Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 761864, 21 pages
http://dx.doi.org/10.1155/2013/761864

Research Article

Composite Iterative Algorithms for Variational Inequality and Fixed Point Problems in Real Smooth and Uniformly Convex Banach Spaces

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Received 30 April 2013; Accepted 6 June 2013

Academic Editor: Wei-Shih Du

We introduce composite implicit and explicit iterative algorithms for solving a general system of variational inequalities and a common fixed point problem of an infinite family of nonexpansive mappings in a real smooth and uniformly convex Banach space. These composite iterative algorithms are based on Korpelevich's extragradient method and viscosity approximation method. We first consider and analyze a composite implicit iterative algorithm in the setting of uniformly convex and 2-uniformly smooth Banach space and then another composite explicit iterative algorithm in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Under suitable assumptions, we derive some strong convergence theorems. The results presented in this paper improve, extend, supplement, and develop the corresponding results announced in the earlier and very recent literatures.

1. Introduction

Let X be a real Banach space whose dual space is denoted by X^*. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \left\{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\right\}, \quad \forall x \in X, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is an immediate consequence of the Hahn-Banach theorem that $J(x)$ is nonempty for each $x \in X$. Let C be a nonempty, closed, and convex subset of X. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for every $x, y \in C$. The set of fixed points of T is denoted by $\mathrm{Fix}(T)$. We use the notation $\rightharpoonup$ to indicate the weak convergence and the one $\to$ to indicate the strong convergence. A mapping $A : C \to X$ is said to be accretive if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \ge 0. \tag{2}$$

It is said to be α-strongly accretive if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \ge \alpha \|x - y\|^2, \tag{3}$$

for some $\alpha \in (0,1)$. The mapping is called β-inverse strongly-accretive if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \ge \beta \|Ax - Ay\|^2, \tag{4}$$

for some $\beta > 0$ and is said to be λ-strictly pseudocontractive if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \le \|x - y\|^2 - \lambda \|x - y - (Ax - Ay)\|^2 \tag{5}$$

for some $\lambda \in (0,1)$.

Let $U = \{x \in X : \| x \| = 1\}$ denote the unite sphere of X. A Banach space X is said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for all $x, y \in U$,

$$\|x - y\| \geq \epsilon \Longrightarrow \frac{\|x + y\|}{2} \leq 1 - \delta. \tag{6}$$

It is known that a uniformly convex Banach space is reflexive and strict convex. A Banach space X is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t} \tag{7}$$

exists for all $x, y \in U$; in this case, X is also said to have a Gâteaux differentiable norm. X is said to have a uniformly Gâteaux differentiable norm if for each $y \in U$, the limit is attained uniformly for $x \in U$. Moreover, it is said to be uniformly smooth if this limit is attained uniformly for $x, y \in U$. The norm of X is said to be the Fréchet differential if for each $x \in U$, this limit is attained uniformly for $y \in U$. In addition, we define a function $\rho : [0, \infty) \to [0, \infty)$ called the modulus of smoothness of X as follows:

$$\begin{aligned} \rho(\tau) = \sup \Big\{ & \frac{1}{2} (\|x + y\| + \|x - y\|) \\ & - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau \Big\}. \end{aligned} \tag{8}$$

It is known that X is uniformly smooth if and only if $\lim_{\tau \to 0} \rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space X is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. As pointed out in [1], no Banach space is q-uniformly smooth for $q > 2$. In addition, it is also known that J is single-valued if and only if X is smooth, whereas if X is uniformly smooth, then the mapping J is norm-to-norm uniformly continuous on bounded subsets of X. If X has a uniformly Gâteaux differentiable norm, then the duality mapping J is norm-to-weak* uniformly continuous on bounded subsets of X.

Very recently, Cai and Bu [2] considered the following general system of variational inequalities (GSVI) in a real smooth Banach space X, which involves finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} \langle \mu_1 B_1 y^* + x^* - y^*, J(x - x^*) \rangle \geq 0, \quad \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, J(x - y^*) \rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{9}$$

where C is a nonempty, closed, and convex subset of X, B_1, and $B_2 : C \to X$ are two nonlinear mappings, and μ_1 and μ_2 are two positive constants. Here the set of solutions of GSVI (9) is denoted by GSVI(C, B_1, B_2). In particular, if $X = H$, a real Hilbert space, then GSVI (9) reduces to the following GSVI of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{10}$$

which μ_1 and μ_2 are two positive constants. The set of solutions of problem (10) is still denoted by GSVI(C, B_1, B_2). It is clear that the problem (10) covers as special case the classical variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{11}$$

The solution set of the VIP (11) is denoted by VI(C, A).

Recently, Ceng et al. [3] transformed problem (10) into a fixed point problem in the following way.

Lemma 1 (see [3]). *For given $\overline{x}, \overline{y} \in C$, $(\overline{x}, \overline{y})$ is a solution of problem* (10) *if and only if $\overline{x}$ is a fixed point of the mapping $G : C \to C$ defined by*

$$\begin{aligned} G(x) = P_C [P_C (x - \mu_2 B_2 x) - \mu_1 B_1 P_C \\ \times (x - \mu_2 B_2 x)], \quad \forall x \in C, \end{aligned} \tag{12}$$

where $\overline{y} = P_C(\overline{x} - \mu_2 B_2 \overline{x})$ and P_C is the the projection of H onto C.

In particular, if the mappings $B_i : C \to H$ is β_i-inverse strongly monotone for $i = 1, 2$, then the mapping G is nonexpansive provided $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$.

Let C be a nonempty, closed, and convex subset of a real smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C, and let $f : C \to C$ be a contraction with coefficient $\rho \in (0, 1)$. In this paper we introduce composite implicit and explicit iterative algorithms for solving GSVI (9) and the common fixed point problem of an infinite family $\{S_n\}$ of nonexpansive mappings of C into itself. These composite iterative algorithms are based on Korpelevich's extragradient method [4] and viscosity approximation method [5]. Let the mapping G be defined by

$$G(x) := \Pi_C (I - \mu_1 B_1) \Pi_C (I - \mu_2 B_2) x, \quad \forall x \in C. \tag{13}$$

We first propose a composite implicit iterative algorithm in the setting of uniformly convex and 2-uniformly smooth Banach space X:

$$\begin{aligned} y_n &= \alpha_n f(y_n) + (1 - \alpha_n) S_n G(x_n), \\ x_{n+1} &= \beta_n x_n + \gamma_n y_n + \delta_n S_n G(y_n), \quad \forall n \geq 0, \end{aligned} \tag{14}$$

where $B_i : C \to X$ is α_i-inverse-strongly accretive with $0 < \mu_i < \alpha_i/\kappa^2$ for $i = 1, 2$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $(0, 1)$ such that $\beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. It is proven that under appropriate conditions, $\{x_n\}$ converges strongly to $q \in F = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i) \cap \Omega$, which solves the following VIP:

$$\langle q - f(q), J(q - p) \rangle \leq 0, \quad \forall p \in F. \tag{15}$$

On the other hand, we also propose another composite explicit iterative algorithm in a uniformly convex Banach space X with a uniformly Gateaux differentiable norm:

$$\begin{aligned} y_n &= \alpha_n G(x_n) + (1 - \alpha_n) S_n G(x_n), \\ x_{n+1} &= \beta_n f(x_n) + \gamma_n y_n + \delta_n S_n G(y_n), \quad \forall n \geq 0, \end{aligned} \tag{16}$$

where $B_i : C \to X$ is λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \geq 1$ for $i = 1, 2$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $(0, 1)$ such that $\beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. It is proven that under mild conditions, $\{x_n\}$ also converges strongly to $q \in F = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i) \cap \Omega$, which solves the VIP (15). The results presented in this paper improve, extend, supplement, and develop the corresponding results announced in the earlier and very recent literatures.

2. Preliminaries

We list some lemmas that will be used in the sequel. Lemma 2 can be found in [6]. Lemma 3 is an immediate consequence of the subdifferential inequality of the function $(1/2)\|\cdot\|^2$.

Lemma 2. *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n) s_n + \alpha_n \beta_n + \gamma_n, \quad \forall n \geq 0, \tag{17}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ satisfy the conditions:

(i) $\{\alpha_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$,

(iii) $\gamma_n \geq 0$, $\forall n \geq 0$, *and* $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\limsup_{n\to\infty} s_n = 0$.

Lemma 3. *In a smooth Banach space X, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle, \quad \forall x, y \in X. \tag{18}$$

Lemma 4 (see [7]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X, and let $\{\alpha_n\}$ be a sequence in $[0, 1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1. \tag{19}$$

Suppose that $x_{n+1} = \alpha_n x_n + (1 - \alpha_n) z_n$, $\forall n \geq 0$, and $\limsup_{n\to\infty}(\| z_{n+1} - z_n \| - \| x_{n+1} - x_n \|) \leq 0$. Then $\lim_{n\to\infty} \| z_n - x_n \| = 0$.

Let D be a subset of C, and let Π be a mapping of C into D. Then Π is said to be sunny if

$$\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x), \tag{20}$$

whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for every $z \in R(\Pi)$ where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D. The following lemma concerns the sunny nonexpansive retraction.

Lemma 5 (see [8]). *Let C be a nonempty, closed, and convex subset of a real smooth Banach space X. Let D be a nonempty subset of C. Let Π be a retraction of C onto D. Then the following are equivalent:*

(i) Π *is sunny and nonexpansive;*

(ii) $\| \Pi(x) - \Pi(y)\|^2 \leq \langle x - y, J(\Pi(x) - \Pi(y))\rangle$, $\forall x, y \in C$;

(iii) $\langle x - \Pi(x), J(y - \Pi(x))\rangle \leq 0$, $\forall x \in C$, $y \in D$.

It is well known that if $X = H$ a Hilbert space, then a sunny nonexpansive retraction Π_C is coincident with the metric projection from X onto C; that is, $\Pi_C = P_C$. If C is a nonempty, closed, and convex subset of a strictly convex and uniformly smooth Banach space X and if $T : C \to C$ is a nonexpansive mapping with the fixed point set $\mathrm{Fix}(T) \neq \emptyset$, then the set $\mathrm{Fix}(T)$ is a sunny nonexpansive retract of C.

Lemma 6 (see [9]). *Given a number $r > 0$. A real Banach space X is uniformly convex if and only if there exists a continuous strictly increasing function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$, such that*

$$\begin{aligned} &\|\lambda x + (1 - \lambda) y\|^2 \\ &\quad \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda) g(\|x - y\|) \end{aligned} \tag{21}$$

for all $\lambda \in [0, 1]$ and $x, y \in X$ such that $\| x \| \leq r$ and $\| y \| \leq r$.

Lemma 7 (see [10]). *Let C be a nonempty, closed, and convex subset of a Banach space X. Let $S_0, S_1, \ldots$, be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_n x - S_{n-1} x\| : x \in C\} < \infty$. Then for each $y \in C$, $\{S_n y\}$ converges strongly to some point of C. Moreover, let S be a mapping of C into itself defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in C$. Then $\lim_{n\to\infty} \sup\{\| Sx - S_n x \| : x \in C\} = 0$.*

Let C be a nonempty, closed, and convex subset of a Banach space X, and let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. As previous, let Ξ_C be the set of all contractions on C. For $t \in (0, 1)$ and $f \in \Xi_C$, let $x_t \in C$ be the unique fixed point of the contraction $x \mapsto tf(x) + (1 - t)Tx$ on C; that is,

$$x_t = tf(x_t) + (1 - t) Tx_t. \tag{22}$$

Lemma 8 (see [11, 12]). *Let X be a uniformly smooth Banach space or a reflexive and strictly convex Banach space with a uniformly Gateaux differentiable norm. Let C be a nonempty, closed, and convex subset of X, let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$, and $f \in \Xi_C$. Then the net $\{x_t\}$ defined by $x_t = tf(x_t) + (1 - t)Tx_t$ converges strongly to a point in $\mathrm{Fix}(T)$. If we define a mapping $Q : \Xi_C \to \mathrm{Fix}(T)$ by $Q(f) := s - \lim_{t\to 0} x_t$, $\forall f \in \Xi_C$, then $Q(f)$ solves the VIP:*

$$\begin{aligned} &\langle (I - f) Q(f), J(Q(f) - p)\rangle \leq 0, \\ &\qquad \forall f \in \Xi_C, \ p \in \mathrm{Fix}(T). \end{aligned} \tag{23}$$

Lemma 9 (see [13]). *Let C be a nonempty, closed, and convex subset of a strictly convex Banach space X. Let $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings on C. Suppose that $\bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=0}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by $Sx = \sum_{n=0}^{\infty} \lambda_n T_n x$ for $x \in C$ is defined well, nonexpansive, and $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ holds.*

3. Implicit Iterative Schemes

In this section, we introduce our implicit iterative schemes and show the strong convergence theorems. We will use the following useful lemmas in the sequel.

Lemma 10 (see [2, Lemma 2.8]). *Let C be a nonempty, closed, and convex subset of a real 2-uniformly smooth Banach space X. Let the mapping $B_i : C \to X$ be α_i-inverse-strongly accretive. Then, one has*

$$\begin{aligned}\|(I-\mu_i B_i)x-(I-\mu_i B_i)y\|^2 &\le \|x-y\|^2+2\mu_i(\mu_i\kappa^2-\alpha_i)\\ &\quad\times\|B_i x-B_i y\|^2, \quad \forall x,y\in C,\end{aligned} \tag{24}$$

for $i=1,2$, where $\mu_i>0$. In particular, if $0<\mu_i\le\alpha_i/\kappa^2$, then $I-\mu_i B_i$ is nonexpansive for $i=1,2$.

Lemma 11 (see [2, Lemma 2.9]). *Let C be a nonempty, closed, and convex subset of a real 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be α_i-inverse-strongly accretive for $i=1,2$. Let $G : C \to C$ be the mapping defined by*

$$\begin{aligned}G(x)=\Pi_C[\Pi_C(x-\mu_2B_2x)\\ -\mu_1B_1\Pi_C(x-\mu_2B_2x)], \quad \forall x\in C.\end{aligned} \tag{25}$$

If $0<\mu_i\le\alpha_i/\kappa^2$ for $i=1,2$, then $G : C \to C$ is nonexpensive.

Lemma 12 (see [2, Lemma 2.10]). *Let C be a nonempty, closed, and convex subset of a real 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $B_1, B_2 : C \to X$ be two nonlinear mappings. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of GSVI (9) if and only if $x^*=\Pi_C(y^*-\mu_1B_1y^*)$ where $y^*=\Pi_C(x^*-\mu_2B_2x^*)$.*

Remark 13. By Lemma 12, we observe that

$$x^*=\Pi_C[\Pi_C(x^*-\mu_2B_2x^*)-\mu_1B_1\Pi_C(x^*-\mu_2B_2x^*)], \tag{26}$$

which implies that x^* is a fixed point of the mapping G.

We now state and prove our first result on the implicit iterative scheme.

Theorem 14. *Let C be a nonempty, closed, and convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be α_i-inverse-strongly accretive for $i=1,2$. Let $f : C \to C$ be a contraction with coefficient $\rho\in(0,1)$. Let $\{S_n\}_{n=0}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself such that $F=\bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n)\cap\Omega\neq\emptyset$, where Ω is the fixed point set of the mapping $G=\Pi_C(I-\mu_1B_1)\Pi_C(I-\mu_2B_2)$. For arbitrarily given $x_0\in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}y_n&=\alpha_n f(y_n)+(1-\alpha_n)S_nG(x_n),\\ x_{n+1}&=\beta_nx_n+\gamma_ny_n+\delta_nS_nG(y_n), \quad \forall n\ge0,\end{aligned} \tag{27}$$

where $0<\mu_i<\alpha_i/\kappa^2$ for $i=1,2$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $(0,1)$ such that $\beta_n+\gamma_n+\delta_n=1$, $\forall n\ge0$. Suppose that the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=0}^{\infty}\alpha_n=\infty$,

(ii) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}(\beta_n+\gamma_n)<1$ *and* $\liminf_{n\to\infty}\gamma_n>0$,

(iii) $\lim_{n\to\infty}|\gamma_n/(1-\beta_n)-\gamma_{n-1}/(1-\beta_{n-1})|=0$.

Assume that $\sum_{n=1}^{\infty}\sup_{x\in D}\|S_nx-S_{n-1}x\|<\infty$ for any bounded subset D of C, and let S be a mapping of C into itself defined by $Sx=\lim_{n\to\infty}S_nx$ for all $x\in C$. Suppose that $\mathrm{Fix}(S)=\bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n)$. Then $\{x_n\}$ converges strongly to $q\in F$, which solves the following VIP:

$$\langle q-f(q),J(q-p)\rangle\le0, \quad \forall p\in F. \tag{28}$$

Proof. Take a fixed $p\in F$ arbitrarily. Then by Lemma 12, we know that $p=G(p)$ and $p=S_np$ for all $n\ge0$. Moreover, by Lemma 11, we have

$$\begin{aligned}\|y_n-p\|&=\|\alpha_n(f(y_n)-p)+(1-\alpha_n)(S_nG(x_n)-p)\|\\ &\le\alpha_n\|f(y_n)-f(p)\|+\alpha_n\|f(p)-p\|\\ &\quad+(1-\alpha_n)\|S_nG(x_n)-p\|\\ &\le\alpha_n\rho\|y_n-p\|+\alpha_n\|f(p)-p\|\\ &\quad+(1-\alpha_n)\|G(x_n)-p\|\\ &\le\alpha_n\rho\|y_n-p\|+\alpha_n\|f(p)-p\|\\ &\quad+(1-\alpha_n)\|x_n-p\|,\end{aligned} \tag{29}$$

which hence implies that

$$\begin{aligned}\|y_n-p\|&\le\left(1-\frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\|x_n-p\|\\ &\quad+\frac{1}{1-\alpha_n\rho}\alpha_n\|f(p)-p\|\\ &\le\|x_n-p\|+\frac{\alpha_n}{1-\alpha_n\rho}\|f(p)-p\|.\end{aligned} \tag{30}$$

Thus, from (27), we have

$$\begin{aligned}\|x_{n+1}-p\|&=\|\beta_n(x_n-p)+\gamma_n(y_n-p)+\delta_n(S_nG(y_n)-p)\|\\ &\le\beta_n\|x_n-p\|+\gamma_n\|y_n-p\|+\delta_n\|G(y_n)-p\|\\ &\le\beta_n\|x_n-p\|+\gamma_n\|y_n-p\|+\delta_n\|y_n-p\|\\ &=\beta_n\|x_n-p\|+(1-\beta_n)\|y_n-p\|\end{aligned}$$

$$\begin{aligned}
&\le \beta_n \|x_n - p\| + (1-\beta_n) \\
&\quad \times \left\{ \left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \|x_n - p\| \right. \\
&\qquad \left. + \frac{1}{1-\alpha_n\rho}\alpha_n \|f(p) - p\| \right\} \\
&= \left[1 - \frac{(1-\beta_n)(1-\rho)}{1-\alpha_n\rho}\alpha_n\right] \|x_n - p\| \\
&\quad + \frac{(1-\beta_n)(1-\rho)}{1-\alpha_n\rho}\alpha_n \frac{\|f(p) - p\|}{1-\rho} \\
&\le \max\left\{ \|x_0 - p\|, \frac{\|f(p) - p\|}{1-\rho} \right\}.
\end{aligned} \tag{31}$$

It immediately follows that $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}$, $\{G(x_n)\}$, and $\{G(y_n)\}$ due to (30) and the nonexpansivity of G.

Let us show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. As a matter of fact, from (27), we have

$$\begin{aligned}
y_n &= \alpha_n f(y_n) + (1-\alpha_n) S_n G(x_n), \\
y_{n-1} &= \alpha_{n-1} f(y_{n-1}) + (1-\alpha_{n-1}) S_{n-1} G(x_{n-1}), \quad \forall n \ge 1.
\end{aligned} \tag{32}$$

Simple calculations show that

$$\begin{aligned}
y_n - y_{n-1} &= \alpha_n (f(y_n) - f(y_{n-1})) \\
&\quad + (\alpha_n - \alpha_{n-1})(f(y_{n-1}) - S_{n-1}G(x_{n-1})) \\
&\quad + (1-\alpha_n)(S_n G(x_n) - S_{n-1}G(x_{n-1})).
\end{aligned} \tag{33}$$

It follows that

$$\begin{aligned}
&\|y_n - y_{n-1}\| \\
&\quad \le \alpha_n \|f(y_n) - f(y_{n-1})\| + |\alpha_n - \alpha_{n-1}| \\
&\qquad \times \|f(y_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\qquad + (1-\alpha_n)\|S_n G(x_n) - S_{n-1}G(x_{n-1})\| \\
&\quad \le \alpha_n\rho \|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \\
&\qquad \times \|f(y_{n-1}) - S_{n-1}G(x_{n-1})\| + (1-\alpha_n) \\
&\qquad \times (\|S_n G(x_n) - S_n G(x_{n-1})\| \\
&\qquad\quad + \|S_n G(x_{n-1}) - S_{n-1}G(x_{n-1})\|) \\
&\quad \le \alpha_n\rho \|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \\
&\qquad \times \|f(y_{n-1}) - S_{n-1}G(x_{n-1})\| + (1-\alpha_n) \\
&\qquad \times (\| x_n - x_{n-1} \| + \|S_n G(x_{n-1}) - S_{n-1}G(x_{n-1})\|),
\end{aligned} \tag{34}$$

which hence yields

$$\begin{aligned}
&\|y_n - y_{n-1}\| \\
&\quad \le \frac{1-\alpha_n}{1-\alpha_n\rho} (\|x_n - x_{n-1}\| \\
&\qquad\qquad + \|S_n G(x_{n-1}) - S_{n-1}G(x_{n-1})\|) \\
&\qquad + \frac{|\alpha_n - \alpha_{n-1}|}{1-\alpha_n\rho} \|f(y_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\quad \le \|x_n - x_{n-1}\| + \|S_n G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\qquad + \frac{|\alpha_n - \alpha_{n-1}|}{1-\alpha_n\rho} \|f(y_{n-1}) - S_{n-1}G(x_{n-1})\|.
\end{aligned} \tag{35}$$

Now, we write $x_n = \beta_{n-1}x_{n-1} + (1-\beta_{n-1})z_{n-1}$, $\forall n \ge 1$, where $z_{n-1} = (x_n - \beta_{n-1}x_{n-1})/(1-\beta_{n-1})$. It follows that for all $n \ge 1$,

$$\begin{aligned}
z_n - z_{n-1} &= \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} - \frac{x_n - \beta_{n-1}x_{n-1}}{1-\beta_{n-1}} \\
&= \frac{\gamma_n y_n + \delta_n S_n G(y_n)}{1-\beta_n} \\
&\quad - \frac{\gamma_{n-1}y_{n-1} + \delta_{n-1}S_{n-1}G(y_{n-1})}{1-\beta_{n-1}} \\
&= \frac{\gamma_n(y_n - y_{n-1}) + \delta_n(S_n G(y_n) - S_{n-1}G(y_{n-1}))}{1-\beta_n} \\
&\quad + \left(\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right) y_{n-1} \\
&\quad + \left(\frac{\delta_n}{1-\beta_n} - \frac{\delta_{n-1}}{1-\beta_{n-1}}\right) S_{n-1}G(y_{n-1}).
\end{aligned} \tag{36}$$

This together with (35) implies that

$$\begin{aligned}
&\|z_n - z_{n-1}\| \\
&\quad \le \frac{\|\gamma_n(y_n - y_{n-1}) + \delta_n(S_n G(y_n) - S_{n-1}G(y_{n-1}))\|}{1-\beta_n} \\
&\qquad + \left|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right| \|y_{n-1}\| \\
&\qquad + \left|\frac{\delta_n}{1-\beta_n} - \frac{\delta_{n-1}}{1-\beta_{n-1}}\right| \|S_{n-1}G(y_{n-1})\| \\
&\quad \le (\gamma_n \|y_n - y_{n-1}\| \\
&\qquad + \delta_n (\|S_n G(y_n) - S_n G(y_{n-1})\| \\
&\qquad\qquad + \|S_n G(y_{n-1}) - S_{n-1}G(y_{n-1})\|))
\end{aligned}$$

$$
\begin{aligned}
&\times (1-\beta_n)^{-1} \\
&+ \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| \|y_{n-1}\| \\
&+ \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| \|S_{n-1}G(y_{n-1})\| \\
\le\ & \frac{\gamma_n \|y_n - y_{n-1}\| + \delta_n \|y_n - y_{n-1}\|}{\gamma_n + \delta_n} \\
&+ \frac{\delta_n}{\gamma_n + \delta_n} \|S_n G(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&+ \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| (\|y_{n-1}\| + \|S_{n-1}G(y_{n-1})\|) \\
\le\ & \|y_n - y_{n-1}\| + \|S_n G(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&+ \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| (\|y_{n-1}\| + \|S_{n-1}G(y_{n-1})\|) \\
\le\ & \|x_n - x_{n-1}\| + \|S_n G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&+ \frac{|\alpha_n - \alpha_{n-1}|}{1-\alpha_n\rho} \|f(y_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&+ \|S_n G(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&+ \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| (\|y_{n-1}\| + \|S_{n-1}G(y_{n-1})\|) \\
\le\ & \|x_n - x_{n-1}\| + \|S_n G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&+ \frac{|\alpha_n - \alpha_{n-1}|}{1-\alpha_n\rho} M + \|S_n G(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&+ \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{\mathrm{n}-1}} \right| M \\
=\ & \|x_n - x_{n-1}\| \\
&+ M \left(\frac{1}{1-\alpha_n\rho} |\alpha_n - \alpha_{n-1}| + \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| \right) \\
&+ \|S_n G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&+ \|S_n G(y_{n-1}) - S_{n-1}G(y_{n-1})\|,
\end{aligned}
\tag{37}
$$

where $\sup_{n\ge 0}\{\|f(y_n)\| + \|S_nG(x_n)\| + \|y_n\| + \|S_nG(y_n)\|\} \le M$ for some $M > 0$. So, from $\alpha_n \to 0$, condition (iii), and the assumption on $\{S_n\}$, it immediately follows that

$$
\limsup_{n\to\infty} (\|z_n - z_{n-1}\| - \|x_n - x_{n-1}\|) \le 0. \tag{38}
$$

In terms of condition (ii) and Lemma 4, we get

$$
\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{39}
$$

Hence we obtain

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n) \|z_n - x_n\| = 0. \tag{40}
$$

Next we show that $\| x_n - G(x_n) \| \to 0$ as $n \to \infty$.

For simplicity, put $q = \Pi_C(p - \mu_2 B_2 p)$, $u_n = \Pi_C(x_n - \mu_2 B_2 x_n)$, and $v_n = \Pi_C(u_n - \mu_1 B_1 u_n)$. Then $v_n = G(x_n)$. From Lemma 10, we have

$$
\begin{aligned}
\|u_n - q\|^2 &= \|\Pi_C(x_n - \mu_2 B_2 x_n) - \Pi_C(p - \mu_2 B_2 p)\|^2 \\
&\le \|x_n - p - \mu_2 (B_2 x_n - B_2 p)\|^2 \\
&\le \|x_n - p\|^2 - 2\mu_2 (\alpha_2 - \kappa^2 \mu_2) \|B_2 x_n - B_2 p\|^2,
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
\|v_n - p\|^2 &= \|\Pi_C(u_n - \mu_1 B_1 u_n) - \Pi_C(q - \mu_1 B_1 q)\|^2 \\
&\le \|u_n - q - \mu_1 (B_1 u_n - B_1 q)\|^2 \\
&\le \|u_n - q\|^2 - 2\mu_1 (\alpha_1 - \kappa^2 \mu_1) \|B_1 u_n - B_1 q\|^2.
\end{aligned}
\tag{42}
$$

Substituting (41) into (42), we obtain

$$
\begin{aligned}
\|v_n - p\|^2 \le\ & \|x_n - p\|^2 - 2\mu_2 (\alpha_2 - \kappa^2 \mu_2) \|B_2 x_n - B_2 p\|^2 \\
&- 2\mu_1 (\alpha_1 - \kappa^2 \mu_1) \|B_1 u_n - B_1 q\|^2.
\end{aligned}
\tag{43}
$$

According to Lemma 3, we have from (27)

$$
\begin{aligned}
&\|y_n - p\|^2 \\
&\quad = \|\alpha_n (f(y_n) - f(p)) + (1-\alpha_n)(S_n v_n - p) \\
&\qquad + \alpha_n (f(p) - p)\|^2 \\
&\quad \le \|\alpha_n (f(y_n) - f(p)) + (1-\alpha_n)(S_n v_n - p)\|^2 \\
&\qquad + 2\alpha_n \langle f(p) - p, J(y_n - p)\rangle \\
&\quad \le \alpha_n \|f(y_n) - f(p)\|^2 + (1-\alpha_n)\|S_n v_n - p\|^2 \\
&\qquad + 2\alpha_n \langle f(p) - p, J(y_n - p)\rangle \\
&\quad \le \alpha_n \rho \|y_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 \\
&\qquad + 2\alpha_n \|f(p) - p\| \|y_n - p\|,
\end{aligned}
\tag{44}
$$

which hence yields

$$
\begin{aligned}
&\|y_n - p\|^2 \\
&\quad \le \left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \|v_n - p\|^2 \\
&\qquad + \frac{2\alpha_n}{1-\alpha_n\rho} \|f(p) - p\| \|y_n - p\|.
\end{aligned}
\tag{45}
$$

This together with (43) and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad= \|\beta_n(x_n-p)+\gamma_n(y_n-p)+\delta_n(S_nG(y_n)-p)\|^2 \\
&\quad\le \beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2+\delta_n\|S_nG(y_n)-p\|^2 \\
&\quad\le \beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2+\delta_n\|y_n-p\|^2 \\
&\quad= \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2 \\
&\quad\le \beta_n\|x_n-p\|^2+(1-\beta_n) \\
&\qquad\times\left\{\left(1-\frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\|v_n-p\|^2\right. \\
&\qquad\quad\left.+\frac{2\alpha_n}{1-\alpha_n\rho}\|f(p)-p\|\|y_n-p\|\right\} \\
&\quad\le \beta_n\|x_n-p\|^2+(1-\beta_n) \\
&\qquad\times\left(1-\frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\|v_n-p\|^2+\alpha_nM_1 \\
&\quad\le \beta_n\|x_n-p\|^2+(1-\beta_n)\left(1-\frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \\
&\qquad\times\left[\|x_n-p\|^2-2\mu_2(\alpha_2-\kappa^2\mu_2)\|B_2x_n-B_2p\|^2\right. \\
&\qquad\quad\left.-2\mu_1(\alpha_1-\kappa^2\mu_1)\|B_1u_n-B_1q\|^2\right]+\alpha_nM_1 \\
&\quad= \left(1-\frac{(1-\beta_n)(1-\rho)}{1-\alpha_n\rho}\alpha_n\right)\|x_n-p\|^2-2(1-\beta_n) \\
&\qquad\times\left(1-\frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \\
&\qquad\times\left[\mu_2(\alpha_2-\kappa^2\mu_2)\|B_2x_n-B_2p\|^2\right. \\
&\qquad\quad\left.+\mu_1(\alpha_1-\kappa^2\mu_1)\|B_1u_n-B_1q\|^2\right]+\alpha_nM_1 \\
&\quad\le \|x_n-p\|^2-2(1-\beta_n)\left(1-\frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \\
&\qquad\times\left[\mu_2(\alpha_2-\kappa^2\mu_2)\|B_2x_n-B_2p\|^2\right. \\
&\qquad\quad\left.+\mu_1(\alpha_1-\kappa^2\mu_1)\|B_1u_n-B_1q\|^2\right]+\alpha_nM_1,
\end{aligned} \tag{46}$$

where $\sup_{n\ge0}\{2(1-\beta_n)/(1-\alpha_n\rho)\|f(p)-p\|\|y_n-p\|\}\le M_1$ for some $M_1>0$. So, it follows that

$$\begin{aligned}
&2(1-\beta_n)\left(1-\frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \\
&\qquad\times\left[\mu_2(\alpha_2-\kappa^2\mu_2)\|B_2x_n-B_2p\|^2\right. \\
&\qquad\quad\left.+\mu_1(\alpha_1-\kappa^2\mu_1)\|B_1u_n-B_1q\|^2\right] \\
&\quad\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_nM_1 \\
&\quad\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+\alpha_nM_1.
\end{aligned} \tag{47}$$

Since $0<\mu_i<\alpha_i/\kappa^2$ for $i=1,2$, from conditions (i), (ii), and (40), we obtain

$$\lim_{n\to\infty}\|B_2x_n-B_2p\|=0,\qquad \lim_{n\to\infty}\|B_1u_n-B_1q\|=0. \tag{48}$$

Utilizing [14, Proposition 1] and Lemma 5, we have

$$\begin{aligned}
&\|u_n-q\|^2 \\
&\quad= \|\Pi_C(x_n-\mu_2B_2x_n)-\Pi_C(p-\mu_2B_2p)\|^2 \\
&\quad\le \langle x_n-\mu_2B_2x_n-(p-\mu_2B_2p),J(u_n-q)\rangle \\
&\quad= \langle x_n-p,J(u_n-q)\rangle \\
&\qquad+\mu_2\langle B_2p-B_2x_n,J(u_n-q)\rangle \\
&\quad\le \frac{1}{2}\left[\|x_n-p\|^2+\|u_n-q\|^2-g_1(\|x_n-u_n-(p-q)\|)\right] \\
&\qquad+\mu_2\|B_2p-B_2x_n\|\|u_n-q\|,
\end{aligned} \tag{49}$$

which implies that

$$\begin{aligned}
\|u_n-q\|^2 \le{}& \|x_n-p\|^2-g_1(\|x_n-u_n-(p-q)\|) \\
&+2\mu_2\|B_2p-B_2x_n\|\|u_n-q\|.
\end{aligned} \tag{50}$$

In the same way, we derive

$$\begin{aligned}
&\|v_n-p\|^2 \\
&\quad= \|\Pi_C(u_n-\mu_1B_1u_n)-\Pi_C(q-\mu_1B_1q)\|^2 \\
&\quad\le \langle u_n-\mu_1B_1u_n-(q-\mu_1B_1q),J(v_n-p)\rangle \\
&\quad= \langle u_n-q,J(v_n-p)\rangle+\mu_1\langle B_1q-B_1u_n,J(v_n-p)\rangle \\
&\quad\le \frac{1}{2}\left[\|u_n-q\|^2+\|v_n-p\|^2\right. \\
&\qquad\left.-g_2(\|u_n-v_n+(p-q)\|)\right] \\
&\qquad+\mu_1\|B_1q-B_1u_n\|\|v_n-p\|,
\end{aligned} \tag{51}$$

which implies that

$$\begin{aligned}
\|v_n-p\|^2 \le{}& \|u_n-q\|^2-g_2(\|u_n-v_n+(p-q)\|) \\
&+2\mu_1\|B_1q-B_1u_n\|\|v_n-p\|.
\end{aligned} \tag{52}$$

Substituting (50) into (52), we get

$$\begin{aligned}
\|v_n-p\|^2 \le{}& \|x_n-p\|^2-g_1(\|x_n-u_n-(p-q)\|) \\
&-g_2(\|u_n-v_n+(p-q)\|) \\
&+2\mu_2\|B_2p-B_2x_n\|\|u_n-q\| \\
&+2\mu_1\|B_1q-B_1u_n\|\|v_n-p\|.
\end{aligned} \tag{53}$$

From (46) and (53), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \alpha_n M_1 + \beta_n \|x_n - p\|^2 \\
&\quad + (1-\beta_n)\left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \\
&\quad \times \left[\|x_n - p\|^2 - g_1(\|x_n - u_n - (p-q)\|)\right. \\
&\qquad - g_2(\|u_n - v_n + (p-q)\|) \\
&\qquad + 2\mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\| \\
&\qquad \left. + 2\mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\|\right] \\
&\le \alpha_n M_1 + \left(1 - \frac{(1-\beta_n)(1-\rho)}{1-\alpha_n\rho}\alpha_n\right)\|x_n - p\|^2 \\
&\quad - (1-\beta_n)\left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \\
&\quad \times [g_1(\|x_n - u_n - (p-q)\|) \\
&\qquad + g_2(\|u_n - v_n + (p-q)\|)] \\
&\quad + 2\mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\| \\
&\quad + 2\mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\| \\
&\le \alpha_n M_1 + \|x_n - p\|^2 - (1-\beta_n) \\
&\quad \times \left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \\
&\quad \times [g_1(\|x_n - u_n - (p-q)\|) \\
&\qquad + g_2(\|u_n - v_n + (p-q)\|)] \\
&\quad + 2\mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\| \\
&\quad + 2\mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\|,
\end{aligned} \tag{54}$$

which implies that

$$\begin{aligned}
&(1-\beta_n)\left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \\
&\quad \times [g_1(\|x_n - u_n - (p-q)\|) \\
&\qquad + g_2(\|u_n - v_n + (p-q)\|)] \\
&\le \alpha_n M_1 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + 2\mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\| \\
&\quad + 2\mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\| \\
&\le \alpha_n M_1 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| \\
&\quad + 2\mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\| \\
&\quad + 2\mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\|.
\end{aligned} \tag{55}$$

Utilizing conditions (i), (ii), from (40) and (48), we have

$$\begin{aligned}
\lim_{n\to\infty} g_1(\|x_n - u_n - (p-q)\|) &= 0, \\
\lim_{n\to\infty} g_2(\|u_n - v_n + (p-q)\|) &= 0.
\end{aligned} \tag{56}$$

Utilizing the properties of g_1 and g_2, we deduce that

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - u_n - (p-q)\| &= 0, \\
\lim_{n\to\infty} \|u_n - v_n + (p-q)\| &= 0.
\end{aligned} \tag{57}$$

From (57), we obtain

$$\begin{aligned}
\|x_n - v_n\| &\le \|x_n - u_n - (p-q)\| \\
&\quad + \|u_n - v_n + (p-q)\| \longrightarrow 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{58}$$

That is,

$$\lim_{n\to\infty} \|x_n - G(x_n)\| = 0. \tag{59}$$

On the other hand, since $\{y_n\}$ and $\{S_n G(y_n)\}$ are bounded, by Lemma 6, there exists a continuous strictly increasing function $g_3 : [0,\infty) \to [0,\infty)$, $g_3(0) = 0$ such that for $p \in F$

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \|\beta_n(x_n - p) + \gamma_n(y_n - p) + \delta_n(S_n G(y_n) - p)\|^2 \\
&\quad = \left\|(\gamma_n + \delta_n)\left[\frac{\gamma_n}{\gamma_n + \delta_n}(y_n - p)\right.\right. \\
&\qquad \left.\left. + \frac{\delta_n}{\gamma_n + \delta_n}(S_n G(y_n) - p)\right] + \beta_n(x_n - p)\right\|^2 \\
&\quad \le (\gamma_n + \delta_n)\left\|\frac{\gamma_n}{\gamma_n + \delta_n}(y_n - p) + \frac{\delta_n}{\gamma_n + \delta_n}(S_n G(y_n) - p)\right\|^2 \\
&\qquad + \beta_n \|x_n - p\|^2 \\
&\quad \le (\gamma_n + \delta_n)\left[\frac{\gamma_n}{\gamma_n + \delta_n}\|y_n - p\|^2 + \frac{\delta_n}{\gamma_n + \delta_n}\|S_n G(y_n) - p\|^2\right. \\
&\qquad \left. - \frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_3(\|y_n - S_n G(y_n)\|)\right] \\
&\qquad + \beta_n \|x_n - p\|^2 \\
&\quad \le \gamma_n \|y_n - p\|^2 + \delta_n \|y_n - p\|^2 \\
&\qquad - \frac{\gamma_n \delta_n}{\gamma_n + \delta_n} g_3(\|y_n - S_n G(y_n)\|) + \beta_n \|x_n - p\|^2 \\
&\quad = (1-\beta_n)\|y_n - p\|^2 - \frac{\gamma_n \delta_n}{\gamma_n + \delta_n} \\
&\qquad \times g_3(\|y_n - S_n G(y_n)\|) + \beta_n \|x_n - p\|^2,
\end{aligned} \tag{60}$$

which together with (30) implies that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad\le (1-\beta_n)\left(\|x_n-p\|+\frac{\alpha_n}{1-\alpha_n\rho}\|f(p)-p\|\right)^2 \\
&\qquad -\frac{\gamma_n\delta_n}{\gamma_n+\delta_n}g_3(\|y_n-S_nG(y_n)\|)+\beta_n\|x_n-p\|^2 \\
&\quad\le \left(\|x_n-p\|+\frac{\alpha_n}{1-\alpha_n\rho}\|f(p)-p\|\right)^2 \\
&\qquad -\frac{\gamma_n\delta_n}{\gamma_n+\delta_n}g_3(\|y_n-S_nG(y_n)\|).
\end{aligned} \tag{61}$$

It immediately follows that

$$\begin{aligned}
&\gamma_n\delta_n g_3(\|y_n-S_nG(y_n)\|) \\
&\quad\le \frac{\gamma_n\delta_n}{\gamma_n+\delta_n}g_3(\|y_n-S_nG(y_n)\|) \\
&\quad\le \left(\|x_n-p\|+\frac{\alpha_n}{1-\alpha_n\rho}\|f(p)-p\|\right)^2 \\
&\qquad -\|x_{n+1}-p\|^2 \\
&\quad\le \Big(\|x_n-p\|+\|x_{n+1}-p\| \\
&\qquad +\frac{\alpha_n}{1-\alpha_n\rho}\|f(p)-p\|\Big) \\
&\qquad \times\left(\|x_n-x_{n+1}\|+\frac{\alpha_n}{1-\alpha_n\rho}\|f(p)-p\|\right).
\end{aligned} \tag{62}$$

According to condition (ii), we get

$$\begin{aligned}
\liminf_{n\to\infty}\delta_n &= \liminf_{n\to\infty}(1-\beta_n-\gamma_n) \\
&= 1-\limsup_{n\to\infty}(\beta_n+\gamma_n)>0.
\end{aligned} \tag{63}$$

Since $\alpha_n \to 0$, $\|x_{n+1}-x_n\| \to 0$, and $\liminf_{n\to\infty}\gamma_n > 0$, we conclude that

$$\lim_{n\to\infty} g_3(\|y_n-S_nG(y_n)\|)=0. \tag{64}$$

Utilizing the property of g_3, we have

$$\lim_{n\to\infty}\|y_n-S_nG(y_n)\|=0. \tag{65}$$

We note that

$$\begin{aligned}
x_{n+1}-x_n+x_n-y_n &= x_{n+1}-y_n \\
&= \beta_n(x_n-y_n)+\delta_n(S_nG(y_n)-y_n).
\end{aligned} \tag{66}$$

So,

$$\begin{aligned}
&(1-\beta_n)\|x_n-y_n\| \\
&\quad= \|\delta_n(S_nG(y_n)-y_n)-(x_{n+1}-x_n)\| \\
&\quad\le \delta_n\|S_nG(y_n)-y_n\|+\|x_{n+1}-x_n\| \\
&\quad\le \|S_nG(y_n)-y_n\| \\
&\qquad +\|x_{n+1}-x_n\| \longrightarrow 0 \quad \text{as } n\longrightarrow\infty.
\end{aligned} \tag{67}$$

That is,

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{68}$$

We observe that

$$\begin{aligned}
&\|G(x_n)-S_nG(x_n)\| \\
&\quad\le \|G(x_n)-x_n\|+\|x_n-y_n\|+\|y_n-S_nG(y_n)\| \\
&\qquad +\|S_nG(y_n)-S_nG(x_n)\| \\
&\quad\le \|G(x_n)-x_n\|+2\|x_n-y_n\|+\|y_n-S_nG(y_n)\|.
\end{aligned} \tag{69}$$

Thus, from (59)–(68), we obtain that

$$\lim_{n\to\infty}\|G(x_n)-S_nG(x_n)\|=0. \tag{70}$$

By (70) and Lemma 7, we have

$$\begin{aligned}
&\|SG(x_n)-G(x_n)\| \\
&\quad\le \|SG(x_n)-S_nG(x_n)\| \\
&\qquad +\|S_nG(x_n)-G(x_n)\| \longrightarrow 0 \quad \text{as } n\longrightarrow\infty.
\end{aligned} \tag{71}$$

In terms of (59) and (71), we have

$$\begin{aligned}
\|x_n-Sx_n\| &\le \|x_n-G(x_n)\|+\|G(x_n)-SG(x_n)\| \\
&\quad +\|SG(x_n)-Sx_n\| \\
&\le 2\|x_n-G(x_n)\| \\
&\quad +\|G(x_n)-SG(x_n)\| \longrightarrow 0 \quad \text{as } n\longrightarrow\infty.
\end{aligned} \tag{72}$$

Define a mapping $Wx=(1-\theta)Sx+\theta G(x)$, where $\theta\in(0,1)$ is a constant. Then by Lemma 9, we have that $\mathrm{Fix}(W)=\mathrm{Fix}(S)\cap\mathrm{Fix}(G)=F$. We observe that

$$\begin{aligned}
\|x_n-Wx_n\| &= \|(1-\theta)(x_n-Sx_n)+\theta(x_n-G(x_n))\| \\
&\le (1-\theta)\|x_n-Sx_n\|+\theta\|x_n-G(x_n)\|.
\end{aligned} \tag{73}$$

From (59) and (72), we obtain

$$\lim_{n\to\infty}\|x_n-Wx_n\|=0. \tag{74}$$

Now, we claim that

$$\limsup_{n\to\infty}\langle f(q)-q, J(x_n-q)\rangle \le 0, \tag{75}$$

where $q = s - \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction

$$x \longmapsto tf(x) + (1-t)Wx. \tag{76}$$

Then x_t solves the fixed point equation $x_t = tf(x_t) + (1-t)Wx_t$. Thus we have

$$\|x_t - x_n\| = \|(1-t)(Wx_t - x_n) + t(f(x_t) - x_n)\|. \tag{77}$$

By Lemma 3, we conclude that

$$\begin{aligned}
&\|x_t - x_n\|^2 \\
&\quad = \|(1-t)(Wx_t - x_n) + t(f(x_t) - x_n)\|^2 \\
&\quad \le (1-t)^2\|Wx_t - x_n\|^2 \\
&\qquad + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle \\
&\quad \le (1-t)^2(\|Wx_t - Wx_n\| + \|Wx_n - x_n\|)^2 \\
&\qquad + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle \\
&\quad \le (1-t)^2(\|x_t - x_n\| + \|Wx_n - x_n\|)^2 \\
&\qquad + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle \\
&\quad = (1-t)^2\left[\|x_t - x_n\|^2 + 2\|x_t - x_n\| \right. \\
&\qquad\qquad \left. \times \|Wx_n - x_n\| + \|Wx_n - x_n\|^2\right] \\
&\qquad + 2t\langle f(x_t) - x_t, J(x_t - x_n)\rangle \\
&\qquad + 2t\langle x_t - x_n, J(x_t - x_n)\rangle \\
&\quad = \left(1 - 2t + t^2\right)\|x_t - x_n\|^2 + f_n(t) \\
&\qquad + 2t\langle f(x_t) - x_t, J(x_t - x_n)\rangle + 2t\|x_t - x_n\|^2,
\end{aligned} \tag{78}$$

where

$$\begin{aligned}
f_n(t) &= (1-t)^2(2\|x_t - x_n\| + \|x_n - Wx_n\|) \\
&\quad \times \|x_n - Wx_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{79}$$

It follows from (78) that

$$\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}f_n(t). \tag{80}$$

Letting $n \to \infty$ in (80) and noticing (79), we derive

$$\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2}M_2, \tag{81}$$

where $M_2 > 0$ is a constant such that $\|x_t - x_n\|^2 \le M_2$ for all $t \in (0,1)$ and $n \ge 0$. Taking $t \to 0$ in (81), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le 0. \tag{82}$$

On the other hand, we have

$$\begin{aligned}
&\langle f(q) - q, J(x_n - q)\rangle \\
&\quad = \langle f(q) - q, J(x_n - q)\rangle - \langle f(q) - q, J(x_n - x_t)\rangle \\
&\qquad + \langle f(q) - q, J(x_n - x_t)\rangle - \langle f(q) - x_t, J(x_n - x_t)\rangle \\
&\qquad + \langle f(q) - x_t, J(x_n - x_t)\rangle - \langle f(x_t) - x_t, J(x_n - x_t)\rangle \\
&\qquad + \langle f(x_t) - x_t, J(x_n - x_t)\rangle \\
&\quad = \langle f(q) - q, J(x_n - q) - J(x_n - x_t)\rangle \\
&\qquad + \langle x_t - q, J(x_n - x_t)\rangle + \langle f(q) - f(x_t), J(x_n - x_t)\rangle \\
&\qquad + \langle f(x_t) - x_t, J(x_n - x_t)\rangle.
\end{aligned} \tag{83}$$

It follows that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle f(q) - q, J(x_n - q)\rangle \\
&\quad \le \limsup_{n\to\infty}\langle f(q) - q, J(x_n - q) - J(x_n - x_t)\rangle \\
&\qquad + \|x_t - q\|\limsup_{n\to\infty}\|x_n - x_t\| \\
&\qquad + \rho\|q - x_t\|\limsup_{n\to\infty}\|x_n - x_t\| \\
&\qquad + \limsup_{n\to\infty}\langle f(x_t) - x_t, J(x_n - x_t)\rangle.
\end{aligned} \tag{84}$$

Taking into account that $x_t \to q$ as $t \to 0$, we have from (82)

$$\begin{aligned}
&\limsup_{n\to\infty}\langle f(q) - q, J(x_n - q)\rangle \\
&\quad = \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(q) - q, J(x_n - q)\rangle \\
&\quad \le \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(q) - q, J(x_n - q) - J(x_n - x_t)\rangle.
\end{aligned} \tag{85}$$

Since X has a uniformly Fréchet differentiable norm, the duality mapping J is norm-to-norm uniformly continuous on bounded subsets of X. Consequently, the two limits are interchangeable, and hence (75) holds. From (68), we get $(y_n - q) - (x_n - q) \to 0$. Noticing that J is norm-to-norm uniformly continuous on bounded subsets of X, we deduce from (75) that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle f(q) - q, J(y_n - q)\rangle \\
&\quad = \limsup_{n\to\infty}(\langle f(q) - q, J(x_n - q)\rangle \\
&\qquad\qquad + \langle f(q) - q, J(y_n - q) - J(x_n - q)\rangle) \\
&\quad = \limsup_{n\to\infty}\langle f(q) - q, J(x_n - q)\rangle \le 0.
\end{aligned} \tag{86}$$

Finally, let us show that $x_n \to q$ as $n \to \infty$. We observe that

$$\begin{aligned}
&\|y_n - q\|^2 \\
&\quad = \|\alpha_n (f(y_n) - f(q)) + (1-\alpha_n) \\
&\qquad \times (S_n G(x_n) - q) + \alpha_n (f(q) - q)\|^2 \\
&\quad \le \|\alpha_n (f(y_n) - f(q)) \\
&\qquad + (1-\alpha_n)(S_n G(x_n) - q)\|^2 \\
&\qquad + 2\alpha_n \langle f(q) - q, J(y_n - q)\rangle \\
&\quad \le \alpha_n \|f(y_n) - f(q)\|^2 \\
&\qquad + (1-\alpha_n)\|S_n G(x_n) - q\|^2 \\
&\qquad + 2\alpha_n \langle f(q) - q, J(y_n - q)\rangle \\
&\quad \le \alpha_n \rho \|y_n - q\|^2 + (1-\alpha_n)\|x_n - q\|^2 \\
&\qquad + 2\alpha_n \langle f(q) - q, J(y_n - q)\rangle,
\end{aligned} \tag{87}$$

which implies that

$$\begin{aligned}
\|y_n - q\|^2 &\le \left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\|x_n - q\|^2 \\
&\quad + \frac{\alpha_n(1-\rho)}{1-\alpha_n\rho} \cdot \frac{2\langle f(q) - q, J(y_n - q)\rangle}{1-\rho}.
\end{aligned} \tag{88}$$

By (27) and the convexity of $\|\cdot\|^2$, we get

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad \le \beta_n \|x_n - q\|^2 + \gamma_n \|y_n - q\|^2 + \delta_n \|S_n G(y_n) - q\|^2 \\
&\quad \le \beta_n \|x_n - q\|^2 + \gamma_n \|y_n - q\|^2 + \delta_n \|y_n - q\|^2 \\
&\quad = \beta_n \|x_n - q\|^2 + (1-\beta_n)\|y_n - q\|^2,
\end{aligned} \tag{89}$$

which together with (88) leads to

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad \le \beta_n \|x_n - q\|^2 + (1-\beta_n) \\
&\qquad \times \left\{ \left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\|x_n - q\|^2 \right. \\
&\qquad \left. + \frac{\alpha_n(1-\rho)}{1-\alpha_n\rho} \cdot \frac{2\langle f(q) - q, J(y_n - q)\rangle}{1-\rho} \right\} \\
&\quad = \left[1 - \frac{(1-\beta_n)(1-\rho)}{1-\alpha_n\rho}\alpha_n\right]\|x_n - q\|^2 \\
&\qquad + \frac{(1-\beta_n)(1-\rho)}{1-\alpha_n\rho}\alpha_n \cdot \frac{2\langle f(q) - q, J(y_n - q)\rangle}{1-\rho}.
\end{aligned} \tag{90}$$

Applying Lemma 2 to (88), we obtain that $x_n \to q$ as $n \to \infty$. This completes the proof. □

Corollary 15. *Let C be a nonempty, closed, and convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be α_i-inverse-strongly accretive for $i = 1, 2$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$. Let S be a nonexpansive mapping of C into itself such that $F = \mathrm{Fix}(S) \cap \Omega \neq \emptyset$, where Ω is the fixed point set of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}
y_n &= \alpha_n f(y_n) + (1-\alpha_n) SG(x_n), \\
x_{n+1} &= \beta_n x_n + \gamma_n y_n + \delta_n SG(y_n), \quad \forall n \ge 0,
\end{aligned} \tag{91}$$

where $0 < \mu_i < \alpha_i/\kappa^2$ for $i = 1, 2$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $(0,1)$ such that $\beta_n + \gamma_n + \delta_n = 1$, $\forall n \ge 0$. Suppose that the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$,

(ii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}(\beta_n + \gamma_n) < 1$ *and* $\liminf_{n\to\infty}\gamma_n > 0$,

(iii) $\lim_{n\to\infty}|(\gamma_n/(1-\beta_n)) - (\gamma_{n-1}/(1-\beta_{n-1}))| = 0$.

Then $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP:

$$\langle q - f(q), J(q - p)\rangle \le 0, \quad \forall p \in F. \tag{92}$$

Further, we illustrate Theorem 14 by virtue of an example, that is, the following corollary.

Corollary 16. *Let C be a nonempty, closed, and convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$. Let T be an η-strictly pseudocontractive mapping of C into itself, and let S be a nonexpansive mapping of C into itself such that $F = \mathrm{Fix}(S) \cap \mathrm{Fix}(T) \neq \emptyset$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}
y_n &= \alpha_n f(y_n) + (1-\alpha_n) S(I - \lambda(I - T)) x_n, \\
x_{n+1} &= \beta_n x_n + \gamma_n y_n + \delta_n S(I - \lambda(I - T)) y_n, \quad \forall n \ge 0,
\end{aligned} \tag{93}$$

where $0 < \lambda < \max\{1, \eta/\kappa^2\}$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $(0,1)$ such that $\beta_n + \gamma_n + \delta_n = 1$, $\forall n \ge 0$. Suppose that the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$,

(ii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}(\beta_n + \gamma_n) < 1$ *and* $\liminf_{n\to\infty}\gamma_n > 0$,

(iii) $\lim_{n\to\infty}|(\gamma_n/(1-\beta_n)) - (\gamma_{n-1}/(1-\beta_{n-1}))| = 0$.

Then $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP:

$$\langle q - f(q), J(q - p)\rangle \le 0, \quad \forall p \in F. \tag{94}$$

Proof. In Corollary 15, put $B_1 = I - T$, $B_2 = 0$, $\mu_1 = \lambda$, and $\alpha_1 = \eta$. Since T is an η-strictly pseudocontractive mapping, it is clear that $B_1 = I - T$ is an η-inverse strongly accretive mapping. Hence, the GSVI (9) is equivalent to the following VIP of finding $x^* \in C$ such that

$$\left(B_1 x^*, J\left(x - x^*\right)\right) \geq 0, \quad \forall x \in C, \tag{95}$$

which leads to $\Omega = \mathrm{VI}(C, B_1)$. In the meantime, we have

$$\begin{aligned} Gx_n &= \Pi_C\left(I - \mu_1 B_1\right)\Pi_C\left(I - \mu_2 B_2\right)x_n \\ &= \Pi_C\left(I - \mu_1 B_1\right)x_n \\ &= \Pi_C\left[(1-\lambda)x_n + \lambda T x_n\right] \\ &= x_n - \lambda\left(I - T\right)x_n. \end{aligned} \tag{96}$$

In the same way, we get $Gy_n = y_n - \lambda(I - T)y_n$. In this case, it is easy to see that (91) reduces to (93). We claim that $\mathrm{Fix}(T) = \mathrm{VI}(C, B_1)$. As a matter of fact, we have, for $\lambda > 0$,

$$\begin{aligned} & u \in \mathrm{VI}\left(C, B_1\right) \\ & \quad \Longleftrightarrow \langle B_1 u, J\left(y - u\right)\rangle \geq 0 \quad \forall y \in C \\ & \quad \Longleftrightarrow \langle u - \lambda B_1 u - u, J\left(u - y\right)\rangle \geq 0 \quad \forall y \in C \\ & \quad \Longleftrightarrow u = \Pi_C\left(u - \lambda B_1 u\right) \\ & \quad \Longleftrightarrow u = \Pi_C\left(u - \lambda u + \lambda T u\right) \\ & \quad \Longleftrightarrow \langle u - \lambda u + \lambda T u - u, J\left(u - y\right)\rangle \geq 0 \quad \forall y \in C \\ & \quad \Longleftrightarrow \langle u - T u, J\left(u - y\right)\rangle \leq 0 \quad \forall y \in C \\ & \quad \Longleftrightarrow u = T u \\ & \quad \Longleftrightarrow u \in \mathrm{Fix}\left(T\right). \end{aligned} \tag{97}$$

So, we conclude that $F = \mathrm{Fix}(S) \cap \Omega = \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$. Therefore, the desired result follows from Corollary 15. □

Remark 17. Theorem 14 improves, extends, supplements, and develops Cai and Bu [2, Theorem 3.1 and Corollary 3.2] and Jung [5, Theorem 3.1] in the following aspects.

(i) The problem of finding a point $q \in \bigcap_n \mathrm{Fix}(S_n) \cap \Omega$ in Theorem 14 is more general and more subtle than the problem of finding a point $q \in \mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$ in Jung [5, Theorem 3.1].

(ii) The iterative scheme in [2, Theorem 3.1] is extended to develop the iterative scheme (27) of Theorem 14 by virtue of the iterative scheme of [5, Theorem 3.1]. The iterative scheme (27) of Theorem 14 is more advantageous and more flexible than the iterative scheme of [2, Theorem 3.1] because it involves several parameter sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$.

(iii) The iterative scheme (27) in Theorem 14 is very different from everyone in both [2, Theorem 3.1] and [5, Theorem 3.1] because the mappings S_n and G in the iterative scheme of [2, Theorem 3.1] and the mapping $SP_C(I - \lambda_n A)$ in the iterative scheme of [5, Theorem 3.1] are replaced by the same composite mapping $S_n G$ in the iterative scheme (27) of Theorem 14.

(iv) The proof in [2, Theorem 3.1] depends on the argument techniques in [3], the inequality in 2-uniformly smooth Banach spaces ([9]), and the inequality in smooth and uniform convex Banach spaces ([14, Proposition 1]). Because the composite mapping $S_n G$ appears in the iterative scheme (27) of Theorem 14, the proof of Theorem 14 depends on the argument techniques in [3], the inequality in 2-uniformly smooth Banach spaces, the inequality in smooth and uniform convex Banach spaces, and the inequality in uniform convex Banach spaces (Lemma 6).

(v) The iterative scheme in [2, Corollary 3.2] is extended to develop the new iterative scheme in Corollary 15 because the mappings S and G are replaced by the same composite mapping SG in Corollary 15.

4. Explicit Iterative Schemes

In this section, we introduce our explicit iterative schemes and show the strong convergence theorems. First, we give several useful lemmas.

Lemma 18. *Let C be a nonempty, closed, and convex subset of a smooth Banach space X, and let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \geq 1$ for $i = 1, 2$. Then, for $\mu_i \in (0, 1]$, we have*

$$\begin{aligned} & \left\|\left(I - \mu_i B_i\right)x - \left(I - \mu_i B_i\right)y\right\| \\ & \quad \leq \left\{\sqrt{\frac{1-\alpha_i}{\lambda_i}} + \left(1 - \mu_i\right)\left(1 + \frac{1}{\lambda_i}\right)\right\}\left\|x - y\right\|, \\ & \qquad \forall x, y \in C, \end{aligned} \tag{98}$$

for $i = 1, 2$. In particular, if $1 - (\lambda_i/(1+\lambda_i))(1 - \sqrt{(1-\alpha_i)/\lambda_i}) \leq \mu_i \leq 1$, then $I - \mu_i B_i$ is nonexpansive for $i = 1, 2$.

Proof. Taking into account the λ_i-strict pseudocontractivity of B_i, we derive for every $x, y \in C$

$$\begin{aligned} & \lambda_i\left\|\left(I - B_i\right)x - \left(I - B_i\right)y\right\|^2 \\ & \quad \leq \left\langle\left(I - B_i\right)x - \left(I - B_i\right)y, J\left(x - y\right)\right\rangle \\ & \quad \leq \left\|\left(I - B_i\right)x - \left(I - B_i\right)y\right\|\left\|x - y\right\|, \end{aligned} \tag{99}$$

which implies that

$$\left\|\left(I - B_i\right)x - \left(I - B_i\right)y\right\| \leq \frac{1}{\lambda_i}\left\|x - y\right\|. \tag{100}$$

Hence,

$$\begin{aligned}\|B_i x - B_i y\| &\le \|(I - B_i)x - (I - B_i)y\| + \|x - y\| \\ &\le \left(1 + \frac{1}{\lambda_i}\right)\|x - y\|.\end{aligned} \tag{101}$$

Utilizing the α_i-strong accretivity and λ_i-strict pseudocontractivity of B_i, we get

$$\begin{aligned}\lambda_i\|(I - B_i)x - (I - B_i)y\|^2 &\le \|x - y\|^2 - \langle B_i x - B_i y, J(x - y)\rangle \\ &\le (1 - \alpha_i)\|x - y\|^2.\end{aligned} \tag{102}$$

So, we have

$$\|(I - B_i)x - (I - B_i)y\| \le \sqrt{\frac{1 - \alpha_i}{\lambda_i}}\|x - y\|. \tag{103}$$

Therefore, for $\mu_i \in (0, 1]$, we have

$$\begin{aligned}\|(I - \mu_i B_i)x - (I - \mu_i B_i)y\| &\le \|(I - B_i)x - (I - B_i)y\| + (1 - \mu_i)\|B_i x - B_i y\| \\ &\le \sqrt{\frac{1 - \alpha_i}{\lambda_i}}\|x - y\| + (1 - \mu_i)\left(1 + \frac{1}{\lambda_i}\right)\|x - y\| \\ &= \left\{\sqrt{\frac{1 - \alpha_i}{\lambda_i}} + (1 - \mu_i)\left(1 + \frac{1}{\lambda_i}\right)\right\}\|x - y\|.\end{aligned} \tag{104}$$

Since $1 - (\lambda_i/(1 + \lambda_i))(1 - \sqrt{(1 - \alpha_i)/\lambda_i}) \le \mu_i \le 1$, it follows immediately that

$$\sqrt{\frac{1 - \alpha_i}{\lambda_i}} + (1 - \mu_i)\left(1 + \frac{1}{\lambda_i}\right) \le 1. \tag{105}$$

This implies that $I - \mu_i B_i$ is nonexpansive for $i = 1, 2$. □

Lemma 19. *Let C be a nonempty, closed, and convex subset of a smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C, and let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \ge 1$ for $i = 1, 2$. Let $G : C \to C$ be the mapping defined by*

$$\begin{aligned}G(x) = \Pi_C[&\Pi_C(x - \mu_2 B_2 x) \\ &- \mu_1 B_1 \Pi_C(x - \mu_2 B_2 x)], \quad \forall x \in C.\end{aligned} \tag{106}$$

If $1-(\lambda_i/(1+\lambda_i))(1-\sqrt{(1 - \alpha_i)/\lambda_i}) \le \mu_i \le 1$, then $G : C \to C$ is nonexpansive.

Proof. According to Lemma 10, we know that $I - \mu_i B_i$ is nonexpansive for $i = 1, 2$. Hence, for all $x, y \in C$, we have

$$\begin{aligned}\|G(x) - G(y)\| &= \|\Pi_C[\Pi_C(x - \mu_2 B_2 x) - \mu_1 B_1 \Pi_C(x - \mu_2 B_2 x)] \\ &\quad - \Pi_C[\Pi_C(y - \mu_2 B_2 y) - \mu_1 B_1 \Pi_C(y - \mu_2 B_2 y)]\| \\ &= \|\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x \\ &\quad - \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)y\| \\ &\le \|(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x \\ &\quad - (I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)y\| \\ &\le \|\Pi_C(I - \mu_2 B_2)x - \Pi_C(I - \mu_2 B_2)y\| \\ &\le \|(I - \mu_2 B_2)x - (I - \mu_2 B_2)y\| \\ &\le \|x - y\|.\end{aligned} \tag{107}$$

This shows that $G : C \to C$ is nonexpansive. This completes the proof. □

Lemma 20. *Let C be a nonempty, closed, and convex subset of a smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C, and let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive for $i = 1, 2$. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of GSVI (9) if and only if $x^* = \Pi_C(y^* - \mu_1 B_1 y^*)$ where $y^* = \Pi_C(x^* - \mu_2 B_2 x^*)$.*

Proof. We can rewrite GSVI (9) as

$$\begin{aligned}&\langle x^* - (y^* - \mu_1 B_1 y^*), J(x - x^*)\rangle \ge 0, \quad \forall x \in C, \\ &\langle y^* - (x^* - \mu_2 B_2 x^*), J(x - y^*)\rangle \ge 0, \quad \forall x \in C,\end{aligned} \tag{108}$$

which is obviously equivalent to

$$\begin{aligned}x^* &= \Pi_C(y^* - \mu_1 B_1 y^*), \\ y^* &= \Pi_C(x^* - \mu_2 B_2 x^*),\end{aligned} \tag{109}$$

because of Lemma 5. This completes the proof. □

Remark 21. By Lemma 20, we observe that

$$x^* = \Pi_C[\Pi_C(x^* - \mu_2 B_2 x^*) - \mu_1 B_1 \Pi_C(x^* - \mu_2 B_2 x^*)], \tag{110}$$

which implies that x^* is a fixed point of the mapping G. Throughout this paper, the set of fixed points of the mapping G is denoted by Ω.

We are now in a position to state and prove our result on the explicit iterative scheme.

Theorem 22. *Let C be a nonempty, closed, and convex subset of a uniformly convex Banach space X which has a uniformly*

Gâteaux differentiable norm. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \geq 1$ for $i = 1, 2$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0, 1)$. Let $\{S_n\}_{n=0}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself such that $F = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \Omega \neq \emptyset$, where Ω is the fixed point set of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} y_n &= \alpha_n G(x_n) + (1 - \alpha_n) S_n G(x_n), \\ x_{n+1} &= \beta_n f(x_n) + \gamma_n y_n + \delta_n S_n G(y_n), \quad \forall n \geq 0, \end{aligned} \tag{111}$$

where $1 - (\lambda_i/(1 + \lambda_i))(1 - \sqrt{(1 - \alpha_i)/\lambda_i}) \leq \mu_i \leq 1$ for $i = 1, 2$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $(0, 1)$ such that $\beta_n + \gamma_n + \delta_n = 1$, $\forall n \geq 0$. Suppose that the following conditions hold:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty} |\alpha_n - \alpha_{n-1}|/\beta_n = 0$,

(iv) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \beta_{n-1}/\beta_n = 1$,

(v) $\sum_{n=1}^{\infty} |(\gamma_n/(1 - \beta_n)) - (\gamma_{n-1}/(1 - \beta_{n-1}))| < \infty$ *or* $\lim_{n\to\infty} (1/\beta_n)|(\gamma_n/(1 - \beta_n)) - (\gamma_{n-1}/(1 - \beta_{n-1}))| = 0$,

(vi) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \| S_n x - S_{n-1} x \| < \infty$ for any bounded subset D of C, and let S be a mapping of C into itself defined by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$. Suppose that $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$. Then $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP:

$$\langle q - f(q), J(q - p) \rangle \leq 0, \quad \forall p \in F. \tag{112}$$

Proof. Take a fixed $p \in F$ arbitrarily. Then by Lemma 20, we know that $p = G(p)$ and $p = S_n p$ for all $n \geq 0$. Moreover, by Lemma 19, we have

$$\begin{aligned} &\|y_n - p\| \\ &\quad \leq \alpha_n \|G(x_n) - p\| + (1 - \alpha_n) \|S_n G(x_n) - p\| \\ &\quad \leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|x_n - p\| \\ &\quad = \|x_n - p\|. \end{aligned} \tag{113}$$

From (113) we obtain

$$\begin{aligned} &\|x_{n+1} - p\| \\ &\quad \leq \beta_n \|f(x_n) - p\| + \gamma_n \|y_n - p\| \\ &\qquad + \delta_n \|S_n G(y_n) - p\| \\ &\quad \leq \beta_n (\|f(x_n) - f(p)\| + \|f(p) - p\|) \\ &\qquad + \gamma_n \|y_n - p\| + \delta_n \|y_n - p\| \\ &\quad \leq \beta_n \rho \|x_n - p\| + \beta_n \|f(p) - p\| \\ &\qquad + (1 - \beta_n) \|y_n - p\| \\ &\quad \leq \beta_n \rho \|x_n - p\| + \beta_n \|f(p) - p\| \\ &\qquad + (1 - \beta_n) \|x_n - p\| \\ &\quad = (1 - \beta_n (1 - \rho)) \|x_n - p\| \\ &\qquad + \beta_n \|f(p) - p\| \\ &\quad = (1 - \beta_n (1 - \rho)) \|x_n - p\| \\ &\qquad + \beta_n (1 - \rho) \cdot \frac{\|f(p) - p\|}{1 - \rho} \\ &\quad \leq \max\left\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1 - \rho}\right\}, \end{aligned} \tag{114}$$

which implies that $\{x_n\}$ is bounded. By Lemma 19 we know from (113) that $\{y_n\}$, $\{G(x_n)\}$, and $\{G(y_n)\}$ are bounded.

Let us show that $\| x_{n+1} - x_n \| \to 0$ and $\| x_n - y_n \| \to 0$ as $n \to \infty$. As a matter of fact, from (113), we have

$$\begin{aligned} y_n &= \alpha_n G(x_n) + (1 - \alpha_n) S_n G(x_n), \\ y_{n-1} &= \alpha_{n-1} G(x_{n-1}) + (1 - \alpha_{n-1}) S_{n-1} G(x_{n-1}), \quad \forall n \geq 1. \end{aligned} \tag{115}$$

Simple calculations show that

$$\begin{aligned} y_n - y_{n-1} &= \alpha_n (G(x_n) - G(x_{n-1})) \\ &\quad + (\alpha_n - \alpha_{n-1}) (G(x_{n-1}) - S_{n-1} G(x_{n-1})) \\ &\quad + (1 - \alpha_n) (S_n G(x_n) - S_{n-1} G(x_{n-1})). \end{aligned} \tag{116}$$

It follows that

$$\begin{aligned} &\|y_n - y_{n-1}\| \\ &\quad \leq \alpha_n \|G(x_n) - G(x_{n-1})\| \\ &\qquad + |\alpha_n - \alpha_{n-1}| \|G(x_{n-1}) - S_{n-1} G(x_{n-1})\| \\ &\qquad + (1 - \alpha_n) \|S_n G(x_n) - S_{n-1} G(x_{n-1})\| \end{aligned}$$

$$\begin{aligned}
&\le \alpha_n \|G(x_n) - G(x_{n-1})\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \|G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\quad + (1-\alpha_n)(\|S_nG(x_n) - S_nG(x_{n-1})\| \\
&\qquad + \|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\|) \\
&\le \alpha_n \|G(x_n) - G(x_{n-1})\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \|G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\quad + (1-\alpha_n)(\|G(x_n) - G(x_{n-1})\| \\
&\qquad + \|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\|) \\
&\le \|G(x_n) - G(x_{n-1})\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \|G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\quad + \|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\le \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \\
&\quad \times \|G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\quad + \|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\|.
\end{aligned} \tag{117}$$

Now, we write $x_n = \beta_{n-1}f(x_{n-1}) + (1-\beta_{n-1})v_{n-1}$, $\forall n \ge 1$, where $v_{n-1} = (x_n - \beta_{n-1}f(x_{n-1}))/(1-\beta_{n-1})$. It follows that, for all $n \ge 1$,

$$\begin{aligned}
&v_n - v_{n-1} \\
&\quad = \frac{x_{n+1} - \beta_n f(x_n)}{1-\beta_n} - \frac{x_n - \beta_{n-1}f(x_{n-1})}{1-\beta_{n-1}} \\
&\quad = \frac{\gamma_n y_n + \delta_n S_n G(y_n)}{1-\beta_n} \\
&\qquad - \frac{\gamma_{n-1}y_{n-1} + \delta_{n-1}S_{n-1}G(y_{n-1})}{1-\beta_{n-1}} \\
&\quad = \frac{\gamma_n(y_n - y_{n-1}) + \delta_n(S_nG(y_n) - S_{n-1}G(y_{n-1}))}{1-\beta_n} \\
&\qquad + \left(\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right) y_{n-1} \\
&\qquad + \left(\frac{\delta_n}{1-\beta_n} - \frac{\delta_{n-1}}{1-\beta_{n-1}}\right) S_{n-1}G(y_{n-1}).
\end{aligned} \tag{118}$$

This together with (117) implies that

$$\begin{aligned}
&\|v_n - v_{n-1}\| \\
&\quad \le \frac{\|\gamma_n(y_n - y_{n-1}) + \delta_n(S_nG(y_n) - S_{n-1}G(y_{n-1}))\|}{1-\beta_n} \\
&\qquad + \left|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right| \|y_{n-1}\| \\
&\qquad + \left|\frac{\delta_n}{1-\beta_n} - \frac{\delta_{n-1}}{1-\beta_{n-1}}\right| \|S_{n-1}G(y_{n-1})\| \\
&\quad \le (\gamma_n \|y_n - y_{n-1}\| \\
&\qquad + \delta_n(\|S_nG(y_n) - S_nG(y_{n-1})\| \\
&\qquad\quad + \|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\|)) \\
&\qquad \times (1-\beta_n)^{-1} \\
&\qquad + \left|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right| \|y_{n-1}\| \\
&\qquad + \left|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right| \|S_{n-1}G(y_{n-1})\| \\
&\quad \le \frac{\gamma_n \|y_n - y_{n-1}\| + \delta_n \|y_n - y_{n-1}\|}{\gamma_n + \delta_n} \\
&\qquad + \frac{\delta_n}{\gamma_n + \delta_n} \|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&\qquad + \left|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right| \\
&\qquad \times (\|y_{n-1}\| + \|S_{n-1}G(y_{n-1})\|) \\
&\quad \le \|y_n - y_{n-1}\| + \|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&\qquad + \left|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right| (\|y_{n-1}\| + \|S_{n-1}G(y_{n-1})\|) \\
&\quad \le \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\qquad + \|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\qquad + \|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&\qquad + \left|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right| (\|y_{n-1}\| + \|S_{n-1}G(y_{n-1})\|).
\end{aligned} \tag{119}$$

Furthermore, we note that

$$\begin{aligned}
x_{n+1} &= \beta_n f(x_n) + (1-\beta_n) v_n, \\
x_n &= \beta_{n-1} f(x_{n-1}) + (1-\beta_{n-1}) v_{n-1}, \quad \forall n \ge 1.
\end{aligned} \tag{120}$$

Also, simple calculations show that

$$\begin{aligned}
x_{n+1} - x_n &= \beta_n(f(x_n) - f(x_{n-1})) + (\beta_n - \beta_{n-1}) \\
&\quad \times (f(x_{n-1}) - v_{n-1}) + (1-\beta_n)(v_n - v_{n-1}).
\end{aligned} \tag{121}$$

This together with (119) implies that

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \le \beta_n \|f(x_n) - f(x_{n-1})\| + |\beta_n - \beta_{n-1}| \\
&\qquad \times \|f(x_{n-1}) - v_{n-1}\| + (1-\beta_n)\|v_n - v_{n-1}\|
\end{aligned}$$

$$
\begin{aligned}
&\le \beta_n \rho \|x_n - x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| \|f(x_{n-1}) - v_{n-1}\| + (1-\beta_n) \\
&\quad \times \Big[\|x_n - x_{n-1}\| \\
&\qquad + |\alpha_n - \alpha_{n-1}| \|G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\qquad + \|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\qquad + \|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&\qquad + \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| \\
&\qquad \times (\|y_{n-1}\| + \|S_{n-1}G(y_{n-1})\|) \Big] \\
&\le (1 - \beta_n(1-\rho)) \|x_n - x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| \|f(x_{n-1}) - v_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \|G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\quad + \|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\quad + \|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&\quad + \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| (\|y_{n-1}\| + \|S_{n-1}G(y_{n-1})\|) \\
&\le (1 - \beta_n(1-\rho)) \|x_n - x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| M + |\alpha_n - \alpha_{n-1}| M \\
&\quad + \|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\quad + \|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&\quad + \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| M \\
&= (1 - \beta_n(1-\rho)) \|x_n - x_{n-1}\| \\
&\quad + M \Big(|\alpha_n - \alpha_{n-1}| \\
&\qquad + |\beta_n - \beta_{n-1}| + \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| \Big) \\
&\quad + \|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\quad + \|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\|,
\end{aligned} \tag{122}
$$

where $\sup_{n\ge 0}\{\| f(x_n) \| + \| v_n \| + \| G(x_n) \| + \| S_nG(x_n) \| + \| y_n \| + \| S_nG(y_n) \|\} \le M$ for some $M > 0$. Utilizing Lemma 2, from conditions (ii)–(v) and the assumption on $\{S_n\}$, we deduce that

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{123}
$$

Taking into account the boundedness of $\{G(x_n)\}$ and $\{S_nG(x_n)\}$, by Lemma 6, we know that there exists a continuous strictly increasing function $g_1 : [0,\infty) \to [0,\infty)$, $g_1(0) = 0$ such that for $p \in F$

$$
\begin{aligned}
&\|y_n - p\|^2 \\
&\quad \le \alpha_n \|G(x_n) - p\|^2 + (1-\alpha_n) \|S_nG(x_n) - p\|^2 \\
&\qquad - \alpha_n(1-\alpha_n) g_1(\|G(x_n) - S_nG(x_n)\|) \\
&\quad \le \alpha_n \|x_n - p\|^2 + (1-\alpha_n) \|x_n - p\|^2 \\
&\qquad - \alpha_n(1-\alpha_n) g_1(\|G(x_n) - S_nG(x_n)\|) \\
&\quad = \|x_n - p\|^2 - \alpha_n(1-\alpha_n) g_1(\|G(x_n) - S_nG(x_n)\|).
\end{aligned} \tag{124}
$$

Since $\{y_n\}$ and $\{S_nG(y_n)\}$ are bounded, by Lemma 6, there exists a continuous strictly increasing function $g_2 : [0,\infty) \to [0,\infty)$, $g_2(0) = 0$ such that for $p \in F$

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \|\beta_n(f(x_n) - p) + \gamma_n(y_n - p) \\
&\qquad + \delta_n(S_nG(y_n) - p)\|^2 \\
&\quad = \Big\| (\gamma_n + \delta_n) \\
&\qquad \times \Big[\frac{\gamma_n}{\gamma_n + \delta_n}(y_n - p) \\
&\qquad\quad + \frac{\delta_n}{\gamma_n + \delta_n}(S_nG(y_n) - p) \Big] \\
&\qquad + \beta_n(f(x_n) - p) \Big\|^2 \\
&\quad \le (\gamma_n + \delta_n)^2 \Big\| \frac{\gamma_n}{\gamma_n + \delta_n}(y_n - p) \\
&\qquad\qquad + \frac{\delta_n}{\gamma_n + \delta_n}(S_nG(y_n) - p) \Big\|^2 \\
&\qquad + 2\beta_n \langle f(x_n) - p, J(x_{n+1} - p) \rangle \\
&\quad \le (\gamma_n + \delta_n)^2 \Big[\frac{\gamma_n}{\gamma_n + \delta_n} \|y_n - p\|^2 \\
&\qquad\qquad + \frac{\delta_n}{\gamma_n + \delta_n} \|S_nG(y_n) - p\|^2 \\
&\qquad\qquad - \frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_2(\|y_n - S_nG(y_n)\|) \Big] \\
&\qquad + 2\beta_n \|f(x_n) - p\| \|x_{n+1} - p\|
\end{aligned}
$$

$$\begin{aligned}
&\le \gamma_n \|y_n - p\|^2 + \delta_n \|y_n - p\|^2 \\
&\quad - \gamma_n \delta_n g_2 (\|y_n - S_n G(y_n)\|) \\
&\quad + 2\beta_n \|f(x_n) - p\| \|x_{n+1} - p\| \\
&\le \|y_n - p\|^2 - \gamma_n \delta_n g_2 (\|y_n - S_n G(y_n)\|) \\
&\quad + 2\beta_n \|f(x_n) - p\| \|x_{n+1} - p\|,
\end{aligned} \tag{125}$$

which together with (124) implies that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad \le \|x_n - p\|^2 - \alpha_n (1 - \alpha_n) g_1 (\|G(x_n) - S_n G(x_n)\|) \\
&\quad - \gamma_n \delta_n g_2 (\|y_n - S_n G(y_n)\|) \\
&\quad + 2\beta_n \|f(x_n) - p\| \|x_{n+1} - p\|.
\end{aligned} \tag{126}$$

It immediately follows that

$$\begin{aligned}
&\alpha_n (1 - \alpha_n) g_1 (\|G(x_n) - S_n G(x_n)\|) \\
&\quad + \gamma_n \delta_n g_2 (\|y_n - S_n G(y_n)\|) \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + 2\beta_n \|f(x_n) - p\| \|x_{n+1} - p\| \\
&\quad \le (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\| \\
&\quad + 2\beta_n \|f(x_n) - p\| \|x_{n+1} - p\|.
\end{aligned} \tag{127}$$

According to condition (vi), we get

$$\liminf_{n\to\infty} \delta_n = \liminf_{n\to\infty} (1 - \beta_n - \gamma_n) = 1 - \limsup_{n\to\infty} (\beta_n + \gamma_n) > 0. \tag{128}$$

Since $\beta_n \to 0$ and $\| x_{n+1} - x_n \| \to 0$, we conclude from conditions (i) and (vi) that

$$\begin{aligned}
&\lim_{n\to\infty} g_1 (G(x_n) - S_n G(x_n)) = 0, \\
&\lim_{n\to\infty} g_2 (\|y_n - S_n G(y_n)\|) = 0.
\end{aligned} \tag{129}$$

Utilizing the properties of g_1 and g_2, we have

$$\begin{aligned}
&\lim_{n\to\infty} \|G(x_n) - S_n G(x_n)\| = 0, \\
&\lim_{n\to\infty} \|y_n - S_n G(y_n)\| = 0.
\end{aligned} \tag{130}$$

Note that

$$\begin{aligned}
&\|y_n - x_n\| \\
&\quad = \|x_{n+1} - x_n - \beta_n (f(x_n) - y_n) \\
&\quad - \delta_n (S_n G(y_n) - y_n)\| \\
&\quad \le \|x_{n+1} - x_n\| + \beta_n \|f(x_n) - y_n\| \\
&\quad + \delta_n \|S_n G(y_n) - y_n\|.
\end{aligned} \tag{131}$$

Thus, from (123), (130), and $\beta_n \to 0$, it follows that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{132}$$

On the other hand, from (130), we get

$$\begin{aligned}
\lim_{n\to\infty} \|y_n - G(x_n)\| &= \lim_{n\to\infty} (1 - \alpha_n) \\
&\quad \times \|S_n G(x_n) - G(x_n)\| = 0.
\end{aligned} \tag{133}$$

This together with (132) implies that

$$\lim_{n\to\infty} \|x_n - G(x_n)\| = 0. \tag{134}$$

By (130) and Lemma 7, we have

$$\begin{aligned}
&\|SG(x_n) - G(x_n)\| \\
&\quad \le \|SG(x_n) - S_n G(x_n)\| \\
&\quad + \|S_n G(x_n) - G(x_n)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{135}$$

In terms of (134) and (135), we have

$$\begin{aligned}
&\|x_n - Sx_n\| \\
&\quad \le \|x_n - G(x_n)\| + \|G(x_n) - SG(x_n)\| \\
&\quad + \|SG(x_n) - Sx_n\| \\
&\quad \le 2\|x_n - G(x_n)\| \\
&\quad + \|G(x_n) - SG(x_n)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{136}$$

Define a mapping $Wx = (1-\theta)Sx + \theta G(x)$, where $\theta \in (0, 1)$ is a constant. Then by Lemma 9, we have that $\mathrm{Fix}(W) = \mathrm{Fix}(S) \cap \mathrm{Fix}(G) = F$. We observe that

$$\begin{aligned}
&\|x_n - Wx_n\| \\
&\quad = \|(1 - \theta)(x_n - Sx_n) + \theta (x_n - G(x_n))\| \\
&\quad \le (1 - \theta) \|x_n - Sx_n\| + \theta \|x_n - G(x_n)\|.
\end{aligned} \tag{137}$$

From (134) and (136), we obtain

$$\lim_{n\to\infty} \| x_n - Wx_n \| = 0. \tag{138}$$

Now, we claim that

$$\limsup_{n\to\infty} \langle f(q) - q, J(x_n - q) \rangle \le 0, \tag{139}$$

where $q = s - \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction

$$x \longmapsto tf(x) + (1 - t) Wx. \tag{140}$$

Then x_t solves the fixed point equation $x_t = tf(x_t) + (1 - t)Wx_t$. Thus we have

$$\|x_t - x_n\| = \|(1 - t)(Wx_t - x_n) + t(f(x_t) - x_n)\|. \tag{141}$$

By Lemma 3, we conclude that

$$\begin{aligned}
&\|x_t - x_n\|^2 \\
&\quad \le (1-t)^2 \|Wx_t - x_n\|^2 \\
&\qquad + 2t \langle f(x_t) - x_n, J(x_t - x_n)\rangle \\
&\quad \le (1-t)^2 (\|Wx_t - Wx_n\| + \|Wx_n - x_n\|)^2 \\
&\qquad + 2t \langle f(x_t) - x_n, J(x_t - x_n)\rangle \\
&\quad \le (1-t)^2 \left[\|x_t - x_n\|^2 + 2\|x_t - x_n\| \right. \\
&\qquad\qquad \left. \times \|Wx_n - x_n\| + \|Wx_n - x_n\|^2\right] \\
&\qquad + 2t \langle f(x_t) - x_t, J(x_t - x_n)\rangle \\
&\qquad + 2t \langle x_t - x_n, J(x_t - x_n)\rangle \\
&\quad = \left(1 - 2t + t^2\right) \|x_t - x_n\|^2 + f_n(t) \\
&\qquad + 2t \langle f(x_t) - x_t, J(x_t - x_n)\rangle + 2t\|x_t - x_n\|^2,
\end{aligned} \tag{142}$$

where

$$\begin{aligned}
f_n(t) &= (1-t)^2 (2\|x_t - x_n\| + \|x_n - Wx_n\|) \\
&\quad \times \|x_n - Wx_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{143}$$

It follows from (142) that

$$\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t} f_n(t). \tag{144}$$

Letting $n \to \infty$ in (144) and noticing (143), we derive

$$\limsup_{n\to\infty} \langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2} M_2, \tag{145}$$

where $M_2 > 0$ is a constant such that $\| x_t - x_n\|^2 \le M_2$ for all $t \in (0,1)$ and $n \ge 0$. Taking $t \to 0$ in (145), we have

$$\limsup_{t\to 0} \limsup_{n\to\infty} \langle x_t - f(x_t), J(x_t - x_n)\rangle \le 0. \tag{146}$$

On the other hand, we have

$$\begin{aligned}
&\langle f(q) - q, J(x_n - q)\rangle \\
&\quad = \langle f(q) - q, J(x_n - q) - J(x_n - x_t)\rangle \\
&\qquad + \langle x_t - q, J(x_n - x_t)\rangle \\
&\qquad + \langle f(q) - f(x_t), J(x_n - x_t)\rangle \\
&\qquad + \langle f(x_t) - x_t, J(x_n - x_t)\rangle.
\end{aligned} \tag{147}$$

Hence it follows that

$$\begin{aligned}
&\limsup_{n\to\infty} \langle f(q) - q, J(x_n - q)\rangle \\
&\quad \le \limsup_{n\to\infty} \langle f(q) - q, J(x_n - q) - J(x_n - x_t)\rangle \\
&\qquad + \|x_t - q\| \limsup_{n\to\infty} \|x_n - x_t\| \\
&\qquad + \rho \|q - x_t\| \limsup_{n\to\infty} \|x_n - x_t\| \\
&\qquad + \limsup_{n\to\infty} \langle f(x_t) - x_t, J(x_n - x_t)\rangle.
\end{aligned} \tag{148}$$

Taking into account that $x_t \to q$ as $t \to 0$, we have from (146)

$$\begin{aligned}
&\limsup_{n\to\infty} \langle f(q) - q, J(x_n - q)\rangle \\
&\quad = \limsup_{t\to 0} \limsup_{n\to\infty} \langle f(q) - q, J(x_n - q)\rangle \\
&\quad \le \limsup_{t\to 0} \limsup_{n\to\infty} \langle f(q) - q, J(x_n - q) - J(x_n - x_t)\rangle.
\end{aligned} \tag{149}$$

Since X has a uniformly Gâteaux differentiable norm, the duality mapping J is norm-to-weak* uniformly continuous on bounded subsets of X. Consequently, the two limits are interchangeable, and hence (139) holds. From (123), we get $(x_{n+1} - q) - (x_n - q) \to 0$. Noticing the norm-to-weak* uniform continuity of J on bounded subsets of X, we deduce from (139) that

$$\begin{aligned}
&\limsup_{n\to\infty} \langle f(q) - q, J(x_{n+1} - q)\rangle \\
&\quad = \limsup_{n\to\infty} (\langle f(q) - q, J(x_{n+1} - q) - J(x_n - q)\rangle \\
&\qquad\qquad + \langle f(q) - q, J(x_{n+1} - q)\rangle) \\
&\quad = \limsup_{n\to\infty} \langle f(q) - q, J(x_n - q)\rangle \le 0.
\end{aligned} \tag{150}$$

Finally, let us show that $x_n \to q$ as $n \to \infty$. We observe that

$$\begin{aligned}
\|y_n - q\| &= \|\alpha_n (G(x_n) - q) \\
&\qquad + (1-\alpha_n)(S_n G(x_n) - q)\| \\
&\le \alpha_n \|x_n - q\| + (1-\alpha_n)\|x_n - q\| \\
&= \|x_n - q\|,
\end{aligned} \tag{151}$$

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \beta_n \langle f(x_n)-f(q)+f(q)-q, J(x_{n+1}-q)\rangle \\
&\quad + \langle \gamma_n (y_n-q) \\
&\qquad + \delta_n (S_n G(y_n)-q), J(x_{n+1}-q)\rangle \\
&\le \beta_n \|f(x_n)-f(q)\| \|x_{n+1}-q\| \\
&\quad + \beta_n \langle f(q)-q, J(x_{n+1}-q)\rangle \\
&\quad + \|\gamma_n (y_n-q)+\delta_n (S_n G(y_n)-q)\| \|x_{n+1}-q\| \\
&\le \beta_n \rho \|x_n-q\| \|x_{n+1}-q\| \\
&\quad + \beta_n \langle f(q)-q, J(x_{n+1}-q)\rangle \\
&\quad + (\gamma_n \|y_n-q\| + \delta_n \|y_n-q\|) \|x_{n+1}-q\| \\
&= \beta_n \rho \|x_n-q\| \|x_{n+1}-q\| \\
&\quad + \beta_n \langle f(q)-q, J(x_{n+1}-q)\rangle \\
&\quad + (\gamma_n+\delta_n) \|y_n-q\| \|x_{n+1}-q\| \\
&\le \beta_n \rho \|x_n-q\| \|x_{n+1}-q\| \\
&\quad + \beta_n \langle f(q)-q, J(x_{n+1}-q)\rangle \\
&\quad + (1-\beta_n) \|x_n-q\| \|x_{n+1}-q\| \\
&= (1-\beta_n(1-\rho)) \|x_n-q\| \|x_{n+1}-q\| \\
&\quad + \beta_n \langle f(q)-q, J(x_{n+1}-q)\rangle \\
&\le \frac{1-\beta_n(1-\rho)}{2} \left(\|x_n-q\|^2 + \|x_{n+1}-q\|^2\right) \\
&\quad + \beta_n \langle f(q)-q, J(x_{n+1}-q)\rangle \\
&\le \frac{1-\beta_n(1-\rho)}{2} \|x_n-q\|^2 + \frac{1}{2}\|x_{n+1}-q\|^2 \\
&\quad + \beta_n \langle f(q)-q, J(x_{n+1}-q)\rangle .
\end{aligned} \tag{152}$$

So, we have

$$\begin{aligned}
&\|x_{n+1}-q\|^2 \\
&\quad \le (1-\beta_n(1-\rho)) \|x_n-q\|^2 \\
&\qquad + 2\beta_n \langle f(q)-q, J(x_{n+1}-q)\rangle \\
&\quad = (1-\beta_n(1-\rho)) \|x_n-q\|^2 \\
&\qquad + \beta_n (1-\rho) \frac{2\langle f(q)-q, J(x_{n+1}-q)\rangle}{1-\rho}.
\end{aligned} \tag{153}$$

Since $\sum_{n=0}^{\infty} \beta_n = \infty$ and $\limsup_{n\to\infty} \langle f(q)-q, J(x_{n+1}-q)\rangle \le 0$, by Lemma 2, we conclude from (153) that $x_n \to q$ as $n \to \infty$. This completes the proof. □

Corollary 23. *Let C be a nonempty, closed, and convex subset of a uniformly convex Banach space X which has a uniformly Gâteaux differentiable norm. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \ge 1$ for $i = 1, 2$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0, 1)$. Let S be a nonexpansive mapping of C into itself such that $F = \mathrm{Fix}(S) \cap \Omega \ne \emptyset$, where Ω is the fixed point set of the mapping $G = \Pi_C(I-\mu_1 B_1)\Pi_C(I-\mu_2 B_2)$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}
y_n &= \alpha_n G(x_n) + (1-\alpha_n) SG(x_n), \\
x_{n+1} &= \beta_n f(x_n) + \gamma_n y_n + \delta_n SG(y_n), \quad \forall n \ge 0,
\end{aligned} \tag{154}$$

where $1-(\lambda_i/(1+\lambda_i))(1-\sqrt{(1-\alpha_i)/\lambda_i}) \le \mu_i \le 1$ for $i = 1, 2$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $(0, 1)$ satisfying the following conditions:

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$,

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty} |\alpha_n - \alpha_{n-1}|/\beta_n = 0$,

(iv) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \beta_{n-1}/\beta_n = 1$,

(v) $\sum_{n=1}^{\infty} |(\gamma_n/(1-\beta_n)) - (\gamma_{n-1}/(1-\beta_{n-1}))| < \infty$ *or* $\lim_{n\to\infty} (1/\beta_n)|(\gamma_n/(1-\beta_n)) - (\gamma_{n-1}/(1-\beta_{n-1}))| = 0$,

(vi) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$.

Then $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP:

$$\langle q - f(q), J(q-p)\rangle \le 0, \quad \forall p \in F. \tag{155}$$

Further, we illustrate Theorem 22 by virtue of an example, that is, the following corollary.

Corollary 24. *Let C be a nonempty, closed, and convex subset of a uniformly convex Banach space X which has a uniformly Gâteaux differentiable norm. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0, 1)$. Let $T : C \to C$ be a self-mapping on C such that $I - T$ is ζ-strictly pseudocontractive and θ-strongly accretive with $\zeta + \theta \ge 1$, and let S be a nonexpansive mapping of C into itself such that $F = \mathrm{Fix}(S) \cap \mathrm{Fix}(T) \ne \emptyset$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}
y_n &= \alpha_n G(x_n) + (1-\alpha_n) S(I-\lambda(I-T)) x_n, \\
x_{n+1} &= \beta_n f(x_n) + \gamma_n y_n + \delta_n S(I-\lambda(I-T)) y_n, \quad \forall n \ge 0,
\end{aligned} \tag{156}$$

where $1-(\zeta/(1+\zeta))(1-\sqrt{(1-\theta)/\zeta}) \le \lambda \le 1$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $(0, 1)$ satisfying the following conditions:

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$,

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty} |\alpha_n - \alpha_{n-1}|/\beta_n = 0$,

(iv) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \beta_{n-1}/\beta_n = 1$,

(v) $\sum_{n=1}^{\infty} |(\gamma_n/(1-\beta_n)) - (\gamma_{n-1}/(1-\beta_{n-1}))| < \infty$ *or* $\lim_{n\to\infty}(1/\beta_n)|(\gamma_n/(1-\beta_n)) - (\gamma_{n-1}/(1-\beta_{n-1}))| = 0$,

(vi) $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1$.

Then $\{x_n\}$ *converges strongly to* $q \in F$, *which solves the following VIP:*

$$\langle q - f(q), J(q-p)\rangle \le 0, \quad \forall p \in F. \tag{157}$$

Proof. Utilizing the arguments similar to those in the proof of Corollary 16, we can obtain the desired result. □

Remark 25. As previous, we emphasize that our composite iterative algorithms (i.e., the iterative schemes (27) and (111)) are based on Korpelevich's extragradient method and viscosity approximation method. It is well known that the so-called viscosity approximation method must contain a contraction f on C. In the meantime, it is worth pointing out that our proof of Theorems 14 and 22 must make use of Lemma 8 for implicit viscosity approximation method; that is, Lemma 8 plays a key role in our proof of Theorems 14 and 22. Therefore, there is no doubt that the contraction f in Theorems 14 and 22 cannot be replaced by a general k-Lipschitzian mapping with constant $k \ge 0$.

Remark 26. Theorem 22 improves, extends, supplements, and develops [2, Theorem 3.1 and Corollary 3.2] and [5, Theorems 3.1] in the following aspects.

(i) The problem of finding a point $q \in \bigcap_n \mathrm{Fix}(S_n) \cap \Omega$ in Theorem 22 is more general and more subtle than the problem of finding a point $q \in \mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$ in Jung [5, Theorem 3.1].

(ii) The iterative scheme in [2, Theorem 3.1] is extended to develop the iterative scheme (111) of Theorem 22 by virtue of the iterative scheme of [5, Theorem 3.1]. The iterative scheme (111) in Theorem 22 is more advantageous and more flexible than the iterative scheme in [2, Theorem 3.1] because it involves several parameter sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$.

(iii) The iterative scheme (111) in Theorem 22 is very different from everyone in both [2, Theorem 3.1] and [5, Theorem 3.1] because the mappings S_n and G in the iterative scheme of [2, Theorem 3.1] and the mapping $SP_C(I - \lambda_n A)$ in the iterative scheme of [5, Theorem 3.1] are replaced by the same composite mapping S_nG in the iterative scheme (111) of Theorem 22.

(iv) The proof in [2, Theorem 3.1] depends on the argument techniques in [3], the inequality in 2-uniformly smooth Banach spaces, and the inequality in smooth and uniform convex Banach spaces. However, the proof of Theorem 22 does not depend on the argument techniques in [3], the inequality in 2-uniformly smooth Banach spaces, and the inequality in smooth and uniform convex Banach spaces. It depends on only the inequality in uniform convex Banach spaces.

(v) The assumption of the uniformly convex and 2-uniformly smooth Banach space X in [2, Theorem 3.1] is weakened to the one of the uniformly convex Banach space X having a uniformly Gateaux differentiable norm in Theorem 22.

(vi) The iterative scheme in [2, Corollary 3.2] is extended to develop the new iterative scheme in Corollary 15 because the mappings S and G are replaced by the same composite mapping SG in Corollary 23.

Finally, we observe that related results can be found in recent papers, for example, [15–24] and the references therein.

Acknowledgments

This research was partially supported by the National Science Foundation of China (11071169), Innovation Program of Shanghai Municipal Education Commission (09ZZ133), and Ph.D. Program Foundation of Ministry of Education of China (20123127110002). This research was partially supported by a Grant from NSC 101-2115-M-037-001.

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