

Research Article

Positive Solutions for Third-Order Boundary-Value Problems with the Integral Boundary Conditions and Dependence on the First-Order Derivatives

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By using a fixed point theorem in a cone and the nonlocal third-order BVP's Green function, the existence of at least one positive solution for the third-order boundary-value problem with the integral boundary conditions $x'''(t) + f(t, x(t), x'(t)) = 0, t \in J$, x(0) = 0, x''(0) = 0, and $x(1) = \int_0^1 g(t)x(t)dt$ is considered, where f is a nonnegative continuous function, J = [0, 1], and $g \in L[0, 1]$. The emphasis here is that f depends on the first-order derivatives.

1. Introduction

Third-order boundary-value problems for differential equation play a very important role in a variety of different areas of applied mathematics and physics. Recently, third-order boundary-value problems have been many scholars' research object. For example, heat conduction, chemical engineering, underground water flow, thermoelasticity, and plasma physics can produce boundary-value problems with integral boundary conditions [1–3]. For more information about the general theory of integral equations and their relation with boundary-value problems, we refer readers to the books of Corduneanu [4] and Agarwal and O'Regan [5].

Moreover, boundary-value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint, and nonlocal boundary-value problems as special cases. Such kind of BVPs in Banach space has been studied by some researchers [6–8].

By the fixed point index theory in cones [9], Zhang et al. [10] investigated the multiplicity of positive solutions for a class of nonlinear boundary-value problems of fourthorder differential equations with integral boundary conditions in ordered Banach spaces. Feng et al. [11] investigated the existence and multiplicity of positive solutions for a class of nonlinear boundary-value problems of second-order differential equations with integral boundary conditions in ordered Banach spaces. Guo et al. [12] investigated the existence of positive solutions for the third-order boundary-value problems with integral boundary conditions and dependence on the second derivatives. In [13], by using the fixed point theorem of cone expansion and compression of norm type, Zhang and Ge proved the existence and multiplicity of symmetric positive solutions for the fourth-order boundaryvalue problems with integral boundary conditions. By using Krasnoselskii's fixed point theorem, Wang et al. [14] investigated the existence and nonexistence of positive solutions for a class of fourth-order nonlinear differential equation with integral boundary conditions

$$x^{(4)}(t) = \omega(t) f(t, x(t), x''(t)), \quad 0 < t < 1,$$

$$x(0) = \int_0^1 h_1(s) x(s) ds,$$

$$x(1) = \int_0^1 k_1(s) x(s) ds,$$

$$x''(0) = \int_0^1 h_2(s) x''(s) ds,$$

$$x''(1) = \int_0^1 k_2(s) x''(s) ds,$$
(1)

where the arguments are based on Krasnoselskii's fixed point theorem for operators on a cone.

However, Zhao et al. [15] investigated the following third-order boundary-value problem with integral boundary conditions:

$$x'''(t) + f(t, x(t)) = \theta, \quad t \in J,$$

$$x(0) = \theta, \qquad x''(0) = \theta,$$

$$x(1) = \int_0^1 g(t) x(t) dt,$$
(2)

under the assumptions

(1) J = [0, 1], and θ is the zero element of *E*,

(2)
$$f : C([0, 1] \times P, P)$$
, and $g \in L[0, 1]$ is nonnegative,

where *P* is a cone in the real Banach *E*.

All the above works were done under the assumption that the first-order derivative x' is not involved explicitly in the nonlinear term f. In this paper, we are concerned with the existence of positive solutions for the third-order boundaryvalue problem with the integral boundary conditions

$$x'''(t) + f(t, x(t), x'(t)) = 0, \quad t \in J,$$

$$x(0) = 0, \qquad x''(0) = 0,$$

$$x(1) = \int_0^1 g(t) x(t) dt.$$
(3)

Throughout, we assume

$$(H_1) J = [0,1], f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^+$$
 is continuous, $g \in L[0,1], g(t) \ge 0$, and $\sigma \in [0,1)$, where $\sigma = \int_0^1 sg(s)ds$.

To show the existence of positive solutions for (3), we define two positive continuous convex functionals. Then, by using the fixed point theorem [16] in a cone and the nonlocal third-order BVP's Green function, we give some new criteria for the existence of positive solutions for (3).

2. Preliminaries

Let Y = C[0, 1] be the Banach space equipped with the norm $||x||_0 = \max_{t \in [0,1]} |x(t)|$.

Lemma 1 (see [15]). Suppose (H_1) holds. Then for any $y(t) \in C[0, 1]$, the problem

$$x'''(t) + y(t) = 0, \quad t \in J,$$

$$x(0) = 0, \qquad x''(0) = 0,$$

$$x(1) = \int_0^1 g(t) x(t) dt$$
(4)

has a unique solution

$$x(t) = \int_{0}^{1} H(t,s) y(s) \, ds, \tag{5}$$

where

$$H(t,s) = G(t,s) + \frac{t}{1-\sigma} \int_0^1 G(\tau,s) g(\tau) d\tau,$$

$$G(t,s) = \begin{cases} \frac{1}{2}t(1-s)^2 - \frac{1}{2}(t-s)^2, & 0 \le s \le t \le 1, \\ \frac{1}{2}t(1-s)^2, & 0 \le t \le s \le 1. \end{cases}$$
(6)

Lemma 2 (see [15]). For $t, s \in [0, 1]$, one has $0 \le G(t, s) \le \max_{0 \le t, s \le 1} G(t, s) \le 1/8$.

Remark 3. When $t, s \in (0, 1)$, it is easy to check that G(t, s) > 0.

In addition, for $0 \le s \le t \le 1$, the maximum of G(t, s) occurs at $t = (1 + s^2)/2$.

Lemma 4 (see [15]). Choose $\delta \in (0, 1/2)$ and $J_{\delta} = [\delta, 1 - \delta]$; then for all $t \in J_{\delta}$, $v, s \in [0, 1]$, one has

$$G(t,s) \ge \rho G(v,s), \qquad (7)$$

where $\rho = 4\delta^2(1-\delta)$.

Remark 5. For $0 \le s \le t \le 1$, denote $G(t, s) = G_1(t, s)$. Notice that $G_1(t, s)$ is concave with respect to *t*; we have

$$\min_{t \in J_{\delta}, 0 \le s \le t} G_1(t, s) = \min \left\{ G_1(\delta, s), G_1(1 - \delta, s) \right\}$$

$$= \frac{1}{2} \delta^2 (1 - \delta).$$
(8)

Lemma 6 (see [15]). Assume that (H_1) holds; then

(i)
$$H(t, s) \le (1/2)\gamma, t \in [0, 1],$$

(ii) $H(t, s) \ge \rho H(v, s), t \in J_{\delta}, v, s \in [0, 1],$

where $\gamma = (1 + \int_0^1 (1 - s)g(s)ds)/(1 - \sigma).$

Lemma 7. If $y \in C[0, 1]$, $y(t) \ge 0$, then the unique solution x(t) of problem (4) satisfies

$$\min_{t \in J_{\delta}} x(t) \ge \rho \|x\|_0.$$
(9)

Proof. By Lemmas 4 and 6 and (5), we get

$$\min_{t \in J_{\delta}} x(t) = \min_{t \in J_{\delta}} \int_{0}^{1} H(t, s) y(s) ds$$
$$\geq \rho \int_{0}^{1} H(v, s) y(s) ds$$
$$\geq \rho x(v).$$
(10)

For $v \in [0, 1]$, we have

$$\min_{t \in J_{\delta}} x(t) \ge \rho x(v) \,. \tag{11}$$

So,

$$\min_{t \in J_{\delta}} x(t) \ge \rho \max_{\nu \in [0,1]} x(\nu) = \rho \max_{\nu \in [0,1]} |x(\nu)| = \rho ||x||_0.$$
(12)

The proof is completed.
$$\Box$$

Let *X* be a Banach space and $K \,\subset X$ a cone. Suppose $\alpha, \beta : X \to R^+$ are two continuous convex functionals satisfying $\alpha(\lambda x) = |\lambda|\alpha(x), \beta(\lambda x) = |\lambda|\beta(x), \text{ for } x \in X, \lambda \in R, ||x|| \le M \max\{\alpha(x), \beta(x)\}, \text{ for } x \in X, \text{ and } \alpha(x) \le \alpha(y) \text{ for } x, y \in K, x \le y, \text{ where } M > 0 \text{ is a constant.}$

Theorem 8 (see [16]). *Let* $r_2 > r_1 > 0$, L > 0 *be constants and*

$$\Omega_{i} = \{ x \in X : \alpha(x) < r_{i}, \beta(x) < L \}, \quad i = 1, 2,$$
(13)

two bounded open sets in X. Set

$$D_i = \{x \in X : \alpha(x) = r_i\}, \quad i = 1, 2.$$
(14)

Assume $T: K \rightarrow K$ is a completely continuous operator satisfying

$$(A_1) \alpha(Tx) < r_1, x \in D_1 \cap K; \alpha(Tx) > r_2, x \in D_2 \cap K;$$

 $(A_2) \beta(Tx) < L, x \in K;$

(A₃) there is a $p \in (\Omega_2 \cap K) \setminus \{0\}$ such that $\alpha(p) \neq 0$ and $\alpha(x + \lambda p) \ge \alpha(x)$, for all $x \in K$ and $\lambda \ge 0$.

Then T has at least one fixed point in $(\Omega_2 \setminus \overline{\Omega}_1) \cap K$ *.*

3. Main Results

Let $X = C^1[0, 1]$ be the Banach space equipped with the norm $||x|| = \max_{t \in [0,1]} |x(t)| + \max_{t \in [0,1]} |x'(t)|$, and $K = \{x \in X : x(t) \ge 0, \min_{t \in J_\delta} x(t) \ge \rho ||x||_0\}$ is a cone in *X*.

Define two continuous convex functionals $\alpha(x) = \max_{t \in [0,1]} |x(t)|$ and $\beta(x) = \max_{t \in [0,1]} |x'(t)|$, for each $x \in X$; then $||x|| \le 2 \max\{\alpha(x), \beta(x)\}$ and $\alpha(\lambda x) = |\lambda|\alpha(x), \beta(\lambda x) = |\lambda|\beta(x)$, for $x \in X, \lambda \in R$; $\alpha(x) \le \alpha(y)$ for $x, y \in K, x \le y$.

In the following, we denote

$$\eta_{0} = \frac{1}{8} + \int_{0}^{1} \left[\frac{1}{1 - \sigma} \int_{0}^{1} G(\tau, s) g(\tau) d\tau \right] ds,$$

$$\eta_{1} = \max_{v \in [0, 1]} \int_{\delta}^{1 - \delta} H(v, s) ds,$$
(15)

$$\eta_{2} = \frac{2}{3} + \int_{0}^{1} \left[\frac{1}{1 - \sigma} \int_{0}^{1} G(\tau, s) g(\tau) d\tau \right] ds.$$

We will suppose that there are $L > b > \rho b > c > 0$ such that f(t, x, y) satisfies the following growth conditions:

$$(H_2) \ f(t, x, y) < c/\eta_0, \text{for } (t, x, y) \in [0, 1] \times [0, c] \times [-L, L],$$

$$(H_3) f(t, x, y) \ge b/\rho\eta_1, \text{ for } (t, x, y) \in [\delta, 1 - \delta] \times [\rho b, b] \times [-L, L],$$

$$(H_4) \ f(t,x,y) < L/\eta_2, \text{for} \ (t,x,y) \in [0,1] \times [0,b] \times [-L,L].$$

Let

$$f^{*}(t, x, y) = \begin{cases} f(t, x, y), (t, x, y) \in [0, 1] \times [0, b] \times (-\infty, \infty), \\ f(t, b, y), (t, x, y) \in [0, 1] \times (b, \infty) \times (-\infty, \infty), \end{cases}$$

$$f_{1}(t, x, y)$$

$$=\begin{cases} f^{*}(t, x, y), (t, x, y) \in [0, 1] \times [0, \infty) \times [-L, L], \\ f^{*}(t, x, -L), (t, x, y) \in [0, 1] \times [0, \infty) \times (-\infty, -L], \\ f^{*}(t, x, L), (t, x, y) \in [0, 1] \times [0, \infty) \times [L, \infty). \end{cases}$$
(16)

We denote

$$(Tx)(t) = \int_{0}^{1} H(t,s) f_{1}(s,x,x') ds,$$

$$(Tx)'(t) = \int_{0}^{1} \frac{\partial H(t,s)}{\partial t} f_{1}(s,x,x') ds.$$
(17)

Lemma 9. Suppose (H_1) holds. Then $T : K \to K$ is completely continuous.

Proof. For $x \in K$, by Lemmas 2 and 4, it is obviously that $Tx \ge 0$.

By Lemma 7, we have

$$\min_{t \in I_{\delta}} Tx(t) \ge \rho \|Tx\|_{0}.$$
(18)

So, we can get $T(K) \in K$.

In the following, we will show that $T : K \rightarrow K$ is completely continuous.

At first we show that $T: K \to K$ is continuous.

Let $x_n, x^* \in K$, it satisfies $||x_n - x^*|| \to 0$, $(n \to \infty)$, and then there is a constant $M_0 > 0$, such that $\max_{t \in [0,1]} \{|x_n(t)|, |x^*(t)|, |x_n'(t)|, |x^{*'}(t)|\} \le M_0$; then

$$\begin{split} \left| (Tx_n)(t) - (Tx^*)(t) \right| \\ &= \left| \int_0^1 H(t,s) f_1(s, x_n, x'_n) ds \right| \\ &- \int_0^1 H(t,s) f_1(s, x^*, x^{*'}) ds \right| \\ &\leq \int_0^1 H(t,s) \left| f_1(s, x_n, x'_n) - f_1(s, x^*, x^{*'}) \right| ds, \\ \left| (Tx_n)'(t) - (Tx^*)'(t) \right| \\ &= \left| \int_0^1 \frac{\partial H(t,s)}{\partial t} f_1(s, x, x'_n) ds \right| \\ &- \int_0^1 \frac{\partial H(t,s)}{\partial t} f_1(s, x^*, x^{*'}) ds \right| \end{split}$$

$$\leq \int_{0}^{1} \left| \frac{\partial H(t,s)}{\partial t} \right| \left| f_{1}\left(s,x,x_{n}'\right) - f_{1}\left(s,x^{*},x^{*'}\right) \right| ds$$

$$< \int_{0}^{1} \left[\frac{1}{2}(1-s)^{2} + (1-s) \right] \\ \times \left| f_{1}\left(s,x,x_{n}'\right) - f_{1}\left(s,x^{*},x^{*'}\right) \right| ds$$

$$+ \int_{0}^{1} \left[\frac{1}{1-\sigma} \int_{0}^{1} G(\tau,s) g(\tau) d\tau \right] \\ \times \left| f_{1}\left(s,x,x_{n}'\right) - f_{1}\left(s,x^{*},x^{*'}\right) \right| ds.$$

(19)

By f which is uniformly continuous on $[0,1]\times [-M_0,M_0]\times [-M_0,M_0],$ we get

$$||Tx_n - Tx^*|| \longrightarrow 0, \quad (n \longrightarrow \infty).$$
 (20)

Next we show that $T: K \rightarrow K$ is compact.

Let $B \subset K$ be bounded; then there is M > 0, such that $||x|| \leq M$. For $x \in B$, we have

$$|(Tx)(t)| = \left| \int_{0}^{1} H(t,s) f_{1}(s,x,x') ds \right|$$

$$\leq \int_{0}^{1} \frac{1}{2} \gamma f_{1}(s,x,x') ds \qquad (21)$$

$$= \frac{1}{2} \int_{0}^{1} \frac{1 + \int_{0}^{1} (1-s) g(s) ds}{1 - \sigma} ds \times C^{*},$$

where $C^* = \max\{|f_1(t, x, x')|; t \in [0, 1], x \in B\}$. Consider

$$\begin{aligned} \left| (Tx)'(t) \right| \\ &= \left| \int_0^1 \frac{\partial H(t,s)}{\partial t} f_1(s,x,x') ds \right| \\ &= \left| \int_0^1 \left[\frac{\partial G(t,s)}{\partial t} + \frac{1}{1-\sigma} \int_0^1 G(\tau,s) g(\tau) d\tau \right] \\ &\times f_1(s,x,x') ds \right| \end{aligned}$$
(22)
$$< \left| \int_0^1 \left[\frac{1}{2} (1-s)^2 + (1-s) \right] ds \\ &+ \int_0^1 \left[\frac{1}{1-\sigma} \int_0^1 G(\tau,s) g(\tau) d\tau \right] ds \right| \times C^* \\ &= \left| \frac{2}{3} + \int_0^1 \left[\frac{1}{1-\sigma} \int_0^1 G(\tau,s) g(\tau) d\tau \right] ds \right| \times C^*. \end{aligned}$$

It is clear that T(B) is a bounded set in K, because H(t, s) is uniformly continuous on $[0, 1] \times [0, 1]$, for $\varepsilon > 0$, there exists $\delta \in (0, \varepsilon)$, such that $|H(t_1, s) - H(t_2, s)| < \varepsilon$, and for t_1 , $t_2 \in [0, 1], |t_1 - t_2| < \delta$.

For $x \in B$, we have

$$\begin{aligned} |(Tx)(t_{1}) - (Tx)(t_{2})| \\ &= \left| \int_{0}^{1} H(t_{1},s) f_{1}(s,x,x') ds \right| \\ &- \int_{0}^{1} H(t_{2},s) f_{1}(s,x,x') ds \right| \\ &\leq \int_{0}^{1} |H(t_{1},s) - H(t_{2},s)| ds \times C^{*} \leq \varepsilon C^{*}, \\ |(Tx)'(t_{1}) - (Tx)'(t_{2})| \\ &= \left| \int_{0}^{1} \frac{\partial H(t,s)}{\partial t} \right|_{t=t_{1}} f_{1}(s,x,x') ds \\ &- \int_{0}^{1} \frac{\partial H(t,s)}{\partial t} \Big|_{t=t_{2}} f_{1}(s,x,x') ds \\ &= \left| \int_{0}^{1} \frac{\partial G(t,s)}{\partial t} \right|_{t=t_{2}} f_{1}(s,x,x') ds \\ &- \int_{0}^{1} \frac{\partial G(t,s)}{\partial t} \Big|_{t=t_{2}} f_{1}(s,x,x') ds \\ &- \int_{0}^{1} \frac{\partial G(t,s)}{\partial t} \Big|_{t=t_{2}} f_{1}(s,x,x') ds \\ &= \left| \int_{0}^{t_{1}} (s-t_{1}) f_{1}(s,x,x') ds \right| \\ &= \left| \int_{0}^{t_{1}} (s-t_{1}) f_{1}(s,x,x') ds \right| \\ &\leq \frac{1}{2} \left| (t_{1}-t_{2}) (t_{1}+t_{2}) \right| \times C^{*} \leq \varepsilon C^{*}. \end{aligned}$$

Therefore T(B) is equicontinuous. Using the Arzela-Ascoli theorem, a standard proof yields $T: K \to K$ which is completely continuous.

Theorem 10. Suppose (H_1) – (H_4) hold. Then BVP (3) has at least one positive solution x(t) satisfying

$$c < \alpha(x) < b, \qquad \beta(x) < L.$$
 (24)

Proof. Take

$$\Omega_{1} = \left\{ x \in X : |x(t)| < c, |x(t)'| < L \right\},
\Omega_{2} = \left\{ x \in X : |x(t)| < b, |x(t)'| < L \right\},$$
(25)

two bounded open sets in X, and

$$D_1 = \{x \in X : \alpha(x) = c\}, \qquad D_2 = \{x \in X : \alpha(x) = b\}.$$

(26)

By Lemma 9, $T : K \to K$ is completely continuous, and there is a $p \in (\Omega_2 \cap K) \setminus \{0\}$ such that $\alpha(p) \neq 0$ and $\alpha(x + \lambda p) \ge \alpha(x)$ for all $u \in K$ and $\lambda \ge 0$.

By
$$(H_2)$$
, for $x \in D_1 \cap K$ and $\alpha(x) = c$, we get

$$\alpha(Tx) = \max_{t \in [0,1]} \left| \int_0^1 H(t,s) f_1(s,x,x') ds \right|$$

$$= \max_{t \in [0,1]} \left| \int_0^1 \left[G(t,s) + \frac{t}{1-\sigma} \int_0^1 G(\tau,s) g(\tau) d\tau \right] \times f_1(s,x,x') ds \right|$$

$$< \int_0^1 \left[\max_{t \in [0,1]} G(t,s) + \frac{t}{1-\sigma} \int_0^1 G(\tau,s) g(\tau) d\tau \right] \times f_1(s,x,x') ds$$

$$< \left[\int_0^1 \frac{1}{8} ds + \int_0^1 \left(\frac{t}{1-\sigma} \int_0^1 G(\tau,s) g(\tau) d\tau \right) ds \right] \times \frac{c}{\eta_0}$$

$$= \left[\frac{1}{8} + \int_0^1 \left(\frac{t}{1-\sigma} \int_0^1 G(\tau,s) g(\tau) d\tau \right) ds \right] \times \frac{c}{\eta_0} = c.$$
(27)

By Lemma 7, for $x \in D_2 \cap K$ and $\alpha(x) = b$, there is $x(t) \ge \rho\alpha(x) = \rho b$, $t \in J_{\delta}$. So, by (H_3) , we get

$$\alpha (Tx) = \max_{t \in [0,1]} \left| \int_0^1 H(t,s) f_1(s,x,x') ds \right|$$

$$> \int_{\delta}^{1-\delta} H(t,s) f_1(s,x,x') ds \qquad (28)$$

$$> \rho \int_{\delta}^{1-\delta} H(v,s) ds \times \frac{b}{\rho \eta_1}.$$

For $v \in [0, 1]$, we have

$$\alpha(Tx) > \rho \int_{\delta}^{1-\delta} H(v,s) \, ds \times \frac{b}{\rho \eta_1}.$$
 (29)

So,

$$\alpha(Tx) > \rho \max_{v \in [0,1]} \int_{\delta}^{1-\delta} H(v,s) \, ds \times \frac{b}{\rho \eta_1} = b. \tag{30}$$

By (H_4) , for $x \in K$, we have

$$\beta(Tx) = \max_{t \in [0,1]} \left| \int_0^1 \frac{\partial H(t,s)}{\partial t} f_1(s, x, x') ds \right|$$

$$< \left| \int_0^1 \left[\frac{\partial G(t,s)}{\partial t} + \frac{1}{1-\sigma} \int_0^1 G(\tau,s) g(\tau) d\tau \right] \right|$$

$$\times f_1(s, x, x') ds$$

$$= \left| \int_{0}^{t} \left(\frac{1}{2} (1-s)^{2} - (t-s) \right) f_{1} \left(s, x, x' \right) ds + \int_{t}^{1} \frac{1}{2} (1-s)^{2} f_{1} \left(s, x, x' \right) ds + \int_{0}^{1} \left[\frac{1}{1-\sigma} \int_{0}^{1} G \left(\tau, s \right) g \left(\tau \right) d\tau \right] \times f_{1} \left(s, x, x' \right) ds \right| \\ < \left[\int_{0}^{1} \left(\frac{1}{2} (1-s)^{2} + (1-s) \right) ds + \int_{0}^{1} \left(\frac{1}{1-\sigma} \int_{0}^{1} G \left(\tau, s \right) g \left(\tau \right) d\tau \right) ds \right] \times \frac{L}{\eta_{2}} \\ = \left[\frac{2}{3} + \int_{0}^{1} \left(\frac{1}{1-\sigma} \int_{0}^{1} G \left(\tau, s \right) g \left(\tau \right) d\tau \right) ds \right] \times \frac{L}{\eta_{2}} \\ \times \frac{L}{\eta_{2}} = L.$$
(31)

Theorem 8 implies there is $x \in (\Omega_2 \setminus \overline{\Omega}_1) \cap K$ such that x = Tx. So, x(t) is a positive solution for BVP (3) satisfying

$$c < \alpha(x) < b, \qquad \beta(x) < L.$$
 (32)

Thus, Theorem 10 is completed. $\hfill \Box$

4. Example

Example 1. Consider the following boundary-value problem

$$x'''(t) + f(t, x(t), x'(t)) = 0, \quad 0 < t < 1,$$

$$x(0) = 0, \qquad x''(0) = 0,$$

$$x(1) = \int_0^1 x(t) dt,$$
(33)

where

$$f(t, x, y) = \begin{cases} \frac{t}{3}x + 2x + |\cos y|, \\ (t, x, y) \in [0, 1] \times [0, 0.5] \times [-3667, 3667], \\ \frac{109t}{3}(x - 0.5) + 25742(x - 0.5) + \frac{t}{6} + 1 + |\cos y|, \\ (t, x, y) \in [0, 1] \times [0.5, 0.6] \times [-3667, 3667], \\ \frac{t}{3}(11 - x) + 222(x + 11) + |\cos y|, \\ (t, x, y) \in [0, 1] \times [0.6, 11] \times [-3667, 3667]. \end{cases}$$

$$(34)$$

In this problem, we know that g(t) = 1; then we can get $\sigma = \int_0^1 sg(s)ds = 1/2$. Choose $\delta = 1/8 \in (0, 1/2)$; then $\rho = 4\delta^2(1-\delta) = 7/128$.

$$\eta_0 = \frac{5}{24}, \qquad \rho \eta_1 = \frac{35}{8192}, \qquad \eta_2 = \frac{3}{4}.$$
 (35)

If we take c = 0.5, b = 11, and L = 3667, then we get $\rho b \approx 0.601 > 0.6$:

$$f(t, x, y) = \frac{t}{3}x + 2x + |\cos y| \le 2.17 < \frac{c}{\eta_0} \approx 2.4, \quad (36)$$

for $(t, x, y) \in [0, 1] \times [0, 0.5] \times [-3667, 3667]$,

$$f(t, x, y) = \frac{t}{3} (11 - x) + 222 (x + 11) + |\cos y| \ge 2575.2 > \frac{b}{\rho \eta_1} \approx 2574.1,$$
(37)

for $(t, x, y) \in [\delta, 1 - \delta] \times [\delta b, 11] \times [-3667, 3667]$,

$$f(t, x, y) \le 4888.8 < \frac{L}{\eta_2} \approx 4889.3,$$
 (38)

for $(t, x, y) \in [0, 1] \times [0, 11] \times [-3667, 3667]$.

Then all the conditions of Theorem 10 are satisfied. Therefore, by Theorem 10 we know that boundary-value problem (33) has at least one positive solution x(t) satisfying

$$0.5 < \alpha(x) < 11, \qquad \beta(x) < 3667.$$
 (39)

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