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Research Article

Eigenvector-Free Solutions to the Matrix Equation $AXB^H = E$ with Two Special Constraints

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The matrix equation $AXB^H = E$ with $SX = XR$ or $PX = sXQ$ constraint is considered, where S, R are Hermitian idempotent, P, Q are Hermitian involutory, and $s = \pm 1$. By the eigenvalue decompositions of S, R, the equation $AXB^H = E$ with $SX = XR$ constraint is equivalently transformed to an unconstrained problem whose coefficient matrices contain the corresponding eigenvectors, with which the constrained solutions are constructed. The involved eigenvectors are released by Moore-Penrose generalized inverses, and the eigenvector-free formulas of the general solutions are presented. By choosing suitable matrices S, R, we also present the eigenvector-free formulas of the general solutions to the matrix equation $AXB^H = E$ with $PX = sXQ$ constraint.

1. Introduction

In [1], Chen has denoted a square matrix X, the reflexive or antireflexive matrix with respect to P by

$$PX = XP \quad \text{or} \quad PX = -XP, \tag{1}$$

where the matrix $P \in \mathscr{C}^{n\times n}$ is Hermitian involutory. He also pointed out that these matrices possessed special properties and had wide applications in engineering and scientific computations [1, 2]. So, solving the matrix equation or matrix equations with these constraints is maybe interesting [3–14]. In this paper, we consider the matrix equation

$$AXB^H = E \tag{2}$$

with constraint

$$PX = sXQ \quad \text{or} \quad SX = XR, \tag{3}$$

where the matrices $A \in \mathscr{C}^{m\times n}$, $B \in \mathscr{C}^{p\times n}$, $E \in \mathscr{C}^{m\times p}$, the Hermitian involutory matrices $P, Q \in \mathscr{C}^{n\times n}$, the Hermitian idempotent matrices $S, R \in \mathscr{C}^{n\times n}$, and the scalars $s = \pm 1$.

Equation (2) with different constraints such as symmetry, skew-symmetry, and $PX = \pm XP$, was discussed in [9–11, 15–21], where existence conditions and the general solutions to the constrained equation were presented. By generalized singular value decomposition (GSVD) [22, 23], the authors of [15–17] simplified the matrix equation by diagonalizing the coefficient matrices and block-partitioned the new variable matrices into several block matrices, then imposed the constrained condition on subblocks, and determined the unknown subblocks separately for (2) with symmetric constraint. A similar strategy was also used in [18]; the authors achieved symmetric, skew-symmetric, and positive semidefinite solutions to (2) by quotient singular value decomposition (QSVD) [24, 25]. Moreover, in [20], CCD [26] was used for establishing a formula of the general solutions to (2) with diagonal constraint.

In [19], we have presented an eigenvector-free solution to the matrix equation (2) with constraint $PX = \pm XP$, where we represented its general solution and existence condition by g-inverses of the matrices A, B, and P. Note that the g-inverses are always not unique, and they can be generalized to the Moore-Penrose generalized inverses. Moreover, the constraint which guarantees the eigenvector-free expressions can be maybe improved further. So, in this paper, we focus on (2) with generalized constraint $PX = sXQ$ or another constraint $SX = XR$; our ideas are based on the following observations.

(1) If we set

$$S = \frac{1}{2}(I + P), \qquad R = \frac{1}{2}(I + sQ), \tag{4}$$

then S and R are both Hermitian idempotent. The above fact implies $PX = sXQ$ is the special case of $SX = XR$. So, we only discuss (2) with $SX = XR$ constraint and construct the $PX = sXQ$ constrained solution by selecting suitable matrices R, Q as (4).

(2) With the eigenvalue decompositions (EVDs) of the Hermitian matrices R, S, matrix X with $SX = XR$ constraint can be rewritten in (lower dimensional) two free variables $\widehat{X}$ and $\widehat{Y}$. And the corresponding constrained problem can be equivalently transformed to an unconstrained equation

$$\widehat{A}_1\widehat{X}\widehat{B}_1^H + \widehat{A}_2\widehat{Y}\widehat{B}_2^H = E, \tag{5}$$

with given coefficient matrices $\widehat{A}_i, \widehat{B}_i, i = 1, 2$ (one can see the details of this discussion in Section 2).

(3) The general solutions and existence conditions of (5) can be represented by the Moore-Penrose generalized inverses of $\widehat{A}_i$, $\widehat{B}_i$, $i = 1, 2$ [15, 20, 27–29]. However, the formulas above are maybe not simpler because the coefficient matrices contain the eigenvectors of S, R. In fact, the Hermitian idempotence of the matrices S, R implies they only have two clusters different eigenvalues, and their corresponding eigenvectors appear in the expression of general solutions, and existence conditions can be easily represented by S, R themselves. So we present a simple and eigenvector-free formulation for the constrained general solution.

The rest of this paper is organized as follows. In Section 2, we give the general solutions and the existence condition to (2) with $SX = XR$ constraint by the EVDs of S, R. In Section 3, we present the corresponding eigenvector-free representations. Equation (2) with $PX = sXQ$ constraint is regarded as the special case of (2) with $SX = XR$ constraint, and its eigenvector-free representation is given in Section 4. Numerical examples are given in Section 5 to display the effectiveness of our theorems.

We will use the following notations in the rest of this paper. Let $\mathscr{C}^{m\times n}$ denote the space of complex $m \times n$ matrix. For a matrix A, A^H and $A^\dagger$ denote its transpose and Moore-Penrose generalized inverse, respectively. Matrix I_n is identity matrix with order n; $O_{m\times n}$ refers to $m \times n$ zero matrix, and O_n is the zero matrix with order n. For any matrix $A \in \mathscr{C}^{m\times n}$, we also denote

$$\mathscr{P}_A = AA^\dagger, \qquad K_A = I_m - \mathscr{P}_A. \tag{6}$$

So,

$$\mathscr{P}_{A^H} = A^\dagger A, \qquad K_{A^H} = I_n - \mathscr{P}_{A^H}. \tag{7}$$

2. Solution to (2) with $SX=RX$ Constraint by the EVDs

For the Hermitian idempotent matrices S, R, let

$$S = U\operatorname{diag}(I_k, O_{n-k})U^H, \qquad R = V\operatorname{diag}(I_l, O_{n-l})V^H \tag{8}$$

be their two eigenvalue decompositions with unitary matrices U, V, respectively. Then $SX = XR$ holds if and only if

$$\operatorname{diag}(I_k, O_{n-k})\widetilde{X} = \widetilde{X}\operatorname{diag}(I_l, O_{n-l}), \tag{9}$$

where $\widetilde{X} = U^H XV$. And the constrained solution X can be expressed in

$$X = U\operatorname{diag}\left(\widehat{X}, \widehat{Y}\right)V^H, \qquad \widehat{X} \in \mathscr{C}^{k\times l}, \qquad \widehat{Y} \in \mathscr{C}^{(n-k)\times(n-l)}. \tag{10}$$

Partitioning $U = [U_1, U_2]$, $V = [V_1, V_2]$ and using the transformations (10), (2) with $SX = XR$ constraint is equivalent to the following unconstrained problem:

$$\widehat{A}_1\widehat{X}\widehat{B}_1^H + \widehat{A}_2\widehat{Y}\widehat{B}_2^H = E, \tag{11}$$

where

$$\widehat{A}_1 = AU_1, \qquad \widehat{B}_1 = BV_1, \qquad \widehat{A}_2 = AU_2, \qquad \widehat{B}_2 = BV_2. \tag{12}$$

For the unconstrained problem (11), we introduce the results about its existence conditions and expression of solutions.

Lemma 1. *Given $A \in \mathscr{C}^{m\times n}$, $B \in \mathscr{C}^{p\times q}$, $C \in \mathscr{C}^{m\times r}$, $D \in \mathscr{C}^{s\times q}$, and $E \in \mathscr{C}^{m\times q}$, the linear matrix equation $AXB + CYD = E$ is consistent if and only if*

$$\mathscr{P}_G K_A E \mathscr{P}_{D^H} = K_A E, \qquad \mathscr{P}_C E K_{B^H} \mathscr{P}_{J^H} = E K_{B^H}, \tag{13}$$

or, equivalently, if and only if

$$\begin{aligned} K_G K_A E &= 0, \qquad K_A E K_{D^H} = 0, \\ K_C E K_{B^H} &= 0, \qquad E K_{B^H} K_{J^H} = 0, \end{aligned} \tag{14}$$

where $G = K_A C$ and $J = DK_{B^H}$. And a representation of the general solution is

$$\begin{aligned} Y &= G^\dagger K_A E D^\dagger + T - \mathscr{P}_{G^H} T \mathscr{P}_D, \\ X &= A^\dagger (E - CYD) B^\dagger + Z - \mathscr{P}_{A^H} Z \mathscr{P}_B, \end{aligned} \tag{15}$$

with

$$T = \left(CK_{G^H}\right)^\dagger \left(I_m - CG^\dagger K_A\right) E K_{B^H} J^\dagger + W - \mathscr{P}_{(CK_{G^H})^H} W \mathscr{P}_J, \tag{16}$$

where the matrices $W \in \mathscr{C}^{r\times s}$ and $Z \in \mathscr{C}^{n\times p}$ are arbitrary.

The lemma is easy to verify; we can turn to [27] for details. The difference between them is that we replace the g-inverse in the theorem of [27] by the corresponding Moore-Penrose generalized inverse, and the expression of solutions is complicated relatively. However, compared with the multiformity of the g-inverses, the Moore-Penrose generalized inverse involved representation is unique and fixed.

Apply Lemma 1 on the unconstrained problem (11), we have the following theorem.

Theorem 2. *The matrix equation $AXB^H = E$ with constraint $SX = XR$ is consistent if and only if*

$$\mathscr{P}_{\widehat{G}} K_{\widehat{A}_1} E \mathscr{P}_{\widehat{B}_2} = K_{\widehat{A}_1} E, \qquad \mathscr{P}_{\widehat{A}_2} E K_{\widehat{B}_1} \mathscr{P}_{\widehat{J}^H} = E K_{\widehat{B}_1}, \quad (17)$$

where

$$\widehat{G} = K_{\widehat{A}_1} \widehat{A}_2, \qquad \widehat{J} = \widehat{B}_2^H K_{\widehat{B}_1}. \quad (18)$$

In the meantime, a general solution is given by

$$\begin{aligned}
\widehat{Y} &= \widehat{G}^\dagger K_{\widehat{A}_1} E \widehat{B}_2^{H^\dagger} + \left(\widehat{A}_2 K_{\widehat{G}^H}\right)^\dagger \left(I_m - \widehat{A}_2 \widehat{G}^\dagger K_{\widehat{A}_1}\right) E K_{\widehat{B}_1} \widehat{J}^\dagger \\
&\quad - \mathscr{P}_{\widehat{G}^H} \left(\widehat{A}_2 K_{\widehat{G}^H}\right)^\dagger \left(I_m - \widehat{A}_2 \widehat{G}^\dagger K_{\widehat{A}_1}\right) E K_{\widehat{B}_1} \widehat{J}^\dagger \mathscr{P}_{\widehat{B}_2^H} \\
&\quad + W - \mathscr{P}_{\widehat{G}^H} W \mathscr{P}_{\widehat{B}_2^H} - \mathscr{P}_{(\widehat{A}_2 K_{\widehat{G}^H})^H} W \mathscr{P}_{\widehat{J}} \\
&\quad + \mathscr{P}_{\widehat{G}^H} \mathscr{P}_{(\widehat{A}_2 K_{\widehat{G}^H})^H} W \mathscr{P}_{\widehat{J}} \mathscr{P}_{\widehat{B}_2^H}, \\
\widehat{X} &= \widehat{A}_1^\dagger \left(E - \widehat{A}_2 \widehat{Y} \widehat{B}_2^H\right) \widehat{B}_1^{H^\dagger} + Z - \mathscr{P}_{\widehat{A}_1^H} Z \mathscr{P}_{\widehat{B}_1^H},
\end{aligned} \quad (19)$$

where the matrices W and Z are arbitrary.

In order to separate $\widehat{Y}$ from $\widehat{X}$ of the second equality in (19), we substitute $\widehat{Y}$ into $\widehat{X}$. Let

$$\begin{aligned}
Y_* &= \widehat{G}^\dagger K_{\widehat{A}_1} E \widehat{B}_2^{H^\dagger} + \left(\widehat{A}_2 K_{\widehat{G}^H}\right)^\dagger \left(I_m - \widehat{A}_2 \widehat{G}^\dagger K_{\widehat{A}_1}\right) E K_{\widehat{B}_1} \widehat{J}^\dagger \\
&\quad - \mathscr{P}_{\widehat{G}^H} \left(\widehat{A}_2 K_{\widehat{G}^H}\right)^\dagger \left(I_m - \widehat{A}_2 \widehat{G}^\dagger K_{\widehat{A}_1}\right) E K_{\widehat{B}_1} \widehat{J}^\dagger \mathscr{P}_{\widehat{B}_2^H}, \\
X_* &= \widehat{A}_1^\dagger E \widehat{B}_1^{H^\dagger} - \widehat{A}_1^\dagger \widehat{A}_2 Y_* \widehat{B}_2^H \widehat{B}_1^{H^\dagger},
\end{aligned} \quad (20)$$

together with

$$\begin{aligned}
\widehat{B}_2^\dagger \widehat{B}_2 \widehat{B}_2^H &= \left(\widehat{B}_2^\dagger \widehat{B}_2\right)^H \widehat{B}_2^H = \widehat{B}_2^H, \\
\widehat{A}_2 K_{\widehat{G}^H} \left(\widehat{A}_2 K_{\widehat{G}^H}\right)^\dagger \widehat{A}_2 K_{\widehat{G}^H} &= \widehat{A}_2 K_{\widehat{G}^H}.
\end{aligned} \quad (21)$$

Then (19) can be rewritten as

$$\begin{aligned}
\widehat{Y} &= Y_* + W - \mathscr{P}_{\widehat{G}^H} W \mathscr{P}_{\widehat{B}_2^H} - \mathscr{P}_{(\widehat{A}_2 K_{\widehat{G}^H})^H} W \mathscr{P}_{\widehat{J}} \\
&\quad + \mathscr{P}_{\widehat{G}^H} \mathscr{P}_{(\widehat{A}_2 K_{\widehat{G}^H})^H} W \mathscr{P}_{\widehat{J}} \mathscr{P}_{\widehat{B}_2^H}, \\
\widehat{X} &= X_* + Z - \mathscr{P}_{\widehat{A}_1^H} Z \mathscr{P}_{\widehat{B}_1^H} - \widehat{A}_1^\dagger \widehat{A}_2 K_{\widehat{G}^H} W K_{\widehat{J}} \widehat{B}_2^H \widehat{B}_1^{H^\dagger}.
\end{aligned} \quad (22)$$

3. Eigenvector-Free Formulas of the General Solutions to (2) with $SX=XR$ Constraint

The existence conditions and the expression of the general solution given in Theorem 2 contain the eigenvector matrices of S, R, respectively. This implies that the eigenvalue decompositions will be included. In this section, we intend to release the involved eigenvectors in detailed expressions. With the first equality in (8), we have

$$\begin{aligned}
U_1 U_1^H &= S, \qquad U_2 U_2^H = I_n - S, \\
V_1 V_1^H &= R, \qquad V_2 V_2^H = I_n - R.
\end{aligned} \quad (23)$$

Note that $U_i(AU_i)^\dagger$ is the Moore-Penrose generalized inverse of $AU_iU_i^H$, which gives

$$\mathscr{P}_{\widehat{A}_i} = \widehat{A}_i \widehat{A}_i^\dagger = \left(AU_iU_i^H\right)\left(AU_iU_i^H\right)^\dagger = A_i A_i^\dagger = \mathscr{P}_{A_i}, \quad (24)$$

where

$$A_1 = AU_1U_1^H = AS, \qquad A_2 = AU_2U_2^H = A\left(I_n - S\right). \quad (25)$$

Then

$$K_{\widehat{A}_i} = I_m - \mathscr{P}_{\widehat{A}_i} = I_m - \mathscr{P}_{A_i} = K_{A_i}, \qquad \widehat{G}U_2^H = K_{A_1} A_2. \quad (26)$$

Set

$$B_1 = BV_1V_1^H = BR, \qquad B_2 = BV_2V_2^H = B\left(I_n - R\right), \quad (27)$$

and denote

$$G = K_{A_1} A_2, \qquad J = B_2^H K_{B_1}. \quad (28)$$

It is not difficult to verify that

$$V_2 \widehat{J} = J, \qquad \widehat{G} U_2^H = G, \quad (29)$$

together with

$$\begin{aligned}
\mathscr{P}_{\widehat{G}} &= \widehat{G}U_2^H \left(\widehat{G}U_2^H\right)^\dagger = \mathscr{P}_G, \\
\mathscr{P}_{\widehat{J}^H} &= \left(V_2 \widehat{J}\right)^\dagger \left(V_2 \widehat{J}\right) = \mathscr{P}_{J^H}.
\end{aligned} \quad (30)$$

Then the first equality of (17) can be rewritten as

$$\mathscr{P}_G K_{A_1} E \mathscr{P}_{B_2} = K_{A_1} E, \quad (31)$$

and the other can be rewritten as

$$\mathscr{P}_{A_2} E K_{B_1} \mathscr{P}_{J^H} = E K_{B_1}. \quad (32)$$

Now, we consider the simplification of the general solution X given by (10), which can be rewritten as

$$X = U_1 \widehat{X} V_1^H + U_2 \widehat{Y} V_2^H. \quad (33)$$

Note that

$$\begin{aligned}
U_2 \widehat{G}^\dagger &= \left(\widehat{G}U_2^H\right)^\dagger = G^\dagger, \qquad K_{\widehat{G}^H} U_2^H = U_2^H K_{G^H}, \\
U_2 \widehat{A}_2^\dagger &= A_2^\dagger.
\end{aligned} \quad (34)$$

Together with (26),

$$\begin{aligned} U_2 Y_* V_2^H = U_2 \Big(& \widehat{G}^\dagger K_{\widehat{A}_1} E \widehat{B}_2^{H^\dagger} + \left(\widehat{A}_2 K_{\widehat{G}^H}\right)^\dagger \\ & \times \left(I_m - \widehat{A}_2 \widehat{G}^\dagger K_{\widehat{A}_1}\right) E K_{\widehat{B}_1} \widehat{J}^\dagger \\ & - \mathscr{P}_{\widehat{G}^H} \left(\widehat{A}_2 K_{\widehat{G}^H}\right)^\dagger \\ & \times \left(I_m - \widehat{A}_2 \widehat{G}^\dagger K_{\widehat{A}_1}\right) E K_{\widehat{B}_1} \widehat{J}^\dagger \mathscr{P}_{\widehat{B}_2^H} \Big) V_2^H \\ = {} & G^\dagger K_{A_1} E B_2^{H^\dagger} + \left(A_2 K_{G^H}\right)^\dagger \\ & \times \left(I_m - A_2 G^\dagger K_{A_1}\right) E K_{B_1} J^\dagger \\ & - \mathscr{P}_{G^H} \left(A_2 K_{G^H}\right)^\dagger \\ & \times \left(I_m - A_2 G^\dagger K_{A_1}\right) E K_{B_1} J^\dagger \mathscr{P}_{B_2^H}, \end{aligned} \tag{35}$$

so we can represent $U_{j_2} Y_* V_2^H$ by a given expression of A_i, B_i, E. Let

$$\begin{aligned} f\left(A_1, A_2, B_1, B_2, E\right) = {} & G^\dagger K_{A_1} E B_2^{H^\dagger} + \left(A_2 K_{G^H}\right)^\dagger \\ & \times \left(I_m - A_2 G^\dagger K_{A_1}\right) E K_{B_1} J^\dagger \\ & - \mathscr{P}_{G^H} \left(A_2 K_{G^H}\right)^\dagger \\ & \times \left(I_m - A_2 G^\dagger K_{A_1}\right) E K_{B_1} J^\dagger \mathscr{P}_{B_2^H}. \end{aligned} \tag{36}$$

Hence, we have

$$\begin{aligned} U_{j_2} Y_* V_2^H &= f\left(A_1, A_2, B_1, B_2, E\right), \\ U_1 X_* V_1^H &= A_1^\dagger E B_1^{H^\dagger} - A_1^\dagger A_2 U_2 Y_* V_2^H B_2^H B_1^{H^\dagger} \\ &= A_1^\dagger E B_1^{H^\dagger} - A_1^\dagger A_2 f\left(A_1, A_2, B_1, B_2, E\right) B_2^H B_1^{H^\dagger}. \end{aligned} \tag{37}$$

Since

$$V_2 K_{\widehat{J}} = V_2 \left(I_{n-l} - \mathscr{P}_{\widehat{J}}\right) = \left(I_p - V_2 \widehat{J} \left(V_2 \widehat{J}\right)^\dagger\right) V_2 = K_J V_2, \tag{38}$$

then

$$\begin{aligned} & U_1 \left(Z - \mathscr{P}_{\widehat{A}_1^H} Z \mathscr{P}_{\widehat{B}_1^H} - \widehat{A}_1^\dagger \widehat{A}_2 K_{\widehat{G}^H} W K_{\widehat{J}} \widehat{B}_2^H \widehat{B}_1^{H^\dagger}\right) V_1^H \\ & \quad = U_1 Z V_1^H - \mathscr{P}_{A_1^H} U_1 Z V_1^H \mathscr{P}_{B_1^H} \\ & \qquad - A_1^\dagger A_2 K_{G^H} U_2 W V_2^H K_J B_2^H B_1^{H^\dagger}, \\ & U_2 \Big(W - \mathscr{P}_{\widehat{G}^H} W \mathscr{P}_{\widehat{B}_2^H} - \mathscr{P}_{(\widehat{A}_2 K_{\widehat{G}^H})^H} W \mathscr{P}_{\widehat{J}} \\ & \qquad + \mathscr{P}_{\widehat{G}^H} \mathscr{P}_{(\widehat{A}_2 K_{\widehat{G}^H})^H} W \mathscr{P}_{\widehat{J}} \mathscr{P}_{\widehat{B}_2^H}\Big) V_2^H \\ & \quad = U_2 W V_2^H - \mathscr{P}_{G^H} U_2 W V_2^H \mathscr{P}_{B_2^H} \\ & \qquad - \mathscr{P}_{(A_2 K_{G^H})^H} U_2 W V_2^H \mathscr{P}_J \\ & \qquad + \mathscr{P}_{G^H} \mathscr{P}_{(A_2 K_{G^H})^H} U_2 W V_2^H P_J \mathscr{P}_{B_2^H}. \end{aligned} \tag{39}$$

Letting

$$U_1 Z V_1^H + U_2 W V_2^H = F, \tag{40}$$

it is not difficult for us to verify $SF = FR$. Together with

$$A_2 U_1 = 0, \qquad A_1 U_2 = 0, \qquad V_2^H B_1^\dagger = 0, \qquad V_1^H B_2^\dagger = 0, \tag{41}$$

the following equality holds:

$$\begin{aligned} & P_{A_1^H} U_1 Z V_1^H \mathscr{P}_{B_1^H} + \mathscr{P}_{G^H} U_2 W V_2^H \mathscr{P}_{B_2^H} \\ & \quad = \left(P_{A_1^H} + \mathscr{P}_{G^H}\right) \left(U_2 W V_2^H + U_1 Z V_1^H\right) \\ & \qquad \times \left(\mathscr{P}_{B_1^H} + \mathscr{P}_{B_2^H}\right) \\ & \quad = \left(P_{A_1^H} + \mathscr{P}_{G^H}\right) F \left(\mathscr{P}_{B_1^H} + \mathscr{P}_{B_2^H}\right). \end{aligned} \tag{42}$$

Note that

$$G U_1 = 0, \qquad A_2 K_{G^H} U_1 = 0. \tag{43}$$

Then

$$A_2 K_{G^H} U_2 W V_2^H = A_2 K_{G^H} \left(U_2 W V_2^H + U_1 Z V_1^H\right) = A_2 K_{G^H} F. \tag{44}$$

Hence,

$$\begin{aligned} & A_1^\dagger A_2 K_{G^H} U_2 W V_2^H K_J B_2^H B_1^{H^\dagger} = A_1^\dagger A_2 K_{G^H} F K_J B_2^H B_1^{H^\dagger}, \\ & \mathscr{P}_{(A_2 K_{G^H})^H} U_2 W V_2^H \mathscr{P}_J - \mathscr{P}_{G^H} \mathscr{P}_{(A_2 K_{G^H})^H} U_2 W V_2^H P_J \mathscr{P}_{B_2^H} \\ & \quad = \mathscr{P}_{(A_2 K_{G^H})^H} F \mathscr{P}_J - \mathscr{P}_{G^H} \mathscr{P}_{(A_2 K_{G^H})^H} F P_J \mathscr{P}_{B_2^H}. \end{aligned} \tag{45}$$

Substituting the expressions above into (33) yields that

$$\begin{aligned} X = {} & A_1^\dagger E B_1^{H^\dagger} + f\left(A_1, A_2, B_1, B_2, E\right) \\ & - A_1^\dagger A_2 f\left(A_1, A_2, B_1, B_2, E\right) B_2^H B_1^{H^\dagger} + F \\ & - \left(P_{A_1^H} + \mathscr{P}_{G^H}\right) F \left(\mathscr{P}_{B_1^H} + \mathscr{P}_{B_2^H}\right) \\ & - A_1^\dagger A_2 K_{G^H} F K_J B_2^H B_1^{H^\dagger} \\ & - \mathscr{P}_{(A_2 K_{G^H})^H} F \mathscr{P}_J + \mathscr{P}_{G^H} \mathscr{P}_{(A_2 K_{G^H})^H} F P_J \mathscr{P}_{B_2^H}. \end{aligned} \tag{46}$$

We have the following theorem.

Theorem 3. *Let*

$$A_1 = AS, \qquad A_2 = A(I_n - S),$$
$$B_1 = BR, \qquad B_2 = B(I_n - R). \tag{47}$$

The matrix equation (2) *with constraint* $SX = XR$ *is consistent if and only if*

$$\mathscr{P}_G K_{A_1} E \mathscr{P}_{B_2} = K_{A_1} E, \qquad \mathscr{P}_{A_2} E K_{B_1} \mathscr{P}_{J^H} = E K_{B_1}, \tag{48}$$

with

$$G = K_{A_1} A_2, \qquad J = B_2^H K_{B_1}. \tag{49}$$

In the meantime, a general solution is given by

$$\begin{aligned} X = {} & A_1^{\dagger} E B_1^{H^{\dagger}} + f(A_1, A_2, B_1, B_2, E) \\ & - A_1^{\dagger} A_2 f(A_1, A_2, B_1, B_2, E) B_2^H B_1^{H^{\dagger}} + F \\ & - \left(P_{A_1^H} + \mathscr{P}_{G^H}\right) F \left(\mathscr{P}_{B_1^H} + \mathscr{P}_{B_2^H}\right) \\ & - A_1^{\dagger} A_2 K_{G^H} F K_J B_2^H B_1^{H^{\dagger}} \\ & - \mathscr{P}_{(A_2 K_{G^H})^H} F \mathscr{P}_J + \mathscr{P}_{G^H} \mathscr{P}_{(A_2 K_{G^H})^H} F P_J \mathscr{P}_{B_2^H}, \end{aligned} \tag{50}$$

where the arbitrary matrix F *satisfies* $SF = FR$ *and* $f(A_1, A_2, B_1, B_2, E)$ *is determined by* (36).

4. Eigenvector-Free Formulas of the General Solutions to (2) with $PX{=}sXQ$ Constraint

For this constraint, if we set S and R as (4), it is not difficult to verify that S, R are Hermitian idempotent, and the constraint $PX = sXQ$ is equivalent to

$$SX = XR. \tag{51}$$

By Theorem 3, we have the following theorem.

Theorem 4. *Let*

$$A_1 = \frac{1}{2} A(I_n + P), \qquad A_2 = \frac{1}{2} A(I_n - P),$$
$$B_1 = \frac{1}{2} B(I_n + sQ), \qquad B_2 = \frac{1}{2} B(I_n - sQ). \tag{52}$$

The matrix equation (2) *with constraint* $PX = sXQ$ *is consistent if and only if*

$$\mathscr{P}_G K_{A_1} E \mathscr{P}_{B_2} = K_{A_1} E, \qquad \mathscr{P}_{A_2} E K_{B_1} \mathscr{P}_{J^H} = E K_{B_1}, \tag{53}$$

with

$$G = K_{A_1} A_2, \qquad J = B_2^H K_{B_1}. \tag{54}$$

In the meantime, a general solution is given by

$$\begin{aligned} X = {} & A_1^{\dagger} E B_1^{H^{\dagger}} + f(A_1, A_2, B_1, B_2, E) \\ & - A_1^{\dagger} A_2 f(A_1, A_2, B_1, B_2, E) B_2^H B_1^{H^{\dagger}} + F \\ & - \left(P_{A_1^H} + \mathscr{P}_{G^H}\right) F \left(\mathscr{P}_{B_1^H} + \mathscr{P}_{B_2^H}\right) \\ & - A_1^{\dagger} A_2 K_{G^H} F K_J B_2^H B_1^{H^{\dagger}} \\ & - \mathscr{P}_{(A_2 K_{G^H})^H} F \mathscr{P}_J + \mathscr{P}_{G^H} \mathscr{P}_{(A_2 K_{G^H})^H} F P_J \mathscr{P}_{B_2^H}, \end{aligned} \tag{55}$$

where the arbitrary matrix F *satisfies* $PF = sFQ$ *and* $f(A_1, A_2, B_1, B_2, E)$ *is determined by* (36).

5. Numerical Examples

In this section, we present some numerical examples to illustrate the effectiveness of Theorems 3 and 4. For simplicity, we set $m = n = p$ and restrict the coefficient matrices A, B and the right-hand-sided matrix E to $\mathscr{R}^{n\times n}$. The coefficient matrices A, B are randomly constructed by

$$A = U \operatorname{diag}(\sigma_1, \dots, \sigma_n) V^T, \tag{56}$$

where the orthogonal matrices U and V are constructed as follows:

$$[U, \text{temp}] = \text{qr}(1 - 2\,\text{rand}(n)),$$
$$[V, \text{temp}] = \text{qr}(1 - 2\,\text{rand}(n)), \tag{57}$$

and the singular values $\{\sigma_i\}$ will be chosen at interval $(0,\ 1)$. For the computational value X of (2) with constraint $PX = sXQ$ or $SX = XR$, the residual error ϵ_X, the PQ-commuting error ϵ_{PQ}, SR-commuting error ϵ_{SR}, and consistent error Cond_{err} are denoted by

$$\begin{aligned} & \epsilon_X = \left\| E - AXB^H \right\|_F, \qquad \epsilon_{PQ} = \| PX - sXQ \|_F, \\ & \epsilon_{SR} = \| SX - XR \|_F, \\ & \text{Cond}_{\text{err}} = \max \left\{ \left\| \mathscr{P}_G K_{A_1} E \mathscr{P}_{B_2} - K_{A_1} E \right\|_F, \right. \\ & \qquad \left. \left\| \mathscr{P}_{A_2} E K_{B_1} \mathscr{P}_{J^H} - E K_{B_1} \right\|_F \right\}. \end{aligned} \tag{58}$$

Example 1. In this example, we test the solutions to (2) with $SX = XQ$ constraint by Theorem 3. The coefficient matrices A, B are constructed as in (56), and the right-hand-sided matrix E is constructed as follows:

$$E = AX_* B^H, \tag{59}$$

where X_* satisfies

$$RX_* = X_* S, \tag{60}$$

and S, R are symmetric idempotent. That implies that the constrained equation (2) is consistent, so the residual error ϵ_X and consistent error Cond_{err} should be zero with the computational value X.

Table 1: Variant matrix sizes n for the solutions to (2) with $SX = XR$ constraint.

n	CPU (s)	ϵ_X	ϵ_{SR}	$\mathrm{Cond}_{\mathrm{err}}$
100	0.38	$1.14 * 10^{-12}$	$6.53 * 10^{-13}$	$7.12 * 10^{-12}$
300	1.34	$3.23 * 10^{-12}$	$4.43 * 10^{-13}$	$5.63 * 10^{-12}$
500	5.62	$4.12 * 10^{-10}$	$4.76 * 10^{-13}$	$2.24 * 10^{-11}$
700	14.55	$3.91 * 10^{-10}$	$7.54 * 10^{-13}$	$5.43 * 10^{-11}$
900	29.63	$2.31 * 10^{-09}$	$3.13 * 10^{-12}$	$1.37 * 10^{-11}$
1100	55.34	$9.36 * 10^{-09}$	$6.64 * 10^{-12}$	$2.19 * 10^{-11}$

Table 2: Variant matrix sizes n for solutions to (2) with $PX = XQ$ constraint.

n	CPU (s)	ϵ_X	ϵ_{PQ}	$\mathrm{Cond}_{\mathrm{err}}$
100	0.42	$6.11 * 10^{-13}$	$5.61 * 10^{-13}$	$2.31 * 10^{-11}$
300	2.83	$2.07 * 10^{-10}$	$9.73 * 10^{-13}$	$4.34 * 10^{-10}$
500	8.21	$5.85 * 10^{-10}$	$1.55 * 10^{-12}$	$3.61 * 10^{-10}$
700	14.53	$1.17 * 10^{-10}$	$2.24 * 10^{-12}$	$5.37 * 10^{-09}$
900	28.54	$2.60 * 10^{-09}$	$4.61 * 10^{-11}$	$8.18 * 10^{-09}$
1100	52.81	$5.35 * 10^{-09}$	$4.92 * 10^{-11}$	$6.53 * 10^{-09}$

For different n, the residual error ϵ_X, SR-commuting error ϵ_{SR}, and consistent errors $\mathrm{Cond}_{\mathrm{err}}$ can reach the precision 10^{-09}, but all of them seem not to depend on the matrix size n very much, and the CPU time also grows quickly as n increases. In Table 1, we list the CPU time, ϵ_X, ϵ_{SR}, and $\mathrm{Cond}_{\mathrm{err}}$, respectively.

Example 2. We test the solutions to (2) with $PX = XQ$ constraint by Theorem 4. The test matrices A, B, and E are constructed as in (56) with X_* satisfying

$$E = AX_*B^H, \tag{61}$$

where X_* satisfies

$$PX_* = X_*Q, \tag{62}$$

and P, Q are symmetric involutory.

For different n, the numerical result is similar to those of Example 1; that is, the residual error ϵ_X, PQ-commuting error ϵ_{PQ}, and consistent errors $\mathrm{Cond}_{\mathrm{err}}$ can all reach the precision 10^{-09}, but it seems that they do not depend on the matrix size n very much. However, the CPU time grows quickly as n increases. In Table 2, we list the CPU time, ϵ_X, ϵ_{PQ}, and $\mathrm{Cond}_{\mathrm{err}}$, respectively.

6. Conclusion

In this paper, we consider (2) with two special constraints $PX = sXQ$ and $SX = XR$, where $P, Q \in \mathscr{C}^{n\times n}$ are Hermitian involutory, $S, R \in \mathscr{C}^{n\times n}$ are Hermitian idempotent, and $s = \pm 1$. We represent the general solutions to the constrained equation by eigenvalue decompositions of P, Q, S, R, release the involved eigenvector by Moore-Penrose generalized inverses, and get the eigenvector-free formulas of the general solutions.

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