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Research Article

Some Results for the Drazin Inverses of the Sum of Two Matrices and Some Block Matrices

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We give a formula of $(P+Q)^D$ under the conditions $P^2Q+QPQ=0$, $P^3Q=0$, and $PQPQ=0$. Then, we apply it to give some expressions for the Drazin inverse of block matrix $M=\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ (A and D are square matrices) under some conditions, generalizing some recent results in the literature. Finally, numerical examples are given to illustrate our results.

1. Introduction

Let $\mathbb{C}^{m\times n}$ denote the set of $m\times n$ complex matrices. The Drazin inverse of $A\in\mathbb{C}^{n\times n}$ is the unique matrix A^D, satisfying the following equation:

$$A^{k+1}A^D=A^k,\qquad A^DAA^D=X,\qquad AA^D=A^DA,\tag{1}$$

where $k=\operatorname{ind}(A)$ is the index of A, the smallest nonnegative integer for which $\operatorname{rank}(A^{k+1})=\operatorname{rank}(A^k)$ (see [1]). In particular, when $\operatorname{ind}(A)=1$, the Drazin inverse of A is called the group inverse of A. If A is nonsingular, it is clear that $\operatorname{ind}(A)=0$ and $A^D=A^{-1}$. Throughout this paper, we denote by $A^\pi=I-AA^D$ and define $A^0=I$, where I is the identity matrix with proper sizes.

The importance of the Drazin inverse and its applications to singular differential equations and difference equations, Morkov chains and iterative methods, cryptography, numerical analysis, to structured matrices, and perturbation bounds for the relative eigenvalue problems can be found in [2–5].

In 1958, Drazin [6] gave a result of $(P+Q)^D$, where P and Q are square matrices, and proved that

$$(P+Q)^D=P^D+Q^D\quad\text{when } PQ=QP=0.\tag{2}$$

In 2001, Hartwig et al. [7] derived a result of $(P+Q)^D$ when $PQ=0$. In 2005, Castro-González [8] provided the representation of $(P+Q)^D$ when $P^DQ=0$, $PQ^D=0$, and $Q^\pi PQP^\pi=0$. In 2008, Castro-González et al. [9] determined the result of $(P+Q)^D$ when $P^2Q=0$ and $PQ^2=0$. In 2009, Martínez-Serrano and Castro-González [10] derived a formula of $(P+Q)^D$ when $P^2Q=0$ and $Q^2=0$. In 2011, Yang and Liu [11] established some expressions of $(P+Q)^D$ when $PQ^2=0$ and $PQP=0$, and in 2012, Bu et al. [12] got the explicit representation of $(P+Q)^D$ when $P^2Q=0$, $Q^2P=0$, $P^3Q=0$, and $QPQ=0$, $QP^2Q=0$, respectively. Other related results have been studied in [4, 13–19].

On the other hand, a related topic is to discuss a representation of the Drazin inverse of block matrix $M=\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\in\mathbb{C}^{n\times n}$, where A and D are square matrices. Campbell and Meyer Jr. [2] first proposed an open problem to find an explicit formula of the Drazin inverse of block matrix $M=\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\in\mathbb{C}^{n\times n}$, (where A and D are square matrices), in terms of A, B, C, and D. To find the Drazin inverse of $(P+Q)$ and M in terms of P, Q, P^D, and Q^D and A, B, C, and D, respectively, without side conditions is quite complicated and it has not been solved till now. However, many papers have been studied some special cases of this open problem and gave the representations for the Drazin inverse of $(P+Q)$ and M under some conditions. Here, we list some cases of Drazin inverse of block matrix M:

(i) $BC=0$, $DC=0$ (or $BD=0$) and D is nilpotent (see [5]);

(ii) $BCA=0$, $BD=0$, and $DC=0$ (or BC is nilpotent) (see [9]);

(iii) $BCA=0$, $BCB=0$, $DCA=0$, and $DCB=0$ (see [11]);

(iv) $BC = 0$, $BD = 0$, and $DC = 0$ (see [20]);

(v) $BC = 0$ and $DC = 0$ (see [21]);

(vi) $BCA = 0$, $BCB = 0$, $ABD = 0$, and $CBD = 0$ (see [22]).

The generalized Schur complement of A in M, which is stated as $S = D - CA^DB$, is very important to find the Drazin inverse of M, when the generalized Schur complement is either zero or nonsingular. Martínez-Serrano and Castro-González [10] gave results under the conditions $A^2A^\pi B = 0$, $CAA^\pi B = 0$, and $BCA^\pi B = 0$ and generalized that Schur complement is equal to zero. Also, they derived the expressions of M^D under the assumptions $A^2A^\pi B = 0$, $CAA^\pi B = 0$, and $CA^\pi B = 0$ and generalized Schur complement is nonsingular. The Drazin inverse of M has been studied in [10, 23], when generalized Schur complement is equal to zero and also has been studied in [5, 10, 24, 25], when the generalized Schur complement is nonsingular. Some representations for the Drazin inverse of M when the generalized Schur complement is nonsingular, including generalizations of the above mentioned results, will be derived in Section 4 under some conditions.

This paper is organized as follows. In Section 2, some helpful lemmas will be given. In Section 3, we give the explicit formula of $(P+Q)^D$ under the conditions $P^3Q = 0$, $P^2Q + QPQ = 0$, and $PQPQ = 0$ and also give a numerical example to demonstrate our result. In Section 4, we use our result to find the Drazin inverse of block matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and also to find the expression for M^D when the generalized Schur complement is nonsingular, which can be regarded as the generalizations of some results given in [5, 20]. Finally, in Section 5, we give two numerical examples to illustrate our results of block matrices.

2. Some Lemmas

In order to prove the main results, first we need the following lemmas.

Lemma 1 (see [1]). *Let $A \in \mathbb{C}^{m\times n}$, and let $B \in \mathbb{C}^{n\times m}$. Then, $(AB)^D = A((BA)^2)^DB$.*

Lemma 2 (see [7]). *Let $P, Q \in \mathbb{C}^{n\times n}$. If $PQ = 0$, then*

$$(P+Q)^D = Q^\pi \sum_{i=0}^{t-1} Q^i \left(P^D\right)^{i+1} + \sum_{i=0}^{t-1} \left(Q^D\right)^{i+1} P^i P^\pi, \tag{3}$$

where $t = \max\{\operatorname{ind}(P), \operatorname{ind}(Q)\}$.

Lemma 3 (see [26]). *Let $M_1 = \left(\begin{smallmatrix} A & 0 \\ C & B \end{smallmatrix}\right)$, and let $M_2 = \left(\begin{smallmatrix} B & C \\ 0 & A \end{smallmatrix}\right)$, where A and B are square matrices with $\operatorname{ind}(A) = r$ and $\operatorname{ind}(B) = s$. Then,*

$$M_1^D = \begin{pmatrix} A^D & 0 \\ X & B^D \end{pmatrix}, \qquad M_2^D = \begin{pmatrix} B^D & X \\ 0 & A^D \end{pmatrix}, \tag{4}$$

where $X = \sum_{i=0}^{r-1} (B^D)^{i+2} CA^i A^\pi + \sum_{i-0}^{s-1} B^\pi B^i C (A^D)^{i+2} - B^D C A^D$.

Lemma 4 (see [24]). *Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in C^{n\times n}$ (A and D are square matrices). If $S = D - CA^DB$ is nonsingular, $A^\pi B = 0$, and $CA^\pi = 0$, then*

$$M^D = \begin{pmatrix} A^D + A^DBS^{-1}CA^D & -A^DBS^{-1} \\ -S^{-1}CA^D & S^{-1} \end{pmatrix}. \tag{5}$$

3. The Drazin Inverse of the Sum of Two Matrices

In this section, we first give the formula for the Drazin inverse of $P + Q$ under some conditions.

Theorem 5. *Let $P, Q \in \mathbb{C}^{n\times n}$. If $P^3Q = 0$, $P^2Q + QPQ = 0$, and $PQPQ = 0$, then*

$$\begin{aligned}(P+Q)^D = {} & P^D + Q^D + QXP + P\left(Q^D\right)^2 \\ & + P^2\left(Q^D\right)^3 + PQ\left(P^D\right)^3 \\ & + QP\left(Q^D\right)^3 + P^2QXP^D + P^2Q^DXP + PQ^2XP^D \\ & + PQQ^DXP + QPQXP^D + QPQ^DXP,\end{aligned} \tag{6}$$

where

$$\begin{aligned} X = {} & \sum_{i=0}^{t-1} \left(Q^D\right)^{2i+4} (P+Q) P^{2i} P^\pi \\ & + \sum_{i=0}^{t-1} Q^\pi Q^{2i} (P+Q) \left(P^D\right)^{2i+4} \\ & - \left(Q^D\right)^2 (P+Q) \left(P^D\right)^2, \\ t = {} & \max\left\{\operatorname{ind}\left(P^2\right), \operatorname{ind}\left(Q^2\right)\right\}. \end{aligned} \tag{7}$$

Proof. Using Lemma 1, $(AB)^D = A((BA)^2)^DB$, we have

$$\begin{aligned}(P+Q)^D &= \left(\begin{pmatrix} I & Q \end{pmatrix}\begin{pmatrix} P \\ I \end{pmatrix}\right)^D = \begin{pmatrix} I & Q \end{pmatrix}\left(\begin{pmatrix} P & PQ \\ I & Q \end{pmatrix}^2\right)^D \begin{pmatrix} P \\ I \end{pmatrix} \\ &= \begin{pmatrix} I & Q \end{pmatrix}\begin{pmatrix} P^2 + PQ & P^2Q + PQ^2 \\ P + Q & Q^2 + PQ \end{pmatrix}^D \begin{pmatrix} P \\ I \end{pmatrix}.\end{aligned} \tag{8}$$

Let

$$M = \begin{pmatrix} P^2 + PQ & P^2Q + PQ^2 \\ P + Q & Q^2 + PQ \end{pmatrix} = E + F, \tag{9}$$

where

$$E = \begin{pmatrix} P^2 & 0 \\ P + Q & Q^2 \end{pmatrix}, \qquad F = \begin{pmatrix} PQ & P^2Q + PQ^2 \\ 0 & PQ \end{pmatrix}. \tag{10}$$

From $P^3Q = 0$, $P^2Q + QPQ = 0$ and $PQPQ = 0$, we get $EF = 0$, and $F^2 = 0$. Then, applying Lemma 2, we obtain

$$M^D = E^D + F\left(E^D\right)^2. \tag{11}$$

By applying Lemma 3, we have

$$E^D = \begin{pmatrix} \left(P^D\right)^2 & 0 \\ X & \left(Q^D\right)^2 \end{pmatrix}, \tag{12}$$

where

$$\begin{aligned} X = &\sum_{i=0}^{t-1}\left(Q^D\right)^{2i+4}(P+Q)P^{2i}P^{\pi} \\ &+\sum_{i=0}^{t-1}Q^{\pi}Q^{2i}(P+Q)\left(P^D\right)^{2i+4} \\ &-\left(Q^D\right)^2(P+Q)\left(P^D\right)^2, \\ t = &\max\left\{\operatorname{ind}\left(P^2\right), \operatorname{ind}\left(Q^2\right)\right\}. \end{aligned} \tag{13}$$

Substituting (12) into (11), we get

$$M^D = \begin{pmatrix} \left(P^D\right)^2 + PQ\left(P^D\right)^4 + P^2QX\left(P^D\right)^2 + P^2Q^DX + PQ^2X\left(P^D\right)^2 + PQQ^DX & P^2\left(Q^D\right)^3 + P\left(Q^D\right)^2 \\ X + PQX\left(P^D\right)^2 + PQ^DX & \left(Q^D\right)^2 + P\left(Q^D\right)^3 \end{pmatrix}. \tag{14}$$

Substituting (14) into (8), we obtain the result. □

Similarly, we give a symmetrical form of Theorem 5.

Theorem 6. *Let $P, Q \in \mathbb{C}^{n\times n}$. If $PQ^3 = 0$, $PQ^2 + PQP = 0$, and $PQPQ = 0$, then*

$$\begin{aligned} (P+Q)^D = &P^D + Q^D + QXP + \left(P^D\right)^2Q \\ &+\left(P^D\right)^3Q^2 + \left(P^D\right)^3QP \\ &+\left(Q^D\right)^3PQ + Q^DXPQ^2 \\ &+Q^DXP^2Q + QXP^DQ^2 \\ &+QXP^DPQ + Q^DXPQP + QXP^DQP, \end{aligned} \tag{15}$$

where

$$\begin{aligned} X = &\sum_{i=0}^{t-1}\left(Q^D\right)^{2i+4}(P+Q)P^{2i}P^{\pi} \\ &+\sum_{i=0}^{t-1}Q^{\pi}Q^{2i}(P+Q)\left(P^D\right)^{2i+4} \\ &-\left(Q^D\right)^2(P+Q)\left(P^D\right)^2, \\ t = &\max\left\{\operatorname{ind}\left(P^2\right), \operatorname{ind}\left(Q^2\right)\right\}. \end{aligned} \tag{16}$$

Next, we present a numerical example to illustrate Theorem 5. This numerical example describes neither the matrices P and Q which do not satisfy the conditions of [10, Theorem 2.2] nor the conditions of [11, Theorem 2.1], but they satisfy the conditions of Theorem 5. Therefore, we can apply the formula given in Theorem 5 to obtain the Drazin inverse of $P + Q$.

Numerical Example. Consider the matrices $P, Q \in \mathbb{C}^{4\times 4}$, where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{17}$$

Since $P^2Q = 0$, $Q^2 \neq 0$ and $PQ^2 = 0$, $PQP \neq 0$, we know that the conditions of Theorem 2.2 in [10] and Theorem 2.1 in [11] do not hold, respectively. But it satisfies $P^3Q = 0$, $P^2Q + QPQ = 0$, and $PQPQ = 0$. Also, we have

$$\begin{gathered} \operatorname{ind}\left(P^2\right) = 1, \qquad \operatorname{ind}\left(Q^2\right) = 1, \\ P^D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Q^D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{gathered} \tag{18}$$

So, applying Theorem 5, we get

$$\begin{aligned} (P+Q)^D &= P^D + Q^D + PQ\left(P^D\right)^3 + QXP \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{19}$$

Remark 7. The above example shows that the conditions given in Theorem 5 are satisfied, but the conditions given in [10, 11] are not satisfied.

4. Drazin Inverse of Some Block Matrices

In this section, we apply our formula to give the representations for the Drazin inverse of block matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ (A and D are square matrices). First, we give the expression of M^D under the conditions $ABC = 0$, $BDC = 0$, $D^2C = 0$, and $CBC = 0$, which generalizes the results in [5, 20].

Theorem 8. *Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ (A and D are square matrices), such that* $\operatorname{ind}(A) = r$ *and* $\operatorname{ind}(D) = s$. *If $ABC = 0$, $BDC = 0$, $D^2C = 0$, and $CBC = 0$, then*

$$M^D = P^D + \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}\left(P^D\right)^2 + \begin{pmatrix} BC & 0 \\ DC & 0 \end{pmatrix}\left(P^D\right)^3, \tag{20}$$

where

$$\left(P^D\right)^i = \begin{pmatrix} \left(A^D\right)^i & \sum_{j=0}^{i-1}\left(A^D\right)^j X\left(D^D\right)^{i-j-1} \\ 0 & \left(D^D\right)^i \end{pmatrix}, \quad \text{for } i \geq 1,$$

$$X = \sum_{i=0}^{s-1}\left(A^D\right)^{i+2} BD^i D^\pi + \sum_{i=0}^{r-1} A^\pi A^i B\left(D^D\right)^{i+2} - A^D BD^D. \tag{21}$$

Proof. Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = P + Q, \tag{22}$$

where

$$P = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}. \tag{23}$$

From $ABC = 0$, $BDC = 0$, $D^2C = 0$, and $CBC = 0$, we have $P^3Q = 0$, $P^2Q + QPQ = 0$, and $PQPQ = 0$. We can see that Q is 2-nilpotent, so we get $Q^D = 0$ and $Q^\pi = I$. Applying Theorem 5, we have

$$M^D = P^D + Q\left(P^D\right)^2 + PQ\left(P^D\right)^3. \tag{24}$$

Using Lemma 3, we obtain the following result:

$$\left(P^D\right)^i = \begin{pmatrix} \left(A^D\right)^i & \sum_{j=0}^{i-1}\left(A^D\right)^j X\left(D^D\right)^{i-j-1} \\ 0 & \left(D^D\right)^i \end{pmatrix}, \quad \text{for } i \geq 1, \tag{25}$$

where $X = \sum_{i=0}^{s-1}(A^D)^{i+2}BD^iD^\pi + \sum_{i=0}^{r-1} A^\pi A^i B(D^D)^{i+2} - A^DBD^D$.

Substituting (25) into (24), we get the result. □

Similarly, we consider another splitting of the block matrix M and state another theorem.

Theorem 9. *Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ (A and D are square matrices), such that* $\operatorname{ind}(A) = r$ *and* $\operatorname{ind}(D) = s$. *If $DCB = 0$, $BCB = 0$, $A^2B = 0$, and $CAB = 0$, then*

$$M^D = P^D + \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}\left(P^D\right)^2 + \begin{pmatrix} 0 & AB \\ 0 & CB \end{pmatrix}\left(P^D\right)^3, \tag{26}$$

where

$$\left(P^D\right)^i = \begin{pmatrix} \left(A^D\right)^i & 0 \\ \sum_{j=0}^{i-1}\left(D^D\right)^j X\left(A^D\right)^{i-j-1} & \left(D^D\right)^i \end{pmatrix}, \quad \text{for } i \geq 1,$$

$$X = \sum_{i=0}^{r-1}\left(D^D\right)^{i+2} CA^i A^\pi + \sum_{i=0}^{s-1} D^\pi D^i C\left(A^D\right)^{i+2} - D^D CA^D. \tag{27}$$

Proof. Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = P + Q, \tag{28}$$

where

$$P = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}. \tag{29}$$

The remaining proof follows directly from Theorem 8. □

Now, we give the representation for M^D when the generalized Schur complement is nonsingular, which generalizes the result in [5].

Theorem 10. *Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ (A and D are square matrices). If $S = D - CA^DB$ is nonsingular, $BCA^\pi B = 0$, $DCA^\pi B = 0$, $A^2A^\pi B = 0$, and $CAA^\pi B = 0$, then*

$$M^D = P^D + \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix}\left(P^D\right)^2 + \begin{pmatrix} 0 & AA^\pi B \\ 0 & CA^\pi B \end{pmatrix}\left(P^D\right)^3, \tag{30}$$

where

$$\left(P^D\right)^j = \left(P_1^D\right)^j + \sum_{i=1}^{k}\left(P_1^D\right)^{i+j}\begin{pmatrix} A^iA^\pi & 0 \\ CA^{i-1}A^\pi & 0 \end{pmatrix},$$

$$P_1^D = \begin{pmatrix} A^D + A^DBS^{-1}CA^D & -A^DBS^{-1} \\ -S^{-1}CA^D & S^{-1} \end{pmatrix}, \quad k = \operatorname{ind}(A). \tag{31}$$

Proof. Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = P + Q, \tag{32}$$

where

$$P = \begin{pmatrix} A & AA^DB \\ C & D \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix}. \tag{33}$$

From $BCA^\pi B = 0$, $DCA^\pi B = 0$, $A^2A^\pi B = 0$, and $CAA^\pi B = 0$, it is obvious that $P^3Q = 0$, $P^2Q + QPQ = 0$, and $PQPQ = 0$. We can see that Q is 2-nilpotent, so we get $Q^D = 0$ and $Q^\pi = I$. Applying Theorem 5, we have

$$M^D = P^D + Q\left(P^D\right)^2 + PQ\left(P^D\right)^3. \tag{34}$$

Let $P = P_1 + P_2$, where

$$P_1 = \begin{pmatrix} A^2A^D & AA^DB \\ CAA^D & D \end{pmatrix}, \qquad P_2 = \begin{pmatrix} AA^\pi & 0 \\ CA^\pi & 0 \end{pmatrix}. \tag{35}$$

Obviously, $P_2P_1 = 0$ and $P_2^{k+1} = 0$, where $k = \operatorname{ind}(A)$. By Lemma 2, we have

$$\left(P^D\right)^j = \left(P_1^D\right)^j + \sum_{i=1}^{k}\left(P_1^D\right)^{i+j}P_2^i, \quad P_2^i = \begin{pmatrix} A^iA^\pi & 0 \\ CA^{i-1}A^\pi & 0 \end{pmatrix}, \quad i \geq 1. \tag{36}$$

For P_1, we get that $S = D - CAA^D(A^2A^D)^DAA^DB = D - CA^DB$ is nonsingular, $(A^2A^D)^\pi AA^DB = 0$, and $CAA^D(A^2A^D)^\pi = 0$. Using Lemma 4, we obtain the following result:

$$P_1^D = \begin{pmatrix} A^D + A^DBS^{-1}CA^D & -A^DBS^{-1} \\ -S^{-1}CA^D & S^{-1} \end{pmatrix}. \tag{37}$$

From the above equation, we obtain the result in Theorem 10. □

In the same way, we consider another splitting of the block matrix M and present next theorem.

Theorem 11. *Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ (A and D are square matrices). If $S = D - CA^DB$ is nonsingular, $ABCA^\pi = 0$, $BDCA^\pi = 0$, $D^2CA^\pi = 0$, and $A^\pi BCA^\pi = 0$, then*

$$M^D = P^D + \begin{pmatrix} 0 & 0 \\ CA^\pi & 0 \end{pmatrix}\left(P^D\right)^2 + \begin{pmatrix} BCA^\pi & 0 \\ DCA^\pi & 0 \end{pmatrix}\left(P^D\right)^3, \tag{38}$$

where

$$\left(P^D\right)^j = \left(P_1^D\right)^j + \sum_{i=1}^{k}\begin{pmatrix} A^iA^\pi & A^{i-1}A^\pi B \\ 0 & 0 \end{pmatrix}\left(P_1^D\right)^{i+j},$$
$$P_1^D = \begin{pmatrix} A^D + A^DBS^{-1}CA^D & -A^DBS^{-1} \\ -S^{-1}CA^D & S^{-1} \end{pmatrix}, \quad k = \operatorname{ind}(A). \tag{39}$$

Proof. Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = P + Q, \tag{40}$$

where

$$P = \begin{pmatrix} A & B \\ CAA^D & D \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 \\ CA^\pi & 0 \end{pmatrix}. \tag{41}$$

The remaining proof is similar to that of Theorem 10. □

5. Numerical Examples

In this section, two numerical examples are given to illustrate Theorems 8 and 10.

Example 1. Consider the block matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbb{C}^{4\times 4}$, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}. \tag{42}$$

Since $BC \neq 0$ and $BCA \neq 0$, the representation for M^D fail to apply in [9, 11, 20–22], respectively. But it satisfies $ABC = 0$, $BDC = 0$, $D^2C = 0$, and $CBC = 0$. Also, we have

$$A^D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A^\pi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$D^D = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad D^\pi = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}. \tag{43}$$

Then, applying Theorem 8, we obtain

$$M^D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}. \tag{44}$$

Remark 12. The above example shows that the conditions given in Theorem 8 are satisfied, but the conditions given in [9, 11, 20–22] are not satisfied.

Example 2. Consider the block matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbb{C}^{5\times 5}$, where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{45}$$

By computing, we get that $S = D - CA^D B$ is nonsingular and $BCA^\pi B = 0$, $DCA^\pi B = 0$, $A^2 A^\pi B = 0$, and $CAA^\pi B = 0$. Also, we have

$$\operatorname{ind}(A) = 1, \qquad A^D = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

$$A^\pi = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}, \qquad S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$P_1^D = \begin{pmatrix} 4 & 4 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{46}$$

$$P^D = \begin{pmatrix} 4 & 4 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Then, applying Theorem 10, we get

$$M^D = \begin{pmatrix} 4 & 4 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}. \tag{47}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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