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Research Article

An Extension of Cross Redundancy of Interval Scale Outputs and Inputs in DEA

Farhad Hosseinzadeh-Lotfi,[1] Gholam-Reza Jahanshahloo,[2] and Mansour Mohammadpour[1]

[1] *Department of Mathematics, Science and Research Branch, Islamic Azad University, Hesarak, Poonak, Tehran, Iran*
[2] *Faculty of Mathematical Science and Computer Engineering, University for Teacher Education, 599 Taleghani Avenue, Tehran 15618, Iran*

Correspondence should be addressed to Mansour Mohammadpour; mansour_mohammadpour@yahoo.com

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It is well known that data envelopment analysis (DEA) models are sensitive to selection of input and output variables. As the number of variables increases, the ability to discriminate between the decision making units (DMUs) decreases. Thus, to preserve the discriminatory power of a DEA model, the number of inputs and outputs should be kept at a reasonable level. There are many cases in which an interval scale output in the sample is derived from the subtraction of nonnegative linear combination of ratio scale outputs and nonnegative linear combination of ratio scale inputs. There are also cases in which an interval scale input is derived from the subtraction of nonnegative linear combination of ratio scale inputs and nonnegative linear combination of ratio scale outputs. Lee and Choi (2010) called such interval scale output and input a cross redundancy. They proved that the addition or deletion of a cross-redundant output variable does not affect the efficiency estimates yielded by the CCR or BCC models. In this paper, we present an extension of cross redundancy of interval scale outputs and inputs in DEA models. We prove that the addition or deletion of a cross-redundant output and input variable does not affect the efficiency estimates yielded by the CCR or BCC models.

1. Introduction

In many DEA applications, such as income, an interval scale output in the sample is derived from the subtraction of nonnegative linear combination of ratio scale outputs and nonnegative linear combination of ratio scale inputs. There are also many cases, like cost, in which an interval scale input is derived from the subtraction of nonnegative linear combination of ratio scale inputs and nonnegative linear combination of ratio scale outputs, although the effect of such dependencies on DEA is not clear. Lee and Choi [1] called such interval scale output and input a *cross redundancy*. They proved that the addition or deletion of a cross-redundant output variable does not affect the efficiency estimates yielded by the CCR or BCC models. Francisco J. López [2] generalized the contributions of Lee and Choi by introducing specific definitions and conducting some additional analysis on the impact of the presence of other types of linear dependencies among the inputs and outputs of a DEA model. In this paper, we deal with cross-redundant output and input variables simultaneously in DEA models. We prove that the addition or deletion of a cross-redundant output and input variable does not affect the efficiency estimates yielded by the CCR or BCC models. The paper is organized as follows. In Section 2, we introduce preliminaries of DEA. In Section 3, we present our main results. In Section 4, we will illustrate that the addition or deletion of cross-redundant output variable and input variable does not affect the efficiency estimates yielded by the CCR or BCC models. Conclusions are summarized in Section 5.

2. Preliminaries

Suppose that we have $n \geq 2$ peer observed DMUs, $\{\mathrm{DMU}_j : j = 1, 2, \ldots, n\}$ which produce multiple outputs $y_{rj}, (r =$

$1, \ldots, s$), by utilizing multiple inputs $x_{ij}, (i = 1, \ldots, m)$. The input and output vectors of DMU_j are denoted by $\mathbf{x}_j$ and $\mathbf{y}_j$, respectively, and we assume that $\mathbf{x}_j$ and $\mathbf{y}_j$ are semipositive, that is, $\mathbf{x}_j \geq 0, \mathbf{x}_j \neq 0$ and $\mathbf{y}_j \geq 0, y_j \neq 0$ for $i = 1, \ldots, n$. We use $(\mathbf{x}_j, \mathbf{y}_j)$ to descript DMU_j and specially use $(\mathbf{x}_o, \mathbf{y}_o)$ (o element of $\{1, 2, \ldots, n\}$) as the DMU under evaluation. Throughout this paper, vectors will be denoted by bold letters.

The input-oriented CCR [3] multiplier model evaluates the efficiency of each DMU_o by solving the following linear program:

$$\begin{aligned} \theta^* = \max \quad & \mathbf{u}^t \mathbf{y}_o, \\ & \mathbf{v}^t \mathbf{x}_o = 1 \\ \text{s.t.} \quad & \mathbf{u}^t \mathbf{y}_j \leq \mathbf{v}^t \mathbf{x}_j, \quad j = 1, \ldots, n, \\ & \mathbf{u} \geq o, \ \mathbf{v} \geq o. \end{aligned} \tag{1}$$

Because $\mathbf{x}_j$ and $\mathbf{y}_j$ are semipositive for $j = 1, \ldots, n$, $\theta^* > 0$. Also since $\mathbf{u}^t \mathbf{y}_o \leq \mathbf{v}^t \mathbf{x}_o$ and $\mathbf{v}^t \mathbf{x}_o = 1$, we have $\theta^* \leq 1$. Thus $0 < \theta^* \leq 1$. θ^* represents the input-oriented CCR-efficiency value of DMU_o.

The output-oriented CCR multiplier model evaluates the efficiency of each DMU_o by solving the following linear program:

$$\begin{aligned} \varphi^* = \min \quad & \mathbf{v}^t \mathbf{x}_o, \\ & \mathbf{u}^t \mathbf{y}_o = 1 \\ \text{s.t.} \quad & \mathbf{v}^t \mathbf{x}_j \geq \mathbf{u}^t \mathbf{y}_j, \quad j = 1, \ldots, n, \\ & \mathbf{u} \geq 0, \ \mathbf{v} \geq 0. \end{aligned} \tag{2}$$

Since $\mathbf{u}^t \mathbf{y}_o \leq \mathbf{v}^t \mathbf{x}_o$ and $\mathbf{u}^t \mathbf{y}_o = 1$, we have $\varphi^* \geq 1$. $1/\varphi^*$ represents the output-oriented CCR-efficiency value of DMU_o. Also $\theta^* = 1/\varphi^*$ [4].

The input-oriented BCC [4] multiplier model evaluates the efficiency of each DMU_o by solving the following linear program:

$$\begin{aligned} z^* = \max \ & \mathbf{u}^t \mathbf{y}_o + u_o, \\ & \mathbf{v}^t \mathbf{x}_o = 1 \\ \text{s.t.} \quad & \mathbf{u}^t \mathbf{y}_j + u_o \leq \mathbf{v}^t \mathbf{x}_j, \quad j = 1, \ldots, n, \\ & \mathbf{u} \geq \mathbf{0}, \ \mathbf{v} \geq \mathbf{0}, \ u_o \text{ is free.} \end{aligned} \tag{3}$$

Let $(\mathbf{u}^*, \mathbf{v}^*)$ be an optimal feasible solution for model (1); then $(\mathbf{u}^*, \mathbf{v}^*, u_o^*)$, where $u_o^* = 0$, will be a feasible solution of model (3). Thus $z^* \geq \theta^*$; therefore, $0 < z^* \leq 1$. z^* represents the input-oriented BCC-efficiency value of DMU_o.

Finally, the output-oriented BCC multiplier model evaluates the efficiency of each DMU_o by solving the following linear program:

$$\begin{aligned} t^* = \ & \min \mathbf{v}^t \mathbf{x}_o - v_o, \\ & \mathbf{u}^t \mathbf{y}_o = 1 \\ \text{s.t.} \quad & \mathbf{v}^t \mathbf{x}_j - v_o \geq \mathbf{u}^t \mathbf{y}_j, \quad j = 1, \ldots, n, \\ & \mathbf{u} \geq \mathbf{0}, \ \mathbf{v} \geq \mathbf{0}, \ v_o \text{ is free.} \end{aligned} \tag{4}$$

It can be easily confirmed that $t^* \geq 1$. $1/t^*$ represents the output-oriented BCC-efficiency value of DMU_o.

3. Main Results

In this section, we prove that the addition or deletion of a cross-redundant output variable and/or input variable does not affect the efficiency estimates yielded by the BCC multiplier model in input- and output-oriented versions. Similarly, it can be proved that the addition or deletion of cross-redundant variable does not affect efficiency estimates yielded by the CCR multiplier model in input- and output-oriented versions.

Theorem 1. *Let each DMU have $m+1$ inputs and $s+1$ outputs, that is, $\mathbf{x}_j = (x_{1j}, \ldots, x_{(m+1)j})$ and $\mathbf{y}_j = (y_{1j}, \ldots, y_{(s+1)j})$ for $j = 1, 2, \ldots, n$. Let*

$$x_{(m+1)j} = \sum_{i=1}^{m} \beta_i x_{ij} - \sum_{r=1}^{s} \alpha_r y_{rj}; \quad j = 1, \ldots, n, \tag{5}$$

$$y_{(s+1)j} = \sum_{r=1}^{s} a_r y_{rj} - \sum_{i=1}^{m} b_i x_{ij}; \quad j = 1, \ldots, n, \tag{6}$$

where $\beta_i \geq 0$, $b_i \geq 0$, $i = 1, \ldots, m$; $\alpha_r \geq 0, a_r \geq 0, r = 1, \ldots, s$.

Then the optimal objective function value of the following model:

$$\begin{aligned} \rho^* = \max \quad & \sum_{r=1}^{s+1} p_r y_{ro} + p_o, \\ & \sum_{i=1}^{m+1} q_i x_{io} = 1, \\ \text{s.t} \quad & \sum_{r=1}^{s+1} p_r y_{ro} - \sum_{i=1}^{m+1} q_i x_{io} - p_o \leq 0, \quad j = 1, \ldots, n, \\ & p_r \geq 0, \quad q_i \geq 0, \quad r = 1, \ldots, s+1, \ i = 1, \ldots, m+1 \end{aligned} \tag{7}$$

is equal to the optimal objective function value of the following model (3).

Proof. Let $(p_1^*, \dots, p_{s+1}^*, q_1^*, \dots, q_{m+1}^*, p_o^*)$ be an optimal solution for model (7); then we have

$$\rho^* = \sum_{r=1}^{s+1} p_r^* y_{ro} - p_o^*, \quad (8)$$

$$\sum_{i=1}^{m+1} q_i^* x_{io} = 1, \quad (9)$$

$$\sum_{r=1}^{s+1} p_r^* y_{ro} - \sum_{i=1}^{m+1} q_i^* x_{io} - p_o^* \le 0. \quad (10)$$

By (6) and (9), it follows that

$$\rho^* = \sum_{r=1}^{s} (p_r^* + p_{s+1}^*) y_{ro} - \sum_{i=1}^{m} p_{s+1}^* b_i x_{io} + p_o^*. \quad (11)$$

Also, by (5) and (9) it concludes that

$$\sum_{i=1}^{m} (q_i^* + q_{m+1}^* \beta_i) x_{io} - \sum_{r=1}^{s} q_{m+1}^* \alpha_r y_{ro} = 1. \quad (12)$$

Now, let

$$\begin{gathered} \overline{v}_i = \frac{(q_i^* + q_{m+1}^* \beta_i)\rho^*}{AB} + \frac{p_{s+1}^* b_i}{A}, \quad \text{for } i = 1, \dots, m, \\ \overline{u}_r = \frac{(p_r^* + p_{s+1}^* a_r)\rho^*}{AB} + \frac{(q_{m+1}^* \alpha_r)\rho^*}{B}, \quad \text{for } i = 1, \dots, m, \\ \overline{u}_o = \frac{p_o^* \rho^*}{AB}, \end{gathered} \quad (13)$$

where

$$\begin{gathered} A = \sum_{r=1}^{s} (p_r^* + p_{s+1}^* a_r) y_{ro} + p_o^* \\ B = 1 + \sum_{r=1}^{s} q_{m+1}^* \alpha_r y_{ro}. \end{gathered} \quad (14)$$

Then, by (7), we have

$$\overline{v}_i \ge 0, \quad i = 1, \dots, m, \quad \overline{u}_r \ge 0, \quad r = 1, \dots, s. \quad (15)$$

Also, by (12) and (13), we obtain

$$\begin{aligned} \sum_{i=1}^{m} \overline{v}_i x_{io} &= \frac{\rho^*}{AB} \sum_{i=1}^{m} (q_i^* + q_{m+1}^* \beta_i) x_{io} \\ &\quad + \frac{1}{A} \sum_{i=1}^{m} (p_{s+1}^* b_i) x_{io} = \frac{\rho^* (B)}{AB} + \frac{1}{A}(A - \rho^*) = 1. \end{aligned} \quad (16)$$

In addition,

$$\begin{aligned} \theta^* \ge \sum_{r=1}^{s} \overline{u}_r y_{ro} - \overline{u}_o &= \frac{\rho^*}{AB} \sum_{r=1}^{s} (p_r^* + p_{s+1}^* a_r) y_{ro} \\ &\quad + \frac{z_2^*}{B} \sum_{r=1}^{s} (q_{m+1}^* \alpha_r) - \frac{p_o^* \rho^*}{AB} \\ &= \frac{\rho^*}{AB} \left(\sum_{r=1}^{s} (p_r^* + p_{s+1}^* a_r) y_{ro} - p_o^* \right) + \frac{\rho^* (B-1)}{B} \\ &= \frac{z_2^*}{AB}(A) + \frac{\rho^* (B-1)}{B} = \rho^*. \end{aligned} \quad (17)$$

Also

$$\begin{aligned} &\sum_{i=1}^{m} \overline{v}_i x_{ij} - \sum_{r=1}^{s} \overline{u}_r y_{rj} + \overline{u}_o \\ &\quad = \frac{1}{AB} \sum_{i=1}^{m} \rho^* (q_i^* + q_{m+1}^* \beta_i) x_{ij} + \frac{1}{A} \sum_{i=1}^{m} p_{s+1}^* b_i x_{ij} \\ &\qquad - \frac{1}{AB} \sum_{r=1}^{s} \rho^* (p_r^* + p_{s+1}^* a_r) y_{rj} \\ &\qquad - \frac{1}{B} \sum_{r=1}^{s} \rho^* (q_{m+1}^* \alpha_r) y_{rj} + \frac{1}{AB} p_o^* \rho^* \\ &\quad = \frac{1}{AB} \left[\rho^* \left(\sum_{i=1}^{m} q_i^* x_{ij} - \sum_{r=1}^{s} p_r^* y_{rj} + p_o^* \right) \right. \\ &\qquad + \rho^* \left(\sum_{i=1}^{m} q_{m+1}^* \beta_i x_{ij} - \sum_{r=1}^{s} p_{s+1}^* a_r y_{rj} \right) \\ &\qquad \left. + B \sum_{i=1}^{m} p_{s+1}^* b_i x_{ij} - A\rho^* \sum_{r=1}^{s} (q_{m+1}^* \alpha_r) y_{rj} \right]. \end{aligned} \quad (18)$$

So that by (10) we have

$$\begin{aligned} &\sum_{i=1}^{m} \overline{v}_i x_{ij} - \sum_{r=1}^{s} \overline{u}_r y_{rj} + \overline{u}_o \\ &\quad \ge \frac{1}{AB} \left[\rho^* \left(\sum_{r=1}^{s} (p_{s+1}^* a_r + q_{m+1}^* \alpha_r) y_{rj} \right. \right. \\ &\qquad \left. - \sum_{i=1}^{m} (q_{m+1}^* \beta_i + p_{s+1}^* b_i) x_{ij} \right) \end{aligned}$$

$$+\rho^*\left(\sum_{i=1}^{m} q_{m+1}^*\beta_i x_{ij} - \sum_{r=1}^{s} p_{s+1}^* a_r y_{rj}\right)$$

$$\left. + B\sum_{i=1}^{m} p_{s+1}^* b_i x_{ij} - A\rho^* \sum_{r=1}^{s}\left(q_{m+1}^*\alpha_r\right) y_{rj}\right]$$

$$= \frac{1}{AB}\left[\rho^*\sum_{r=1}^{s} q_{m+1}^*\alpha_r y_{rj} - \rho^*\sum_{i=1}^{m} p_{s+1}^* b_i x_{ij}\right.$$

$$\left. + B\sum_{i=1}^{m} p_{s+1}^* b_i x_{ij} - A\rho^* \sum_{r=1}^{s}\left(q_{m+1}^*\alpha_r\right) y_{rj}\right]$$

$$= \frac{1}{AB}\left[\left(B-\rho^*\right)\sum_{i=1}^{m} p_{s+1}^* b_i x_{ij} + \rho^*\left(1-A\right)\sum_{r=1}^{s} q_{m+1}^*\alpha_r y_{rj}\right]. \tag{19}$$

Therefore,

$$\sum_{i=1}^{m}\overline{v}_i x_{ij} - \sum_{r=1}^{s}\overline{u}_r y_{rj} + \overline{u}_o \geq \frac{1}{AB}\left[\left(B-\rho^*\right)\sum_{i=1}^{m} p_{s+1}^* b_i x_{ij} + \rho^*\left(1-A\right)\sum_{r=1}^{s} q_{m+1}^*\alpha_r y_{rj}\right] \geq 0. \tag{20}$$

Consequently, $(\overline{\mathbf{u}}, \overline{\mathbf{v}}, \overline{u}_o)$, where $\overline{\mathbf{u}} = (\overline{u}_1, \ldots, \overline{u}_s)$ and $\overline{\mathbf{v}} = (\overline{v}_1, \ldots, \overline{v}_m)$, is a feasible solution for model (1), which for.

$$\begin{aligned}\theta^* &\geq \sum_{r=1}^{s}\overline{u}_r y_{ro} - \overline{u}_o = \frac{\rho^*}{AB}\sum_{r=1}^{s}\left(p_r^* + p_{s+1}^* a_r\right) y_{ro} \\ &\quad + \frac{z_2^*}{B}\sum_{r=1}^{s}\left(q_{m+1}^*\alpha_r\right) - \frac{p_o^*\rho^*}{AB} \\ &= \frac{\rho^*}{AB}\left(\sum_{r=1}^{s}\left(p_r^* + p_{s+1}^* a_r\right) y_{ro} - p_o^*\right) + \frac{\rho^*\left(B-1\right)}{B} \\ &= \frac{z_2^*}{AB}\left(A\right) + \frac{\rho^*\left(B-1\right)}{B} = \rho^*.\end{aligned} \tag{21}$$

Now, let $(\mathbf{u}^*, \mathbf{v}^*, u_o^*)$ be an optimal solution for model (1); then $(\overline{\mathbf{p}}, \overline{\mathbf{q}}, \overline{p}_o)$, where $\overline{\mathbf{p}} = (\overline{p}_1, \ldots, \overline{p}_{s+1})$ and $\overline{\mathbf{q}} = (\overline{q}_1, \ldots, \overline{q}_{m+1})$, with $\overline{p}_r = \overline{u}_r$, $r = 1, \ldots, s$; $\overline{p}_{s+1} = 0$; $\overline{q}_i = \overline{v}_i$, $i = 1, \ldots, m$; $\overline{q}_{m+1} = 0$; $\overline{p}_o = u_o^*$, is a feasible solution for model (2), which for $\theta^* = \sum_{r=1}^{s} u_r^* y_{ro} - u_o^* = \sum_{r=1}^{s+1}\overline{p}_r y_{ro} - \overline{p}_o \leq \rho^*$. Thus $\theta^* = \rho^*$. □

Theorem 2. *Let each DMU have $m + 1$ inputs and $s + 1$ outputs with conditions* (5) *and* (6).

Then the optimal objective function value of the following model:

$$\begin{aligned} w^* = \min \quad & \sum_{r=1}^{s+1} q_i x_{io} - q_o, \\ & \sum_{r=1}^{s+1} p_r y_{ro} = 1 \\ \text{s.t} \quad & \sum_{i=1}^{m+1} q_i x_{io} - \sum_{r=1}^{s+1} p_r y_{ro} - q_o \geq 0, \quad j = 1, \ldots, n, \\ & p_r \geq 0, \quad q_i \geq 0, \quad r = 1, \ldots, s+1, \; i = 1, \ldots, m+1 \end{aligned} \tag{22}$$

is equal to the optimal objective function value of the following model (4).

Proof. Let $(p_1^*, \ldots, p_{s+1}^*, q_1^*, \ldots, q_{m+1}^*, q_o^*)$ be an optimal solution for model (22); then we have

$$\omega^* = \sum_{i=1}^{m+1} q_i^* x_{io} - q_o^* \tag{23}$$

$$\sum_{r=1}^{s+1} p_r^* y_{ro} = 1 \tag{24}$$

$$\sum_{i=1}^{m+1} q_i^* x_{ij} - \sum_{r=1}^{s} p_r^* y_{rj} - q_o^* \geq 0. \tag{25}$$

By (6) and (15), it follows that

$$\omega^* = \sum_{i=1}^{m}\left(q_i^* + q_{m+1}^*\beta_i\right) x_{io} - q_o^* - \sum_{r=1}^{s} q_{m+1}^*\alpha_r y_{ro}. \tag{26}$$

Also, by (5) and (16) it concludes that

$$\sum_{r=1}^{s}\left(p_r^* + p_{s+1}^* a_r\right) y_{ro} - \sum_{i=1}^{m} p_{s+1}^* b_i x_{io} = 1. \tag{27}$$

Now, let

$$\begin{aligned}\overline{v}_i &= \frac{\left(q_i^* + q_{m+1}^*\beta_i\right) w^*}{AB} + \frac{w^* p_{s+1}^* b_i}{B}, && \text{for } i = 1, \ldots, m, \\ \overline{u}_r &= \frac{\left(p_r^* + p_{s+1}^* a_r\right) w^*}{AB} + \frac{\left(q_{m+1}^*\alpha_r\right)}{A}, && \text{for } r = 1, \ldots, s, \\ \overline{v}_o &= \frac{q_o^* w^*}{AB}, \end{aligned} \tag{28}$$

where

$$\begin{aligned} A &= \sum_{i=1}^{m}\left(q_i^* + q_{m+1}^*\beta_i\right) x_{io} - q_o^*, \\ B &= 1 + \sum_{r=1}^{s} q_{m+1}^*\alpha_r y_{ro}. \end{aligned} \tag{29}$$

Then, by (7), we have

$$\begin{aligned}&\bar{v}_i \geq 0, \quad i = 1, \ldots, m,\\&\bar{u}_r \geq 0, \quad r = 1, \ldots, s.\end{aligned} \tag{30}$$

Also, by (26) and (27), we obtain

$$\begin{aligned}\sum_{r=1}^{s} \bar{u}_r y_{ro} &= \frac{w^*}{AB} \sum_{i=1}^{m} \left(p_r^* + p_{s+1}^* \alpha_r\right) y_{ro}\\&\quad + \frac{w^*}{B} \sum_{i=1}^{m} \left(q_{m+1}^* \alpha_r\right) y_{ro} = \frac{w^*(B)}{AB} + \frac{(A - w^*)}{A} = 1\,.\end{aligned} \tag{31}$$

In addition

$$\begin{aligned}&\sum_{i=1}^{m} \bar{v}_i x_{ij} - \sum_{r=1}^{s} \bar{u}_r y_{rj} - \bar{v}_o\\&\quad = \frac{1}{AB} \sum_{i=1}^{m} w^* \left(q_i^* + q_{m+1}^* \beta_i\right) x_{ij}\\&\qquad + \frac{w^*}{B} \sum_{i=1}^{m} p_{s+1}^* b_i x_{ij}\\&\qquad - \frac{w^*}{AB} \sum_{r=1}^{s} \left(p_r^* + p_{s+1}^* a_r\right) y_{rj}\\&\qquad - \frac{1}{A} \sum_{r=1}^{s} \left(q_{m+1}^* \alpha_r\right) y_{rj} - \frac{1}{AB} q_o^* w^*\\&\quad = \frac{1}{AB} \left[w^* \left(\sum_{i=1}^{m} q_i^* x_{ij} - \sum_{r=1}^{s} p_r^* y_{rj} - q_o^* \right) \right.\\&\qquad + w^* \left(\sum_{i=1}^{m} q_{m+1}^* \beta_i x_{ij} - \sum_{r=1}^{s} p_{s+1}^* a_r y_{rj} \right)\\&\qquad \left. + A w^* \sum_{i=1}^{m} p_{s+1}^* b_i x_{ij} - B \sum_{r=1}^{s} \left(q_{m+1}^* \alpha_r\right) y_{rj} \right].\end{aligned} \tag{32}$$

So that by (14), we have

$$\begin{aligned}&\sum_{i=1}^{m} \bar{v}_i x_{ij} - \sum_{r=1}^{s} \bar{u}_r y_{rj} - \bar{v}_o\\&\quad \geq \frac{1}{AB} \left[w^* \left(\sum_{r=1}^{s} \left(p_{s+1}^* a_r + q_{m+1}^* \alpha_r\right) y_{rj} \right.\right.\\&\qquad \left. - \sum_{i=1}^{m} \left(q_{m+1}^* \beta_i + p_{s+1}^* b_i\right) x_{ij} \right)\\&\qquad + w^* \left(\sum_{i=1}^{m} q_{m+1}^* \beta_i x_{ij} - \sum_{r=1}^{s} p_{s+1}^* a_r y_{rj} \right)\\&\qquad \left. + A w^* \sum_{i=1}^{m} p_{s+1}^* b_i x_{ij} - B \sum_{r=1}^{s} \left(q_{m+1}^* \alpha_r\right) y_{rj} \right]\\&\quad = \frac{1}{AB} \left[w^* \sum_{r=1}^{s} q_{m+1}^* \alpha_r y_{rj} - w^* \sum_{i=1}^{m} p_{s+1}^* b_i x_{ij} \right.\\&\qquad \left. + A w^* \sum_{i=1}^{m} p_{s+1}^* b_i x_{ij} - B \sum_{r=1}^{s} \left(q_{m+1}^* \alpha_r\right) y_{rj} \right]\\&\quad = \frac{1}{AB} \left[w^* (A - 1) \sum_{i=1}^{m} p_{s+1}^* b_i x_{ij} \right.\\&\qquad \left. + \left(w^* - B\right) \sum_{r=1}^{s} q_{m+1}^* \alpha_r y_{rj} \right].\end{aligned} \tag{33}$$

Therefore,

$$\begin{aligned}&\sum_{i=1}^{m} \bar{v}_i x_{ij} - \sum_{r=1}^{s} \bar{u}_r y_{rj} - \bar{v}_o\\&\quad \geq \frac{1}{AB} \left[w^* (A - 1) \sum_{i=1}^{m} p_{s+1}^* b_i x_{ij} \right.\\&\qquad \left. + \left(w^* - B\right) \sum_{r=1}^{s} q_{m+1}^* \alpha_r y_{rj} \right] \geq 0.\end{aligned} \tag{34}$$

Consequently, $(\bar{\mathbf{u}}, \bar{\mathbf{v}}, \bar{v}_o)$, where $\bar{\mathbf{u}} = (\bar{u}_1, \ldots, \bar{u}_s)$ and $\bar{\mathbf{v}} = (\bar{v}_1, \ldots, \bar{v}_m)$, is a feasible solution for model (4), which for

$$\begin{aligned}z^* &\leq \sum_{i=1}^{m} \bar{v}_i x_{io} - \bar{v}_o\\&= \frac{w^*}{AB} \sum_{i=1}^{m} \left(q_i^* + q_{m+1}^* \beta_r\right) x_{io}\\&\quad + \frac{w^*}{B} \sum_{r=1}^{s} \left(p_{s+1}^* b_i\right) x_{io} - \frac{q_o^* w^*}{AB}\\&= \frac{w^*}{AB} (A) + \frac{w^* (B - 1)}{B} = w^*.\end{aligned} \tag{35}$$

Now let $(\mathbf{u}^*, \mathbf{v}^*, v_o^*)$ be an optimal solution for model (4), and then $(\bar{\mathbf{p}}, \bar{\mathbf{q}}, \bar{p}_o)$, where $\bar{\mathbf{p}} = (\bar{p}_1, \ldots, \bar{p}_{s+1})$ and $\bar{\mathbf{q}} = (\bar{q}_1, \ldots, \bar{q}_{m+1})$, with $\bar{p}_r = \bar{u}_r$, $r = 1, \ldots, s$; $\bar{p}_{s+1} = 0$; $\bar{q}_i = \bar{v}_i$, $i = 1, \ldots, m$; $\bar{q}_{m+1} = 0$; $\bar{p}_o = v_o^*$, is a feasible solution for model (22), which for $\omega^* \geq \sum_{r=1}^{s+1} \bar{p}_r y_{ro} - \bar{p}_o = \sum_{r=1}^{s} u_r^* y_{ro} - u_o^* = z^*$. Thus $z^* = w^*$. □

Theorem 3. *Let each DMU have $m+1$ inputs and $s+1$ outputs with conditions* (5) *and* (6).

Then, the optimal objective function value of the following model:

$$\begin{aligned} \tilde{\rho} = \max \quad & \sum_{r=1}^{s+1} p_r y_{ro}, \\ & \sum_{i=1}^{m+1} q_i x_{io} = 1 \\ \text{s.t} \quad & \sum_{r=1}^{s+1} p_r y_{rj} - \sum_{i=1}^{m+1} q_i x_{ij} \le 0, \quad j = 1, \ldots, n \\ p_r \ge 0, \quad q_i \ge 0, \quad & r = 1, \ldots, s+1, \; i = 1, \ldots, m+1 \end{aligned} \tag{36}$$

is equal to the optimal objective function value of the following model (1).

Proof. This proof is similar to the proof of Theorem 1. □

Theorem 4. *Let each DMU have $m+1$ inputs and $s+1$ outputs with conditions* (5) *and* (6).

Then, the optimal objective function value of the following model:

$$\begin{aligned} \tilde{w} = \min \quad & \sum_{r=1}^{s+1} q_i x_{io}, \\ & \sum_{r=1}^{s+1} p_r y_{ro} = 1 \\ \text{s.t} \quad & \sum_{i=1}^{m+1} q_i x_{ij} - \sum_{r=1}^{s+1} p_r y_{rj} \ge 0, \quad j = 1, \ldots, n, \\ p_r \ge 0, \quad q_i \ge 0, \quad & r = 1, \ldots, s+1, \; i = 1, \ldots, m+1 \end{aligned} \tag{37}$$

is equal to the optimal objective function value of the following model (2).

Proof. This proof is similar to the proof of Theorem 2. □

4. Illustrative Example

In this section, we use the data recorded in Table 1 to illustrate that the addition or deletion of a cross-redundant output variable and input variable does not affect the efficiency estimates yielded by the CCR or BCC models. These correspond to 20 DMUs, whose efficiency is assessed using four inputs and four outputs where

$$\begin{aligned} x_{4j} = & \left(x_{1j} + x_{2j} + 2x_{3j}\right) \\ & - \left(0.5y_{1j} + 0.5y_{2j} + 0.5y_{3j}\right); \\ & j = 1, \ldots, n, \\ y_{4j} = & \left(0.5y_{1j} + 0.5y_{2j} + 0.5y_{3j}\right) - 0.5x_{3j}; \\ & j = 1, \ldots, n. \end{aligned} \tag{38}$$

Table 1: Dataset.

	Inp 1	Inp 2	Inp 3	Inp 4	Out 1	Out 1	Out 3	Out 4
Unit 1	7	1	4	12.75	1	2.5	3	1.25
Unit 2	3	7	4	14.75	2.5	1	3	1.25
Unit 3	6	6	3	14.25	2.5	2	3	2.25
Unit 4	3	1	3	1.75	4	5.5	7	6.75
Unit 5	6	0.5	3	5.25	5	3.5	6	5.75
Unit 6	4	0.5	3	3.5	2	6	6	5.5
Unit 7	1.5	2.5	3	1.5	6	4	7	7
Unit 8	0.5	4	4	6.25	1.5	5	6	4.25
Unit 9	2.75	1.75	4	4	8	3	6	6.5
Unit 10	1	3	3	1	8	3	7	7.5
Unit 11	2	2	3	1.25	5.5	5	7	7.25
Unit 12	2.5	1.5	3	2	7	3	6	6.5
Unit 13	4.5	1.5	6	13	4	2	4	2
Unit 14	2	4	7	16.25	1.5	2	4	0.25
Unit 15	4	3	6	12.25	6.5	3.5	3.5	3.75
Unit 16	2	5	4	8.75	5	3.5	4	4.25
Unit 17	1.5	6	4	8.5	4.5	4.5	5	5
Unit 18	0.5	4	3	3	3.5	5.5	6	6
Unit 19	3.5	0.75	3	2.5	7.5	2.5	6	6.5
Unit 20	6	3.5	4	11	3.5	3.5	6	4.5

Table 2: Example results.

	θ^*	$\tilde{\rho}$	z^*	ρ^*
Unit 1	0.3461538	0.3461538	0.7500000	0.7500000
Unit 2	0.3214286	0.3214286	0.7500000	0.7500000
Unit 3	0.4285714	0.4285714	1.0000000	1.0000000
Unit 4	1.0000000	1.0000000	1.0000000	1.0000000
Unit 5	1.0000000	1.0000000	1.0000000	1.0000000
Unit 6	1.0000000	1.0000000	1.0000000	1.0000000
Unit 7	1.0000000	1.0000000	1.0000000	1.0000000
Unit 8	1.0000000	1.0000000	1.0000000	1.0000000
Unit 9	0.9973190	0.9973190	1.0000000	1.0000000
Unit 10	1.0000000	1.0000000	1.0000000	1.0000000
Unit 11	1.0000000	1.0000000	1.0000000	1.0000000
Unit 12	1.0000000	1.0000000	1.0000000	1.0000000
Unit 13	0.4444444	0.4444444	0.6666667	0.6666667
Unit 14	0.3809524	0.3809524	0.6666667	0.6666667
Unit 15	0.5607702	0.5607702	0.5714286	0.5594240
Unit 16	0.6052279	0.6052279	0.7500000	0.7500000
Unit 17	0.6847156	0.6847156	0.7500000	0.7500000
Unit 18	1.0000000	1.0000000	1.0000000	1.0000000
Unit 19	1.0000000	1.0000000	1.0000000	1.0000000
Unit 20	0.6428571	0.6428571	0.7500000	0.7500000

In other words, the forth input and the forth output are cross-redundant variables. In Table 2, θ^*, $\tilde{z}$, ρ^*, and $\tilde{\rho}$, respectively, record the efficiency measure provided by model (1), model (3), model (7), and model (36). It is evident from Table 2 that the addition or deletion of cross-redundant output variable

and/or input variable does not affect the efficiency estimates yielded by the input-oriented CCR or BCC multiplier models.

5. Conclusions

In this paper, we have studied the effect of the cross redundancy between interval scale input and output variables on the efficiency estimates yielded by the CCR multiplier model in input- and output-oriented versions and the BCC multiplier model in input- and output-oriented versions. We proved that the addition or deletion of a cross-redundant output variable and input variable does not affect the efficiency estimates yielded by the input-oriented BCC multiplier model and the output-oriented BCC multiplier model. Similarly, it can be proved that the addition or deletion of cross-redundant variable does not affect efficiency estimates yielded by the CCR multiplier model in input- and output-oriented versions.

References

[1] K. Lee and K. Choi, "Cross redundancy and sensitivity in DEA models," *Journal of Productivity Analysis*, vol. 34, no. 2, pp. 151–165, 2010.

[2] F. J. López, "Generalizing cross redundancy in data envelopment analysis," *European Journal of Operational Research*, vol. 214, no. 3, pp. 716–721, 2011.

[3] A. Charnes, W. W. Cooper, and E. Rhodes, "Measuring the efficiency of decision making units," *European Journal of Operational Research*, vol. 2, no. 6, pp. 429–444, 1978.

[4] R. D. Banker, A. Charnes, and W. W. Cooper, "Some models for estimating technical and scale inefficiencies in data envelopment analysis," *Management Science*, vol. 30, no. 9, pp. 1078–1092, 1984.