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Research Article

Exact Finite-Difference Schemes for d-Dimensional Linear Stochastic Systems with Constant Coefficients

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The authors attempt to construct the exact finite-difference schemes for linear stochastic differential equations with constant coefficients. The explicit solutions to Itô and Stratonovich linear stochastic differential equations with constant coefficients are adopted with the view of providing exact finite-difference schemes to solve them. In particular, the authors utilize the exact finite-difference schemes of Stratonovich type linear stochastic differential equations to solve the Kubo oscillator that is widely used in physics. Further, the authors prove that the exact finite-difference schemes can preserve the symplectic structure and first integral of the Kubo oscillator. The authors also use numerical examples to prove the validity of the numerical methods proposed in this paper.

1. Introduction

Certainty barely exists as a natural phenomenon in isolation. On the contrary, the phenomenon of certainty is an integral part of a complex environment full of interrelations and interactions. Physics is the general analysis of nature, including elements of matter, motion, space, and time, all of which are relevant to the concept of certainty and uncertainty. In order to capture a clear understanding of physical movements in the natural world by using modeling techniques, it is inevitable to take the effects, as the result of the uncertainty upon any development course of any entities, into full consideration. The stochastic differential equations are one of the best known techniques to depict physical movement in precision. For instance, statistical physics is based on the ergodicity assumption of the development of a system and stochastic differential equations can give perfect solutions to such scenarios. Moreover, the stochastic differential equations and relevant applications have been applied in such a variety of scientific fields such as stochastic control, stochastic neural network, financial economic studies, species dynamics, and electronic engineering [1–5]. In this regard, it is of crucial importance to explore stochastic differential equations. Major research findings of the stochastic differential equations can be found in [4, 5].

Although Mao [4] and Kloeden and Platen [6] have managed to calculate the expression of the explicit solution to certain types of stochastic differential equations, it is not always possible to derive the explicit solution to equations of such kind. Strong nonlinearity and coupling characteristics of stochastic differential equations are the major reason and, in this sense, it is of significant importance to establish an effective numerical method [6–15] that is able to make rational judgment by using computer simulations. In saying so, a new branch of computational mathematics is emerging as the numerical solutions to stochastic differential equations. The fundamental theories in this area can be found in [6, 15].

In recent years, the exact finite-difference schemes of ordinary differential equations have captured more and more attention from the academia. Such exact finite-difference schemes are the same to the solutions to the original equations, whilst being able to retain the characteristics of the solutions. Further, such exact finite-difference schemes can be utilized to rationalize the construction of nonstandardized finite-difference schemes [16], for example, the exponentially fitted method [17–19]. As is well known, the linear ordinary differential equations with constant coefficients involve an exact finite-difference scheme (see [16, 20, 21]) and some scholars have discussed exact discretizations of other systems, for example, Vigo-Aguiar and Ferràndiz [22], Cieśliński [23, 24], Sakamoto et al. [25], Mickens and Washington [26], Roeger et al. [27–29], and so on. However, there are only a few published papers [30–32] discussing whether an exact finite-difference scheme exists for widely applied linear stochastic differential equations with constant coefficients. In order to bridge the gap, this paper aims to calculate the exact finite-difference schemes of general d-dimensional linear stochastic differential equations with constant coefficients and hence apply the results to the Kubo oscillator so as to prove the structure-preserving property of the exact finite-difference schemes, which is a stochastic sample from [24].

The following part of the paper is organized as follows. In Section 2, the authors will discuss the exact finite-difference schemes for Itô and Stratonovich type stochastic differential equations. In Section 3, the authors demonstrate how to apply the findings generalized from Section 2 to the Kubo oscillator. Finally, the authors will use numerical examples to illustrate the validity of the findings.

2. The Exact Finite-Difference Schemes for Linear Stochastic Differential Equations with Constant Coefficients

Let (Ω, F, P) be a complete probability space with a filtration $\{F_t\}$. The filtration $\{F_t\}$ is increasing and right continuous, and F_0 contains all P-null sets. Let $B(t) = (B_1(t), B_2(t), \ldots, B_m(t))^T$ be a standard m-dimensional Brownian motion defined on (Ω, F, P), whose increment $\Delta B_k = B_k(t+h) - B_k(t)$ $(k = 1, 2, \ldots, m)$ is a Gaussian random variable $N(0, h)$.

Then we discuss numerical methods for strong solutions to stochastic differential equations:

$$\begin{aligned} dx(t) &= f(x(t))\,dt \\ &\quad + \sum_{k=1}^{m} g_k(x(t))\,dB_k(t), \quad t \geq 0, \\ x(0) &= x_0 \in R^d, \end{aligned} \tag{1}$$

where the deterministic term $f(x)$ is the drift coefficient and the stochastic terms $g_k(x)$ $(k = 1, 2, \ldots, m)$ are diffusion coefficients. The solution of (1) can be written as

$$\begin{aligned} x(t) &= x_0 + \int_0^t f(x(s))\,ds \\ &\quad + \sum_{k=1}^{m} \int_{t_0}^t g_k(x(s))\,dB_k(s). \end{aligned} \tag{2}$$

Given an equidistant discretization of the interval $[0, t]$ with grid points t_n $(n = 1, 2, \ldots, N)$ and letting $\xi_n = \theta t_n + (1-\theta) t_{n-1}$ $(\theta \in [0, 1])$, the stochastic integrals in (2) can be calculated by the limit of the approximating sums in the mean square sense, as $N \to \infty$,

$$\sum_{n=1}^{N} g_k(x(\xi_n))(B_k(t_n) - B_k(t_{n-1})) \quad (k = 1, 2, \ldots, m). \tag{3}$$

The values of stochastic integrals depend on the choice of θ and the Itô integrals when $\theta = 0$. The corresponding Itô type stochastic differential equations are the equations using the usual notation (1) and the Stratonovich integrals when $\theta = 1/2$. The corresponding Stratonovich type stochastic differential equations are denoted by

$$\begin{aligned} dx(t) &= f_1(x(t))\,dt \\ &\quad + \sum_{k=1}^{m} g_k(x(t)) \circ dB_k(t), \quad t \geq 0, \\ x(0) &= x_0 \in R^d. \end{aligned} \tag{4}$$

The relationships between these two types of stochastic differential equations are given by

$$\begin{aligned} f_1(x(t)) &= f(x(t)) \\ &\quad - \frac{1}{2} \sum_{k=1}^{m} g_k'(x(t))\, g_k(x(t)). \end{aligned} \tag{5}$$

The exact finite-difference schemes of d-dimensional linear stochastic differential equations with constant coefficients of Itô type can be discussed as

$$\begin{aligned} dx(t) &= Fx(t)\,dt + \sum_{k=1}^{m} G_k x(t)\,dB_k(t), \quad t \geq 0, \\ x(0) &= x_0 \in R^d, \end{aligned} \tag{6}$$

where F and G_k $(k = 1, 2, \ldots, m)$ are $d \times d$ matrices and $B_k(t)$ $(k = 1, 2, \ldots, m)$ are independent one-dimensional Brownian motion. Suppose that the matrices $F, G_1, G_2, \ldots, G_m$ are commutative; that is,

$$FG_k = G_k F, \quad G_k G_j = G_j G_k, \quad k, j = 1, 2, \ldots, m. \tag{7}$$

Let h be the time step and $t_n = nh$. $\Delta B_n^k = B_k(t_{n+1}) - B_k(t_n)$ $(k = 1, 2, \ldots, m)$ denotes the increment of Brownian motion. Then the following theorem can be derived.

Theorem 1. *Equation* (6) *admits the exact finite-difference schemes given by*

$$x_{n+1} = \exp\left\{\left(F - \frac{1}{2}\sum_{k=1}^{m} G_k^2\right)h + \sum_{k=1}^{m} G_k \Delta B_n^k\right\} x_n, \tag{8}$$

where x_n is the exact value of solution $x(t)$ to (6) *at $t = t_n$.*

Proof. Using the theories in [4, Chapter 3], the general solution of (6) can be expressed in terms of $F, G_1, G_2, \ldots, G_m$ as follows:

$$x(t) = \exp\left\{\left(F - \frac{1}{2}\sum_{k=1}^{m} G_k^2\right) t + \sum_{k=1}^{m} G_k B_k(t)\right\} x_0. \quad (9)$$

Inserting t_n and t_{n+1} into (9), it can be deduced that

$$\begin{aligned}
x_n &= x(t_n) = \exp\left\{\left(F - \frac{1}{2}\sum_{k=1}^{m} G_k^2\right) t_n + \sum_{k=1}^{m} G_k B_k(t_n)\right\} x_0, \\
x_{n+1} &= x(t_{n+1}) \\
&= \exp\left\{\left(F - \frac{1}{2}\sum_{k=1}^{m} G_k^2\right) t_{n+1} + \sum_{k=1}^{m} G_k B_k(t_{n+1})\right\} x_0 \\
&= \exp\left\{\left(F - \frac{1}{2}\sum_{k=1}^{m} G_k^2\right)(h + t_n) \right. \\
&\quad \left. + \sum_{k=1}^{m} G_k \left(\Delta B_n^k + B_k(t_n)\right)\right\} x_0 \\
&= \exp\left\{\left(F - \frac{1}{2}\sum_{k=1}^{m} G_k^2\right) h + \sum_{k=1}^{m} G_k \Delta B_n^k\right\} x_n,
\end{aligned} \quad (10)$$

which ends the proof. □

Consider a d-dimensional linear stochastic differential equation with constant coefficient of Stratonovich type driven by m independent one-dimensional Brownian motion:

$$\begin{aligned}
dy(t) &= Ay(t)\,dt + \sum_{k=1}^{m} C_k y(t) \circ dB_k(t), \quad t \geq 0, \\
y(0) &= y_0 \in R^d,
\end{aligned} \quad (11)$$

where A and C_k $(k = 1, 2, \ldots, m)$ are $d \times d$ matrices. It is assumed that the matrices $A, C_1, C_2, \ldots, C_m$ are commutative; that is,

$$AC_k = C_k A, \quad C_k C_j = C_j C_k, \qquad k, j = 1, 2, \ldots, m. \quad (12)$$

The exact finite-difference schemes can be constructed by using the general solution of (11).

Theorem 2. *Equation* (11) *admits the exact finite-difference schemes given by*

$$y_{n+1} = \exp\left\{Ah + \sum_{k=1}^{m} C_k \Delta B_n^k\right\} y_n, \quad (13)$$

where y_n is the exact value of solution $y(t)$ to (11) *at $t = t_n$.*

Proof. Firstly, we prove that the general solution to (11) is given by

$$y(t) = \exp\left\{At + \sum_{k=1}^{m} C_k B_k(t)\right\} y_0. \quad (14)$$

Equation (14) follows (9) by a known relation (5) between Itô and Stratonovich type stochastic differential equations. Taking into account that $y_n = y(t_n)$ and, in particular, $y(0) = y_0$, it can be obtained that

$$\begin{aligned}
y_n &= y(t_n) = \exp\left\{At_n + \sum_{k=1}^{m} C_k B_k(t_n)\right\} y_0, \\
y_{n+1} &= y(t_{n+1}) \\
&= \exp\left\{At_{n+1} + \sum_{k=1}^{m} C_k B_k(t_{n+1})\right\} y_0 \\
&= \exp\left\{A(h + t_n) + \sum_{k=1}^{m} C_k \left(\Delta B_n^k + B_k(t_n)\right)\right\} y_0 \\
&= \exp\left\{Ah + \sum_{k=1}^{m} C_k \Delta B_n^k\right\} y_n.
\end{aligned} \quad (15)$$

This completes the proof of the theorem. □

Remark 3. Although the existence of the exact solution of (6) and (11) is known, it is impossible to compute the exact value of the solution at a fixed time, because of the randomness of Brownian motion. The author of [33] has used a Matlab program to simulate the exact solution of one-dimensional linear stochastic differential equations. The codes in the program produce a discretized Brownian path W_i $(i = 1, 2, \ldots, N)$ with the help of the computer. The value of Brownian motion $B(t)$ at $t = t_k$ is obtained by $B(t_k) = \sum_{l=1}^{k-1} W_l$. Then it is possible to derive the value of exact solution $x(t)$ at $t = t_k$. Such sum must be calculated in every step of the iteration, increasing the amount of computation needed. However, the scheme proposed in this paper can overcome such a problem.

3. An Application to the Kubo Oscillator

Consider the Kubo oscillator [34]:

$$\begin{aligned}
dp(t) &= -aq(t)\,dt - bq(t) \circ dB(t), \quad p(0) = p_0 \in R, \\
dq(t) &= ap(t)\,dt + bp(t) \circ dB(t), \quad q(0) = q_0 \in R,
\end{aligned} \quad (16)$$

where a and b are constants and $B(t)$ is a standard one-dimensional Brownian motion. The small circle "$\circ$" before $dB(t)$ denotes stochastic differential equations of Stratonovich type. Using (14), the exact solution to (16) is illustrated as follows.

Theorem 4. *The exact solution of* (16) *is given by*

$$\begin{aligned}
p(t) &= p_0 \cos(at + bB(t)) - q_0 \sin(at + bB(t)), \\
q(t) &= p_0 \sin(at + bB(t)) + q_0 \cos(at + bB(t)).
\end{aligned} \quad (17)$$

Further, the exact finite-difference scheme for (16) *can be constructed by applying* (17). *Assume h is the time increment* *and* $\Delta B_n = B(t_{n+1}) - B(t_n)$ *is independent* $N(0,h)$*-distributed Gaussian random variables.* P_n *and* Q_n $(n = 0,1,2,\ldots)$ *are the exact discrete values to* $p(t)$ *and* $q(t)$ *when* $t = t_n = nh$. *The exact finite-difference scheme of* (16) *is given by*

$$\begin{aligned}
P_{n+1} &= p(t_{n+1}) \\
&= p_0 \cos(at_{n+1} + bB(t_{n+1})) \\
&\quad - q_0 \sin(at_{n+1} + bB(t_{n+1})) \\
&= p_0 \cos(at_n + bB(t_n) + ah + b\Delta B_n) \\
&\quad - q_0 \sin(at_n + bB(t_n) + ah + b\Delta B_n) \\
&= p_0 \cos a_n \cos b_n - p_0 \sin a_n \sin b_n \\
&\quad - q_0 \sin a_n \cos b_n - q_0 \cos a_n \sin b_n \\
&= (p_0 \cos a_n - q_0 \sin a_n) \cos b_n \\
&\quad - (p_0 \sin a_n + q_0 \cos a_n) \sin b_n \\
&= P_n \cos b_n - Q_n \sin b_n, \\
Q_{n+1} &= q(t_{n+1}) \\
&= p_0 \sin(at_{n+1} + bB(t_{n+1})) \\
&\quad + q_0 \cos(at_{n+1} + bB(t_{n+1})) \\
&= p_0 \sin(at_n + bB(t_n) + ah + b\Delta B_n) \\
&\quad + q_0 \cos(at_n + bB(t_n) + ah + b\Delta B_n) \\
&= p_0 \sin a_n \cos b_n + p_0 \cos a_n \sin b_n \\
&\quad + q_0 \cos a_n \cos b_n - q_0 \sin a_n \sin b_n \\
&= (p_0 \sin a_n + q_0 \cos a_n) \cos b_n \\
&\quad + (p_0 \cos a_n - q_0 \sin a_n) \sin b_n \\
&= Q_n \cos b_n + P_n \sin b_n,
\end{aligned} \tag{18}$$

where $a_n = at_n + bB(t_n)$ *and* $b_n = ah + b\Delta B_n$.

Due to

$$\begin{aligned}
&\frac{d\gamma(p(t), q(t))}{dt} \\
&\quad = \frac{d\left(p(t)^2 + q(t)^2\right)}{dt} \\
&\quad = 2p(t)\dot{p}(t) + 2q(t)\dot{q}(t) \\
&\quad = 2p(t)\left(-aq(t) - bq(t) \circ \dot{B}(t)\right) \\
&\qquad + 2q(t)\left(ap(t) + bp(t) \circ \dot{B}(t)\right) \\
&\quad = 0,
\end{aligned} \tag{19}$$

then $\gamma(p,q) = p^2 + q^2$ *is conservative along the phase flow of* (16). *That is,* $\gamma(p,q)$ *is a first integral of* (16), *indicating that a phase trajectory of* (16) *is a circle with the center at* $(0,0)$ *and with the radius* $\sqrt{H(p_0,q_0)}$. *It can be proved that scheme* (18) *can preserve the first integral* $\gamma(p,q)$ *exactly.*

Theorem 5. *Scheme* (18) *for solving* (16) *has the property* $\gamma(P_{n+1}, Q_{n+1}) = \gamma(P_n, Q_n)$ *for any* $n = 0,1,2,\ldots$.

Proof. Substitution of (18) into $\gamma(p,q)$ yields

$$\begin{aligned}
\gamma(P_{n+1}, Q_{n+1}) &= P_{n+1}^2 + Q_{n+1}^2 \\
&= (P_n \cos b_n - Q_n \sin b_n)^2 \\
&\quad + (Q_n \cos b_n + P_n \sin b_n)^2 \\
&= P_n^2 \cos^2 b_n + Q_n^2 \sin^2 b_n \\
&\quad + Q_n^2 \cos^2 b_n + P_n^2 \sin^2 b_n \\
&= P_n^2 + Q_n^2.
\end{aligned} \tag{20}$$

This completes the proof. □

It is obvious that (16) is a stochastic Hamiltonian system with $H(p,q) = a(p^2 + q^2)/2$ and $H_1(p,q) = b(p^2 + q^2)/2$, and thus the phase flow of (16) preserves the symplectic structure $dp(t) \wedge dq(t) = dp_0 \wedge dq_0$ for all $t \geq 0$. A good analytical and numerical study of stochastic Hamiltonian systems can be found in [34, 35]. The following theorem elicits whether scheme (18) is symplectic.

Theorem 6. *Scheme* (18) *for solving* (16) *preserves the symplectic structure; that is,* $dP_{n+1} \wedge dQ_{n+1} = dP_n \wedge dQ_n$.

Proof. Differentiating (18), it is known that

$$\begin{aligned}
dP_{n+1} &= \cos b_n dP_n - \sin b_n dQ_n, \\
dQ_{n+1} &= \cos b_n dQ_n + \sin b_n dP_n.
\end{aligned} \tag{21}$$

From the exterior product, it can be derived as

$$\begin{aligned}
&dP_{n+1} \wedge dQ_{n+1} \\
&\quad = \cos^2 b_n dP_n \wedge dQ_n - \sin^2 b_n dQ_n \wedge dP_n \\
&\quad = dP_n \wedge dQ_n.
\end{aligned} \tag{22}$$

This completes the proof. □

4. Numerical Experiments

A linear stochastic differential equation of Itô type can be written as

$$\begin{aligned}
dx(t) &= 2x(t)\,dt + x(t)\,dB(t), \quad t \in [0,1], \\
x(0) &= 1.
\end{aligned} \tag{23}$$

The exact solution to (23) is given by

$$x(t) = \exp(1.5t + B(t)). \tag{24}$$

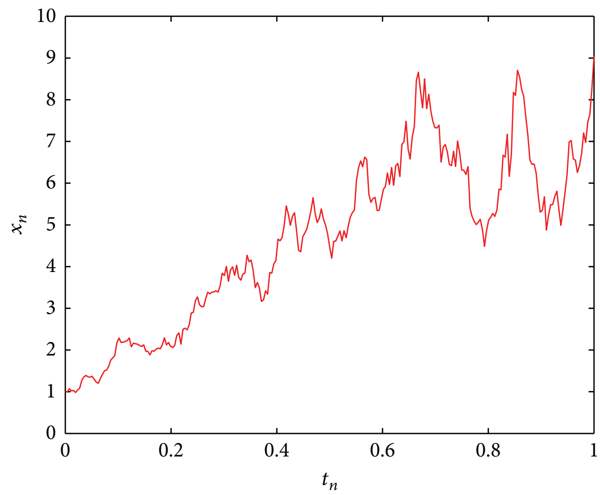

Figure 1: The exact solution of (23) simulated by (25) with fixed step size $h = 2^{-8}$.

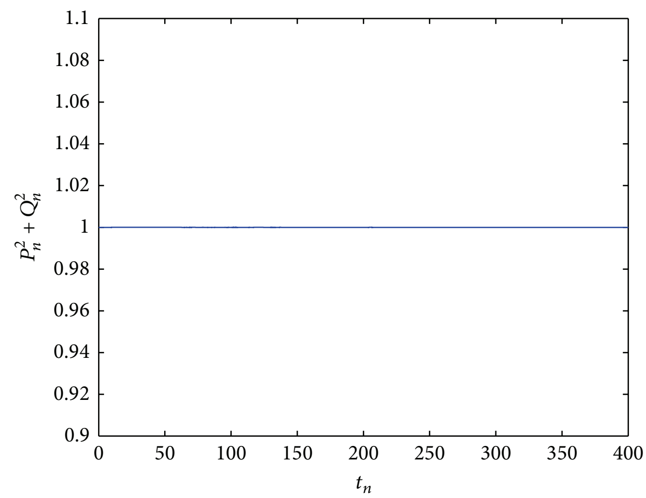

Figure 3: Preservation of first integral $\gamma(p, q)$ by numerical solutions produced by (18).

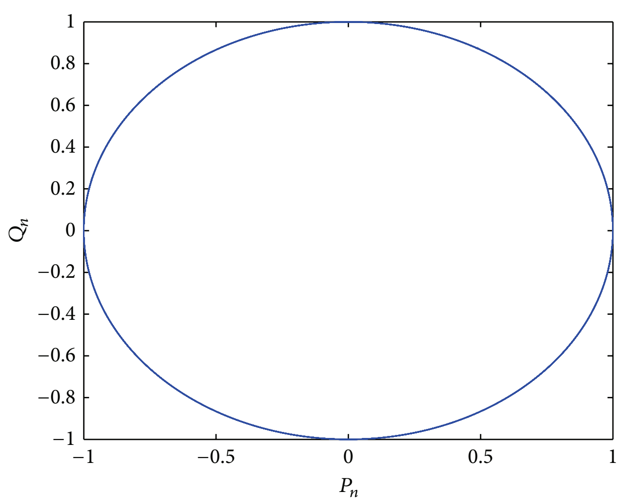

Figure 2: Applying (18) to simulate (16) with fixed step size $h = 0.02$.

By (8), the exact finite-difference scheme of (23) is

$$x_{n+1} = \exp\left(1.5h + \Delta B_n\right) x_n. \tag{25}$$

Figure 1 exhibits the exact solution of a sample phase trajectory of (23) simulated by (25).

Next, the exact finite-difference scheme (18) can be applied to solve the Kubo oscillator (16). The coefficients of (16) are chosen as $a = 0.2$, $b = 0.01$, $p_0 = 1$, $q_0 = 0$, $h = 0.02$, and $t \in [0, 400]$. Figure 2 exhibits the numerical solutions of a sample phase trajectory of (16) simulated by (18). Figure 3 shows that the numerical solutions created by (18) could preserve the first integral $\gamma(p, q)$ of (16).

5. Conclusions

In this paper, the authors extend the exact finite-difference schemes to linear stochastic differential equations with constant coefficients. The exact finite-difference schemes have been calculated for general d-dimensional Itô and Stratonovich type stochastic differential equations. By using the exact finite-difference schemes to solve the Kubo oscillator, the authors have proven that the findings illustrated in this paper can preserve the symplectic structure and first integral. Numerical examples demonstrate the validity of the exact finite-difference schemes in this paper.

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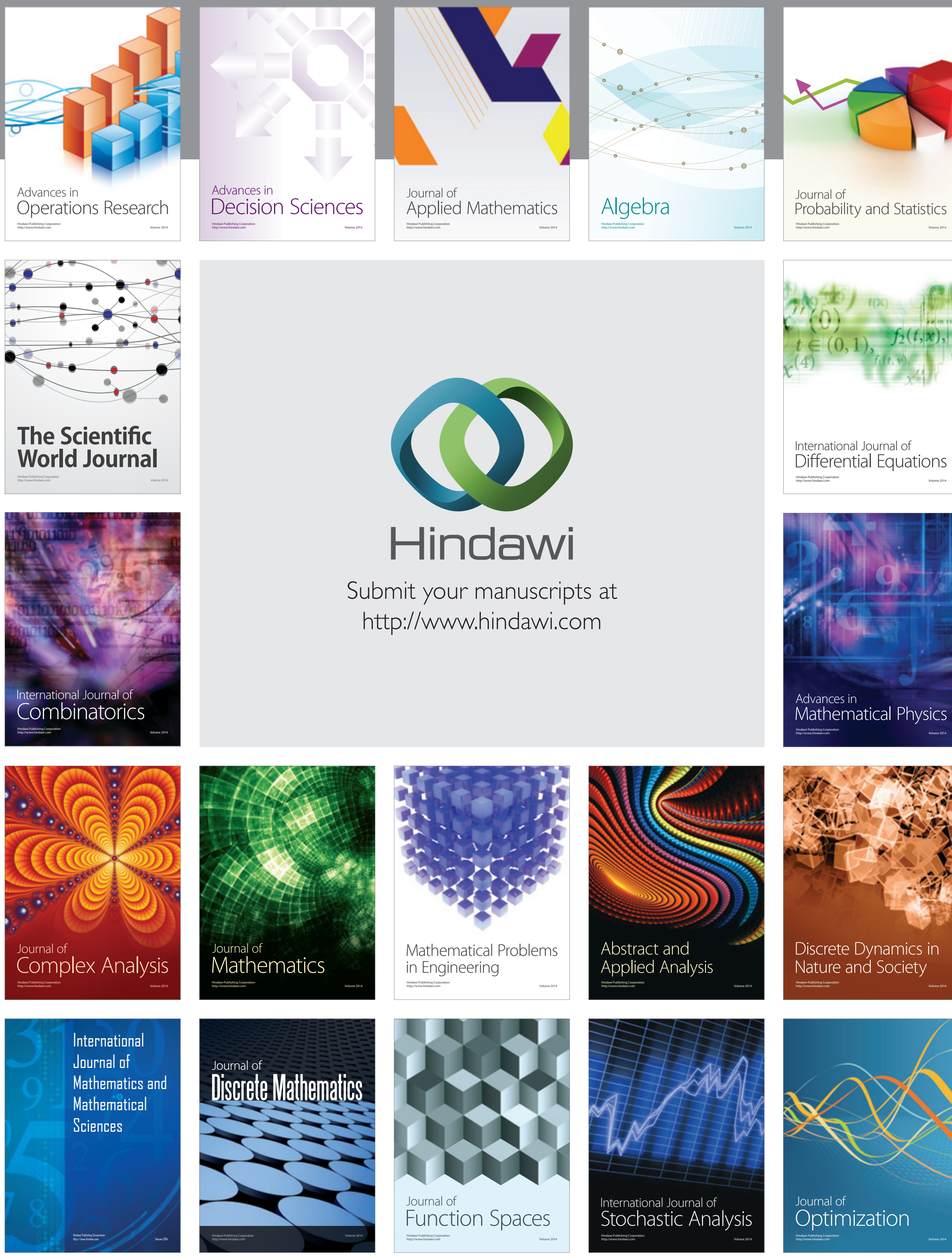
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