+$,

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \widetilde{B}_{k,n}^{(\alpha)}(x \mid q)\, \widetilde{B}_{k,m}^{(\alpha)}(x \mid q)\, d\mu_{q^\beta}(x) \\
&= \binom{n}{k} \binom{m}{k} \int_{\mathbb{Z}_p} [x]_{q^\alpha}^{2k} [1-x]_{q^{-\alpha}}^{m+n-2k} d\mu_{q^\beta}(x) \\
&= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{m+n-2k} \binom{m+n-2k}{l} (-1)^l \int_{\mathbb{Z}_p} [x]_{q^\alpha}^{2k+l} d\mu_{q^\beta}(x) \\
&= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{m+n-2k} \binom{m+n-2k}{l} (-1)^l \beta_{2k+l,q}^{(\alpha,\beta)}.
\end{aligned} \tag{50}$$

Also,

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \widetilde{B}_{k,n}^{(\alpha)}(x \mid q)\, \widetilde{B}_{k,m}^{(\alpha)}(x \mid q)\, d\mu_{q^\beta}(x) \\
&= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{2k} (-1)^l \binom{2k}{l} \int_{\mathbb{Z}_p} [1-x]_{q^{-\alpha}}^{m+n-2k} d\mu_{q^\beta}(x) \\
&= \binom{n}{k} \binom{m}{k} \sum_{l=0}^{2k} (-1)^l \binom{2k}{l} \beta_{m+n-l}^{(\alpha,\beta)}(2)
\end{aligned}$$

$$= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{2k}(-1)^l\binom{2k}{l} \times \left\{q^{2\beta}\beta_{m+n-l}^{(\alpha,\beta)} + 1 - q^{\beta} + (m+n-l)\, q^{\beta-\alpha}\frac{\alpha}{\beta}\left[\frac{\beta}{\alpha}\right]_{q^{-\alpha}}\right\}. \tag{51}$$

Hence, one has the theorem as below.

Theorem 6. *For $\alpha, \beta \in \mathbb{Z}$, $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $n, k \in \mathbb{N}$, one has*

$$\begin{aligned}\int_{\mathbb{Z}_p} \widetilde{B}_{k,n}^{(\alpha)}(x \mid q)\, \widetilde{B}_{k,m}^{(\alpha)}(x \mid q)\, d\mu_{q^{\beta}}(x) &= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{m+n-2k}\binom{m+n-2k}{l}(-1)^l\beta_{2k+l,q}^{(\alpha,\beta)} \\ &= \binom{n}{k}\binom{m}{k}\sum_{l=0}^{2k}(-1)^l\binom{2k}{l}\beta_{m+n-l}^{(\alpha,\beta)}(2).\end{aligned} \tag{52}$$

Also, from Theorem 6, we get the identity as below:

$$\sum_{l=0}^{m+n-2k}\binom{m+n-2k}{l}(-1)^l\beta_{2k+l,q}^{(\alpha,\beta)} = \sum_{l=0}^{2k}(-1)^l\binom{2k}{l}\beta_{m+n-l}^{(\alpha,\beta)}(2). \tag{53}$$

Continuing this process, we obtain the following theorem.

Theorem 7. *For $\alpha, \beta \in \mathbb{Z}$, $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $n, k \in \mathbb{N}$, one has*

$$\begin{aligned}\int_{\mathbb{Z}_p} \widetilde{B}_{k,n_1}^{(\alpha)}(x \mid q)\, \widetilde{B}_{k,n_2}^{(\alpha)}(x \mid q)\cdots \widetilde{B}_{k,n_s}^{(\alpha)}(x \mid q)\, d\mu_{q^{\beta}}(x) &= \prod_{i=1}^{s}\binom{n_i}{k}\sum_{l=0}^{N-ks}\binom{N-ks}{l}(-1)^l\beta_{ks+l,q}^{(\alpha,\beta)}, \\ \int_{\mathbb{Z}_p} \widetilde{B}_{k,n_1}^{(\alpha)}(x \mid q)\, \widetilde{B}_{k,n_2}^{(\alpha)}(x \mid q)\cdots \widetilde{B}_{k,n_s}^{(\alpha)}(x \mid q)\, d\mu_{q^{\beta}}(x) &= \prod_{i=1}^{s}\binom{n_i}{k}\sum_{l=0}^{ks}\binom{ks}{l}(-1)^{ks-l}\beta_{N-l,q^{-1}}^{(\alpha,\beta)}(2),\end{aligned} \tag{54}$$

where $N = n_1 + n_2 + \cdots + n_s$.

Hence, from Theorem 7, we get the corollary as below.

Corollary 8. *For $\alpha, \beta \in \mathbb{Z}$, $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $n, k \in \mathbb{N}$, one has*

$$\sum_{l=0}^{N-ks}\binom{N-ks}{l}(-1)^l\beta_{ks+l,q}^{(\alpha,\beta)} = \sum_{l=0}^{ks}\binom{ks}{l}(-1)^{ks-l}\beta_{N-l,q^{-1}}^{(\alpha,\beta)}(2), \tag{55}$$

where $N = n_1 + n_2 + \cdots + n_s$.

5. Extension of q-Bernstein Polynomials with Weight α

In this section, we introduce the extended Kim's q-Bernstein polynomials of order n and the extended Kim's q-Bernstein operator of order n. Also, we define the extended Kim's q-Bernstein polynomials of order n with weight α and the extended Kim's q-Bernstein operator of order n with weight α and we investigate the properties of these. Finally, we investigate the relation of q-Bernoulli numbers with weight (α, β) and extension of q-Bernstein polynomials with weight α. We assume that $q \in \mathbb{R}$ with $0 < q < 1$. Let $C[0,1]$ be the set of continuous function on $[0,1]$ and $f \in C[0,1]$. For $n, k \in \mathbb{Z}_+$ and $x_1, x_2 \in [0,1]$, the extended Kim's q-Bernstein polynomials of order n are as below:

$$\widetilde{B}_{k,n}(x_1, x_2 \mid q) = \binom{n}{k}[x_1]_q^k[1-x_2]_{q^{-1}}^{n-k}. \tag{56}$$

For $n, k \in \mathbb{Z}_+$, $x_1, x_2 \in [0,1]$ and $f \in C[0,1]$, the extended Kim's q-Bernstein operator of order n is as below:

$$\begin{aligned}\widetilde{\mathbb{B}}_{n,q}(f \mid x_1, x_2) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}[x_1]_q^k[1-x_2]_{q^{-1}}^{n-k} \\ &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\widetilde{B}_{k,n}(x_1, x_2 \mid q).\end{aligned} \tag{57}$$

For $n, k \in \mathbb{Z}_+$, $x_1, x_2 \in [0,1]$, $f \in C[0,1]$ and $\alpha \in \mathbb{N}$, we define the extended Kim's q-Bernstein polynomials of order n with weight α and the extended Kim's q-Bernstein operator of order n with weight α as below:

$$\begin{aligned}\widetilde{B}_{k,n}^{(\alpha)}(x_1, x_2 \mid q) &= \binom{n}{k}[x_1]_{q^{\alpha}}^k[1-x_2]_{q^{-\alpha}}^{n-k}, \\ \widetilde{\mathbb{B}}_{n,q}^{(\alpha)}(f \mid x_1, x_2) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}[x_1]_{q^{\alpha}}^k[1-x_2]_{q^{-\alpha}}^{n-k} \\ &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\widetilde{B}_{k,n}^{(\alpha)}(x_1, x_2 \mid q).\end{aligned} \tag{58}$$

Properties of $\widetilde{B}_{k,n}^{(\alpha)}(x_1, x_2 \mid q)$ and $\widetilde{\mathbb{B}}_{n,q}^{(\alpha)}(f \mid x_1, x_2)$ are same to $\widetilde{B}_{k,n}(x_1, x_2 \mid q)$ and $\widetilde{\mathbb{B}}_{n,q}(fx_1, x_2)$, respectively. So we will enumerate these properties without proof as below.

For $\alpha, q, n \in \mathbb{N}$, $x_1, x_2 \in [0,1]$,

(1) $\widetilde{B}^{(\alpha)}_{n-k,n}(1-x_2, 1-x_1q^{-1}) = \widetilde{B}^{(\alpha)}_{k,n}(x_1, x_2q)$,

(2) $[1-x_2]_{q^{-\alpha}}\widetilde{B}^{(\alpha)}_{k,n-1}(x_1, x_2 \mid q) + [x_1]_{q^{\alpha}}\widetilde{B}^{(\alpha)}_{k-1,n-1}(x_1, x_2q) = \widetilde{B}^{(\alpha)}_{k,n}(x_1, x_2q)$,

(3) $(\partial/\partial x_1)\widetilde{B}^{(\alpha)}_{k,n}(x_1, x_2q) = (\alpha \log q/(q^{\alpha}-1))q^{\alpha x_1} n\widetilde{B}^{(\alpha)}_{k-1,n-1}(x_1, x_2q)$,

(4) $(\partial/\partial x_2)\widetilde{B}^{(\alpha)}_{k,n}(x_1, x_2q) = \alpha \log q/(1-q^{\alpha})q^{\alpha x_2} n\widetilde{B}^{(\alpha)}_{k,n-1}(x_1, x_2q)$,

(5) $(1/(1+[x_1]_{q^{\alpha}}-[x_2]_{q^{\alpha}})^{n-j}) \sum_{k=j}^{n}(\binom{k}{j}/\binom{n}{j})\widetilde{B}^{(\alpha)}_{k,n}(x_1, x_2 \mid q) = [x_1]^{j}_{q^{\alpha}}$.

Taking double bosonic p-adic q-integral on $\mathbb{Z}_p$, we get from (26) and (27) that

$$\iint_{\mathbb{Z}_p} \widetilde{B}^{(\alpha)}_{k,n}(x_1, x_2 \mid q)\, d\mu_{q^{\beta}}(x_1)\, d\mu_{q^{\beta}}(x_2) = \binom{n}{k}\beta^{(\alpha,\beta)}_{n,q} \times \left(q^{2\beta}\beta^{(\alpha,\beta)}_{n-k,q^{-1}} + 1 - q^{\beta} + (n-k)\, q^{\beta-\alpha}\frac{\alpha}{\beta}\left[\frac{\beta}{\alpha}\right]_{q^{-\alpha}}\right). \tag{59}$$

Thus, from (59), we have the following theorem.

Theorem 9. *For $\alpha, \beta \in \mathbb{Z}$, $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $n, k \in \mathbb{N}$, one has*

$$\iint_{\mathbb{Z}_p} \widetilde{B}^{(\alpha)}_{k,n}(x_1, x_2 \mid q)\, d\mu_{q^{\beta}}(x_1)\, d\mu_{q^{\beta}}(x_2) = \begin{cases} \binom{n}{k}\beta^{(\alpha,\beta)}_{k,q}\left(q^{2\beta}\beta^{(\alpha,\beta)}_{n-k,q^{-1}} + 1 - q^{\beta} + (n-k)\, q^{\beta-\alpha}\frac{\alpha}{\beta}\left[\frac{\beta}{\alpha}\right]_{q^{-\alpha}}\right) & \text{if } n > k+1 \\ (k+1)\,\beta^{(\alpha,\beta)}_{k,q}\left(q^{2\beta}\beta^{(\alpha,\beta)}_{1,q^{-1}} + 1 - q^{\beta} + q^{\beta-\alpha}\frac{\alpha}{\beta}\left[\frac{\beta}{\alpha}\right]_{q^{-\alpha}}\right) & \text{if } n = k+1 \\ 0 & \text{if } n < k \\ \beta^{(\alpha,\beta)}_{k,q} & \text{if } n = k \\ 1 & \text{if } n = k = 0. \end{cases} \tag{60}$$

We get from the q-symmetric properties of the q-Bernstein polynomials that for $n, k \in \mathbb{Z}_+$

$$\begin{aligned} &\iint_{\mathbb{Z}_p} \widetilde{B}^{(\alpha)}_{k,n}(x_1, x_2 \mid q)\, d\mu_{q^{\beta}}(x_1)\, d\mu_{q^{\beta}}(x_2) \\ &= \iint_{\mathbb{Z}_p} \binom{n}{n-k}[1-x_2]^{n-k}_{q^{-\alpha}}[x_1]^{k}_{q^{\alpha}} d\mu_{q^{\beta}}(x_1)\, d\mu_{q^{\beta}}(x_2) \\ &= \binom{n}{k}\int_{\mathbb{Z}_p}[1-x_2]^{n-k}_{q^{-\alpha}} d\mu_{q^{\beta}}(x_2) \\ &\quad \times \left(\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\int_{\mathbb{Z}_p}[1-x_1]^{k-l}_{q^{-\alpha}} d\mu_{q^{\beta}}(x_1)\right) \\ &= \binom{n}{k}\beta^{(\alpha)}_{n-k,q^{-1}}(2) \\ &\quad \times \left(1 - k\beta^{(\alpha,\beta)}_{1,q^{-1}}(2) + \sum_{l=0}^{k-2}\binom{k}{l}(-1)^{k+l}\beta^{(\alpha,\beta)}_{k-l,q^{-1}}(2)\right). \end{aligned} \tag{61}$$

Note that for $m, n, k \in \mathbb{Z}_+$

$$\iint_{\mathbb{Z}_p} \widetilde{B}^{(\alpha)}_{k,n}(x_1, x_2 \mid q)\, \widetilde{B}^{(\alpha)}_{k,m}(x_1, x_2 \mid q)\, d\mu_{q^{\beta}}(x_1)\, d\mu_{q^{\beta}}(x_2) = \binom{n}{k}\binom{m}{k}\beta^{(\alpha,\beta)}_{2k,q}\int_{\mathbb{Z}_p}[1-x_2]^{m+n-2k}_{q^{-\alpha}} d\mu_{q^{\beta}}(x_2). \tag{62}$$

Note also that for $m, n, k \in \mathbb{Z}_+$

$$\begin{aligned} &\iint_{\mathbb{Z}_p} \widetilde{B}^{(\alpha)}_{k,n}(x_1, x_2 \mid q)\, \widetilde{B}^{(\alpha)}_{k,m}(x_1, x_2 \mid q)\, d\mu_{q^{\beta}}(x_1)\, d\mu_{q^{\beta}}(x_2) \\ &= \binom{n}{k}\binom{m}{k}\int_{\mathbb{Z}_p}[1-x_2]^{m+n-2k}_{q^{-\alpha}} d\mu_{q^{\beta}}(x_2) \\ &\quad \times \sum_{l=0}^{2k}\binom{2k}{l}(-1)^{2k-l} \\ &\quad \times \int_{\mathbb{Z}_p}[1-x_1]^{2k-l}_{q^{-\alpha}} d\mu_{q^{\beta}}(x_1) \\ &= \binom{n}{k}\binom{m}{k}\int_{\mathbb{Z}_p}[1-x_2]^{m+n-2k}_{q^{-\alpha}} d\mu_{q^{\beta}}(x_2) \\ &\quad \times \left(1 - 2k\beta^{(\alpha,\beta)}_{1,q}(2) + \sum_{l=0}^{2k-2}\binom{2k}{l}(-1)^{2k-l}\beta^{(\alpha,\beta)}_{2k-l,q}(2)\right). \end{aligned} \tag{63}$$

Therefore, from (63) and (65), we have the following theorem.

Theorem 10. *For $\alpha, \beta \in \mathbb{Z}$, $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $n, k \in \mathbb{N}$, one has*

$$\beta_{2k,q}^{(\alpha,\beta)} = 1 - 2k\beta_{1,q}^{(\alpha,\beta)}(2) + \sum_{l=0}^{2k-2} \binom{2k}{l} (-1)^{2k-l} \beta_{2k-l,q}^{(\alpha,\beta)}(2). \tag{64}$$

Note that for $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ and $s \in \mathbb{N}$

$$\begin{aligned} &\iint_{\mathbb{Z}_p} \prod_{i=1}^{s} \widetilde{B}_{k,n_i}^{(\alpha)}(x_1, x_2 \mid q)\, d\mu_{q^\beta}(x_1)\, d\mu_{q^\beta}(x_2) \\ &\quad = \iint_{\mathbb{Z}_p} \prod_{i=1}^{s} \binom{n_i}{k} [x_1]_{q^\alpha}^{sk} [1-x_2]_{q^{-\alpha}}^{M-sk} d\mu_{q^\beta}(x_2) \\ &\quad = \prod_{i=1}^{s} \binom{n_i}{k} \beta_{sk,q}^{(\alpha,\beta)} \int_{\mathbb{Z}_p} [1-x_2]_{q^{-\alpha}}^{M-sk} d\mu_{q^\beta}(x_2), \end{aligned} \tag{65}$$

where $n_1 + n_2 + \cdots + n_s = M$.

Note also from the binomial theorem that $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ and $s \in \mathbb{N}$,

$$\begin{aligned} &\iint_{\mathbb{Z}_p} \prod_{i=1}^{s} \widetilde{B}_{k,n_i}^{(\alpha)}(x_1, x_2 \mid q)\, d\mu_{q^\beta}(x_1)\, d\mu_{q^\beta}(x_2) \\ &\quad = \prod_{i=1}^{s} \binom{n_i}{k} \int_{\mathbb{Z}_p} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk-l} [1-x_1]_{q^{-\alpha}}^{sk-l} d\mu_{q^\beta}(x_1) \\ &\qquad \times \int_{\mathbb{Z}_p} [1-x_2]_{q^{-\alpha}}^{M-sk} d\mu_{q^\beta}(x_2), \end{aligned} \tag{66}$$

where $n_1 + n_2 + \cdots + n_s = M$.

Hence, from (65) and (66), we see that the following theorem holds.

Theorem 11. *For $\alpha, \beta \in \mathbb{Z}$, $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $n, k \in \mathbb{N}$, one has*

$$\beta_{sk,q}^{(\alpha,\beta)} = 1 - sk\beta_{1,q}^{(\alpha,\beta)}(2) + \sum_{l=0}^{sk-2} \binom{sk}{l} (-1)^{sk-l} \beta_{sk-l,q}^{(\alpha,\beta)}(2). \tag{67}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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