Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 918513, 7 pages
http://dx.doi.org/10.1155/2013/918513

Research Article

Asymptotic Estimates for Second Kind Generalized Stirling Numbers

Cristina B. Corcino[1] and Roberto B. Corcino[2]

[1] *Institute of Mathematics, University of the Philippines, Diliman, 1101 Quezon City, Philippines*
[2] *Department of Mathematics, Mindanao State University Main Campus, 9700 Marawi City, Philippines*

Correspondence should be addressed to Roberto B. Corcino; rcorcino@yahoo.com

Received 5 June 2013; Accepted 1 August 2013

Academic Editor: Zhijun Liu

Asymptotic formulas for the generalized Stirling numbers of the second kind with integer and real parameters are obtained and ranges of validity of the formulas are established. The generalizations of Stirling numbers considered here are generalizations along the line of Hsu and Shuie's unified generalization.

1. Introduction

Asymptotic formulas of the classical Stirling numbers have been done by many authors like Temme [1], Moser, and Wyman [2, 3] due to the importance of the formulas in computing values of the numbers under consideration when parameters become large. The Stirling numbers and their generalization, on the other hand, are important due to their applications in statistics, life science, and physics.

The (r,β)-Stirling numbers [4], the r-Whitney numbers of the second kind [5], and the numbers considered by Ruciński and Voigt [6] are exactly the same numbers which can be classified as generalization of the classical Stirling numbers of the second kind. This generalization is in line with the generalization of Hsu and Shuie [7]. For brevity, we can use $S_{\beta,r}(n,m)$ to denote these numbers. These numbers satisfy the following exponential generating function:

$$\frac{1}{\beta^m (m!)} e^{rz}\left(e^{\beta z}-1\right)^m = \sum_{n=m}^{\infty} S_{\beta,r}(n,m)\frac{z^n}{n!}, \tag{1}$$

where n and m are positive integers. When $\beta = 1$, (1) reduces to a generating function of r-Stirling numbers [8] of the second kind and further reduces to a generating function of the classical Stirling numbers of the second when $r = 0$. Combinatorial interpretation and probability distribution involving $S_{\beta,r}(n,m)$ are discussed in [9].

The behavior of the numbers $S_{\beta,r}(n,m)$ was shown to be asymptotically normal in [4, 6]. That means that the distribution of these numbers when the parameters n and m are large will follow a bell-shaped distribution. The unimodality of this distribution was also discussed in [4]. Moreover, the bound for the index in which the maximum value of these numbers occurs has been established in [10]. With these properties of the numbers $S_{\beta,r}(n,m)$, it is necessary to consider the asymptotic formula for the numbers $S_{\beta,r}(n,m)$ to be able to compute large values of the numbers.

Applying Cauchy Integral Formula to (1), the following integral representation is obtained:

$$S_{\beta,r}(n,m) = \frac{n!}{2\pi i \beta^m (m!)} \int_C \frac{e^{rz}\left(e^{\beta z}-1\right)^m}{z^{n+1}} dz, \tag{2}$$

where C is a circle about the origin.

The primary purpose of the present paper is to investigate if the analysis in [3] can be extended to the generalized Stirling numbers of the second kind. The authors are motivated by the work of Chelluri et al. [11] which proved the asymptotic equivalence of the formulas obtained by Temme [1] and those obtained by Moser and Wyman in [2, 3]. Moreover, it was also shown in [11] that the formulas obtained apply not only for integral values of the parameters n and m, but for real values as well. To be able to do a similar investigation with that in [11] for the generalized Stirling numbers of the first and second kinds, it is necessary to come up with asymptotic formulas for each kind generalized Stirling numbers which may have used a method similar to that in [2, 3]. The necessary asymptotic formulas for the generalized Stirling numbers of the first kind can be found in [12, 13]. In this paper, an asymptotic formula for the generalized Stirling numbers of the second kind $S_{\beta,r}(n,m)$ with integral values of m and n are obtained using a similar analysis as that in [3]. The formula is proved to be valid when $m > (1/4)n\mu$ and $\lim_{n\to\infty}(n-m)=\infty$, where $\mu = r/\beta$. Moreover, other asymptotic formulas are obtained for $S_{\beta,r}(n,m)$, where n and m are real numbers under the conditions $n-m \geq n^{\alpha}$ and $n-m \geq n^{1/3}$.

2. Preliminary Results

The integral representation in (2) can be written in the form

$$S_{\beta,r}(n,m) = \frac{n!}{2\pi i \beta^{m-n} m!} \int_C \frac{e^{\mu w}\left(e^{w}-1\right)^{m}}{w^{n+1}} dw, \quad (3)$$

where $\mu = r/\beta$, $w = \beta z$.

Using the representation $w = Re^{i\theta}$, for the circle C, where $-\pi \leq \theta \leq \pi$ and R is a positive real number, (3) becomes

$$S_{\beta,r}(n,m) = \frac{n!}{2\pi \beta^{m-n} R^{n}(m!)} \int_{-\pi}^{\pi} \frac{e^{\mu Re^{i\theta}}\left(e^{Re^{i\theta}}-1\right)^{m}}{e^{in\theta}} d\theta. \quad (4)$$

Multiplying and dividing the right-hand side of (4) by the correction constant

$$\left(e^{R}-1\right)^{m} e^{\mu R}, \quad (5)$$

it reduces to

$$S_{\beta,r}(n,m) = A \int_{-\pi}^{\pi} \exp h(\theta,R)\, d\theta, \quad (6)$$

where

$$A = \frac{n! e^{\mu R}\left(e^{R}-1\right)^{m}}{2\pi \beta^{m-n} R^{n} m!}, \quad (7)$$

and $h(\theta,R)$ is the function,

$$\begin{aligned} h(\theta,R) = {} & \mu Re^{i\theta} + m\log\left(e^{Re^{i\theta}}-1\right) \\ & - m\log\left(e^{R}-1\right) - \mu R - in\theta. \end{aligned} \quad (8)$$

Observe that $h(\theta,R)$ can be written in the form

$$h(\theta,R) = \mu Re^{i\theta} - \mu R + mg(\theta,R), \quad (9)$$

where $g(\theta,R)$ is the same function $g(\theta,R)$ which appeared in [14], except for the value of R.

The Maclaurin expansion of $h(\theta,R)$ is

$$h(\theta,R) = iB\theta + mRH(i\theta)^{2} + m\sum_{k=3}^{\infty} D_k(R)(i\theta)^{k}, \quad (10)$$

where

$$B = R\left(\mu + \frac{m}{1-e^{-R}}\right) - n, \quad (11)$$

$$H = \frac{1}{2}\left[\frac{\mu}{m} + \frac{e^{R}\left(e^{R}-R-1\right)}{\left(e^{R}-1\right)^{2}}\right], \quad (12)$$

$$D_k(R) = \frac{\mu R}{k!m} + \frac{1}{k!}\Delta^{k}\log\left(e^{R}-1\right), \quad (13)$$

where Δ is the operator $R(d/dR)$.

Lemma 1. *There is a unique $0 < R < n/m$ such that*

$$R\left(\mu + \frac{m}{1-e^{-R}}\right) - n = 0. \quad (14)$$

Proof. Equation (14) can be written in the form

$$R\left(1-e^{-R}\right)^{-1} = \frac{n}{m} - \frac{\mu}{m}R. \quad (15)$$

Let $f_1(R) = R(1-e^{-R})^{-1}$ and $f_2(R) = n/m - (\mu/m)R$. Note that $f_1(R)$ is a continuous function on $(0,\infty)$ and $\lim_{R\to\infty} f_1(R) = \infty$ while $\lim_{R\to 0} f_1(R) = 1$. On the other hand, $f_2(R)$ is a line with x intercept at $R = n/\mu$ and y intercept at $f_2(0) = n/m$ and is decreasing when m and μ are positive real numbers. Thus, f_1 and f_2, as functions of R, surely intersect at some point. The value of R at the intersection point is the desired solution. Moreover, it can be seen graphically that $R < n/\mu$. □

Lemma 2. *H defined in (12) obeys the inequalities*

$$\frac{1}{4} \leq H \leq \frac{1}{2} + \frac{\mu}{2}. \quad (16)$$

Proof. This lemma follows from Lemma 2.2 in [3]. □

Lemma 3. *There exists a constant M independent of k and R such that*

$$\left|\frac{D_k(R)}{R}\right| \leq M. \quad (17)$$

Proof. $D_k(R)$ defined in (13) can be written as

$$D_k(R) = \frac{\mu R}{k!m} + C_k(R), \quad (18)$$

where $C_k(R)$ is the same as that defined in (16) in [3]. Dividing both sides of the preceding equation by R and taking the absolute value, we have

$$\left|\frac{D_k(R)}{R}\right| \leq \left|\frac{\mu}{k!m}\right| + \left|\frac{C_k(R)}{R}\right|. \quad (19)$$

With μ a fixed parameter and $m \to \infty$ as $n \to \infty$ and using Lemma 3.2 in [3], the desired result is obtained.

Define $\epsilon = (mR)^{-3/8}$ and consider the integral J defined by

$$J = \int_{\epsilon}^{\pi} \exp h(\theta,R)\, d\theta. \quad (20)$$

□

Lemma 4. *A constant $k > 0$ exists such that*

$$|J| \leq \pi \exp\left(-k(mR)^{1/4}\right). \tag{21}$$

Proof. We claim that

$$|h(\theta, R)| \leq \left|\left(e^{Re^{i\theta}} - 1\right)\right| \cdot \left|\left(e^{R} - 1\right)^{-1}\right|^{m}. \tag{22}$$

Using the definition of $h(\theta, R)$ given by (9), we have

$$\left|\exp h(\theta, R)\right| = \left|\exp\left[\mu R\left(e^{i\theta} - 1\right)\right]\right| \cdot \left|\frac{e^{Re^{i\theta}} - 1}{e^{R} - 1}\right|^{m} \cdot \left|e^{-in\theta}\right|. \tag{23}$$

Because $|e^{in\theta}| = 1$, it remains to show that $|\exp[\mu R(e^{i\theta} - 1)]| \leq 1$.

Note that $|\exp[\mu R(e^{i\theta-1})]| = e^{\mu R\cos\theta} \cdot e^{-\mu R}$. The claim now follows from the fact that $-1 \leq \cos\theta \leq 1$ and $\mu R \geq 0$.

From (9),

$$h(\theta, R) = \mu R\left(e^{i\theta} - \mu R\right) + mg(\theta, R). \tag{24}$$

Thus,

$$\left|\exp h(\theta, R)\right| = \left|\exp\left[\mu R\left(e^{i\theta} - 1\right)\right]\right| \left|\exp mg(\theta, R)\right|. \tag{25}$$

The result now follows from the previous claim and the fact that $g(\theta, R)$ is the same function $g(\theta, R)$ that appeared in [3] except for the value of R. □

Observe that when $mR \to \infty$ as $n \to \infty$, Lemma 4 will imply that $J \to 0$ as $n \to \infty$. Indeed, $mR \to \infty$ as $n \to \infty$ under some conditions, as shown in the following lemma.

Lemma 5. *$mR \to \infty$ as $n - m \to \infty$ provided $m > (1/4)n\mu$, $n > 4$.*

Proof. Write (14) in the form

$$\frac{mR}{1 - e^{-R}} = n - \mu R. \tag{26}$$

Now the application of the mean value theorem to the function $f(R) = 1 - e^{-R}$ over the interval $(0, R)$ will yield

$$1 - e^{-R} = Re^{-\varsigma}, \tag{27}$$

where ς is a number within the interval $(0, R)$. Then,

$$\frac{m}{n - \mu R} = e^{-\varsigma}. \tag{28}$$

Consequently,

$$\begin{gathered} 1 - e^{-R} = R\left(\frac{m}{n - \mu R}\right), \\ 1 - \frac{mR}{n - \mu R} = e^{-R} \leq e^{-\varsigma} = \frac{m}{n - \mu R}, \\ 1 - \frac{mR}{n - \mu R} \leq \frac{m}{n - \mu R}. \end{gathered} \tag{29}$$

The last inequality will yield

$$mR \geq (n - \mu R) - m, \tag{30}$$

provided $n - \mu R > 0$. The previous inequality can be written in the form

$$mR \geq n - (m - \mu R) \geq n - \left(m + \frac{n\mu}{m}\right) \geq n - (m + 4), \tag{31}$$

when $m \geq (1/4)n\mu$. The second inequality previous follows if $R < n/m$. Indeed, $R < n/m$ when $m \geq (1/4)n\mu$ and $n > 4$ because $f_1(n/m) > f_2(n/m)$, where f_1 and f_2 are the functions in the proof of Lemma 1. Hence, R which is the x-coordinate of the point of intersection of the two functions must be less than n/m. The last inequality shows that $mR \to \infty$ as $n - m \to \infty$ under the given restriction of m.

Returning to the condition that $n - \mu R > 0$, note that $R < n/m$. Thus,

$$n - \mu R > n - \mu\frac{n}{m} > n - 4, \tag{32}$$

under the restriction that $m \geq (1/4)n\mu$. So that $n - \mu R > 0$ whenever $n > 4$. □

For example, when $n = 100$ and $\mu = 4/7$, m in Lemma 5 must satisfy $m \geq (1/4)100(4/7) = 14.2857$. Thus, $m \geq 15$.

3. Asymptotic Formula with Integral Parameters

In the discussion that follows, R denotes the unique positive solution to (14). It will be seen later that our final asymptotic formula is expressed in terms of powers of $1/mR$. Anticipating this result and in view of Lemma 4, we write

$$S_{\beta,r}(m, n) \sim A\int_{-\epsilon}^{\epsilon} \exp h(\theta, R)\, d\theta. \tag{33}$$

Substituting (10) for $h(\theta, R)$ and noting that $B(R) = 0$, (33) becomes

$$S_{\beta,r}(n, m) \sim A\int_{-\epsilon}^{\epsilon} \exp\left[-mRH\theta^2 + m\sum_{k=3}^{\infty} D_k(R)(i\theta)^k\right] d\theta. \tag{34}$$

The change of variable $\phi = (mRH)^{1/2}\theta$ will yield $d\phi = (mRH)^{1/2} d\theta$,

$$\theta = \frac{\phi}{(mRH)^{1/2}}, \qquad d\theta = \frac{d\phi}{(mRH)^{1/2}}. \tag{35}$$

Now, (34) becomes

$$S_{\beta,r}(n, m) \sim \frac{A}{(mRH)^{1/2}}\int_{-\alpha}^{\alpha} e^{-\phi^2} \exp\left[f(z, R, \phi)\right] d\phi, \tag{36}$$

where $\alpha = (mRH)^{1/2}\epsilon$, $z = (mRH)^{-1/2}$,

$$f(z, R, \phi) = \sum_{k=1}^{\infty} D_{k+2}(R)\frac{(i\phi)^{k+2}}{RH^{(k+2)/2}}(mR)^{-k/2}. \tag{37}$$

The equation in (37) can be written in the from

$$f(z, R, \phi) = \sum_{k=1}^{\infty} a_k z^k, \tag{38}$$

where

$$a_k = D_{k+2}(R) \frac{(i\phi)^{k+2}}{RH^{(k+2)/2}}. \tag{39}$$

Note here that $z = (mR)^{-1/2}$ is within the radius of convergence of the series in (38). On the other hand, the Maclaurin series of $\exp f(z, R, \phi)$ is

$$e^{f(z,R,\phi)} = \sum_{k=0}^{\infty} b_k(R, \phi) z^k, \tag{40}$$

where b_{2k} is a polynomial in ϕ containing only even powers of ϕ, and b_{2k+1} is a polynomial in ϕ containing only odd powers of ϕ. Now, (36) can be written in the form

$$S_{\beta,r}(n, m) \sim Q \int_{-\alpha}^{\alpha} e^{-\phi^2} \left(\sum_{k=0}^{\infty} b_k(R, \phi) z^k \right) d\phi, \tag{41}$$

where $Q = A/\sqrt{mRH}$ and $z = (mR)^{-1/2}$.

We write (41) as

$$S_{\beta,r}(n, m) \sim Q \left[\sum_{k=0}^{s-1} z^k \left(\int_{-\alpha}^{\alpha} e^{-\phi^2} b_k d\phi \right) + U_s \right], \tag{42}$$

where

$$U_s = \sum_{k=s}^{\infty} \left(\int_{-\alpha}^{\alpha} e^{-\phi^2} b_k d\phi \right) z^k. \tag{43}$$

In view of Lemma 2, $H \geq 1/4$; hence,

$$\alpha = (mRH)^{1/2} \epsilon = (mRH)^{1/2} (mR)^{-3/8} \geq \frac{(mR)^{1/8}}{2}. \tag{44}$$

Since b_k is a polynomial in ϕ, we can replace α with ∞ in (41). Following the discussion in [3], it can be shown that $U_s = O(z^s)$; hence, we have the asymptotic formula

$$S_{\beta,r}(n, m) \sim Q \left[\sum_{k=0}^{\infty} \left(\int_{-\infty}^{\infty} e^{-\phi^2} b_{2k} d\phi \right) (mR)^{-k} \right]. \tag{45}$$

Note that

$$\int_{-\infty}^{\infty} e^{-\phi^2} b_{2k+1} d\phi = 0. \tag{46}$$

This is why no odd subscript of b_k appears in (45).

An approximation is obtained by taking the first two terms of the sum in (45). Thus,

$$S_{\beta,r}(n, m) \approx Q \left[\int_{-\infty}^{\infty} e^{-\phi^2} b_0 d\phi + (mR)^{-1} \int_{-\infty}^{\infty} e^{-\phi^2} b_2 d\phi \right]. \tag{47}$$

It can be computed that

$$\begin{gathered} b_0 = 1, \\ \int_{-\infty}^{\infty} e^{-\phi^2} d\phi = \sqrt{\pi}, \\ b_2 = a_2(R, \phi) + \frac{1}{2}\left[a_1(R, \phi)\right]^2, \\ a_2 = D_4(R) \frac{(i\phi)^4}{RH^2}, \\ D_4(R) = \frac{\mu R}{24m} + C_4(R), \end{gathered} \tag{48}$$

where

$$\begin{aligned} C_4(R) = {} & \frac{1}{24}\left[R + \left(R - 7R^2 + 6R^3 - R^4\right)\left(e^R - 1\right)^{-1}\right] \\ & + \frac{1}{24}\left[\left(18R^3 - 7R^2 - 7R^4\right)\left(e^R - 1\right)^{-2} \right. \\ & \left. + \left(12R^3 - 12R^4\right)\left(e^R - 1\right)^{-3}\right] \\ & - \frac{1}{24}\left[6R^4\left(e^R - 1\right)^{-4}\right], \end{aligned} \tag{49}$$

while

$$a_1(R, \phi) = D_3 \frac{(i\phi)^3}{RH^{3/2}} = \left(\frac{\mu R}{3!m} + C_3 \right) \frac{(i\phi)^3}{RH^{3/2}}, \tag{50}$$

where

$$\begin{aligned} C_3 = {} & \frac{1}{6}\left[R + \left(R - 3R^2 + R^3\right)\left(e^R - 1\right)^{-1} \right. \\ & \left. + \left(3R^3 - 3R^2\right)\left(e^R - 1\right)^{-2}\right] \\ & + \frac{1}{6}\left[2R^3\left(e^R - 1\right)^{-3}\right]. \end{aligned} \tag{51}$$

Thus, b_2 can be written as follows:

$$b_2 = \left[\frac{\mu R}{24m} + C_4 \right] \frac{\phi^4}{RH^2} - \frac{1}{2}\left(\frac{\mu R}{6m} + C_3 \right)^2 \frac{\phi^6}{R^2 H^3}. \tag{52}$$

Let

$$I = \int_{-\infty}^{\infty} e^{-\phi^2} b_2 d\phi. \tag{53}$$

Then,

$$\begin{aligned}
I &= \int_{-\infty}^{\infty} e^{-\phi^2} \frac{(\mu R/24m + C_4)}{RH^2} \phi^4 d\phi \\
&\quad - \frac{1}{2}\int_{-\infty}^{\infty} e^{-\phi^2} \frac{(\mu R/6m + C_3)^2}{R^2H^3} \phi^6 d\phi \\
&= \frac{\mu R/24m + C_4}{RH^2} \int_{-\infty}^{\infty} e^{-\phi^2} \phi^4 d\phi \\
&\quad - \frac{(\mu R/6m + C_3)^2}{2R^2H^3} \int_{-\infty}^{\infty} e^{-\phi^2} \phi^6 d\phi \\
&= \frac{\mu R/24m + C_4}{RH^2} \cdot \frac{3\sqrt{\pi}}{4} - \frac{(\mu R/6m + C_3)^2}{R^2H^3} \cdot \frac{15\sqrt{\pi}}{16}.
\end{aligned} \tag{54}$$

Finally, we obtain the approximation formula

$$S_{\beta,r}(n,m) \approx \frac{n!}{m!} \frac{e^{\mu R}(e^R - 1)^m}{2\beta^{m-n} R^n \sqrt{\pi m R H}} \left[1 + \frac{I}{mR\sqrt{\pi}}\right]. \tag{55}$$

In view of Lemmas 4 and 5 and (33), the previous asymptotic approximation is valid for $m \geq (1/4)n\mu$. The formula obtained in the previous discussion is formally stated in the following theorem.

Theorem 6. *The formula*

$$S_{\beta,r}(n,m) \sim Q\left[\sqrt{\pi} + \frac{I}{mR}\right] \tag{56}$$

behaves as an asymptotic approximation for the generalized Stirling numbers of the second kind $S_{\beta,r}(n,m)$ with positive integral parameters for $m > (1/4)n\mu$, $n > 4$ and such that $n - m \to \infty$ as $n \to \infty$.

Table 1 displays the exact values and approximate values for $n = 100$, $r = 4$, $\beta = 7$. The exact values are obtained using recurrence formula while the approximate values are obtained using Theorem 6. In view of Lemma 5, the formula is valid for $m \geq 15$.

Values on the table affirm that the asymptotic formula in Theorem 6 gives a good approximation for $S_{7,4}(100,m)$ when $m \geq 15$.

4. Asymptotic Formula with Real Parameters

Recently, the (r,β)-Stirling numbers for complex arguments were defined in [15] parallel to the definition of Flajolet and Prodinger as

$$S_{\beta,r}(x,y) = \frac{x!}{y!\beta^y 2\pi i} \int_{\mathscr{H}} e^{rz}(e^{\beta z} - 1)^y \frac{dz}{z^{x+1}}, \tag{57}$$

where $x! := \Gamma(x+1)$ and $\mathscr{H}$ is a Hankel contour that starts from $-\infty$ below the negative axis surrounds the origin counterclockwise and returns to $-\infty$ in the half plane $\mathfrak{I}z > 0$.

Table 1

	Exact value	Approximate value	Relative error
$S_{7,4}(100,5)$	5.685×10^{152}	6.335×10^{152}	0.11428
$S_{7,4}(100,10)$	7.728×10^{171}	8.169×10^{171}	0.05713
$S_{7,4}(100,15)$	8.411×10^{178}	8.731×10^{178}	0.03804
$S_{7,4}(100,30)$	5.604×10^{174}	5.706×10^{174}	0.01824
$S_{7,4}(100,60)$	7.399×10^{122}	7.446×10^{122}	0.00641
$S_{7,4}(100,80)$	1.275×10^{70}	1.279×10^{70}	0.00263
$S_{7,4}(100,90)$	2.208×10^{38}	2.211×10^{38}	0.00123

Note that by change of variable, say $w = \beta z$, we can express $S_{\beta,r}(x,y)$ as follows:

$$S_{\beta,r}(x,y) = \beta^{x-y} \left\{ \begin{matrix} x + \dfrac{r}{\beta} \\ y + \dfrac{r}{\beta} \end{matrix} \right\}_{r/\beta}, \tag{58}$$

where

$$\left\{ \begin{matrix} x+r \\ y+r \end{matrix} \right\}_r = \frac{x!}{y! 2\pi i} \int_{\mathscr{H}} e^{rz}(e^z - 1)^y \frac{dz}{z^{x+1}}. \tag{59}$$

The numbers $\left\{ \begin{smallmatrix} x+r \\ y+r \end{smallmatrix} \right\}_r$ are certain generalization of r-Stirling numbers of the second kind [8] in which the parameters involved are complex numbers. These numbers satisfy the following properties.

Theorem 7. *For nonnegative real number r, one has*

$$\left\{ \begin{matrix} x+r \\ y+r \end{matrix} \right\}_r = \left\{ \begin{matrix} x+r-1 \\ y+r-1 \end{matrix} \right\}_r + (y+r) \left\{ \begin{matrix} x+r-1 \\ y+r \end{matrix} \right\}_r. \tag{60}$$

Theorem 8. *For nonnegative real numbers r and β and $k \in \mathbb{N}$, one has*

$$\left\{ \begin{matrix} x+r \\ k+r \end{matrix} \right\}_r = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (j+r)^x. \tag{61}$$

Furthermore, when $k = y$, a complex number

$$\left\{ \begin{matrix} x+r \\ y+r \end{matrix} \right\}_r = \frac{1}{y!} \sum_{j=0}^{\infty} (-1)^{y-j} \binom{y}{j} (j+r)^x. \tag{62}$$

The Bernoulli polynomial can be expressed using the explicit formula in [16] and the first formula in Theorem 8, with $w = r$ and $x = n$, as

$$\overline{B}_n(r) = \sum_{k=0}^{\infty} \frac{(-1)^k k!}{k+1} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r. \tag{63}$$

Moreover, it is known that, for $n = 1, 2, \ldots$, the Hurwitz zeta function $\zeta(1-n,x) = -\overline{B}_n(x)/n$. Note that, as $y \to 0$, $(y-1)! \sim 1/y$. By Cauchy's integral formula

$$\left\{ \begin{matrix} n+r \\ y+r-1 \end{matrix} \right\}_r \sim yn! \left[z^{n+1}\right] \frac{e^{rz}}{e^z - 1} = y \frac{\overline{B}_{n+1}(r)}{n+1}, \tag{64}$$

which further gives, using a result in [17], the following relation:

$$\left[\frac{d}{dy}\left\{ {n+r \atop y+r} \right\}_r\right]_{y=-1} = -\zeta(-n, r) = \frac{n!}{2\pi i}\int_{\mathscr{H}} e^{rz}\left(e^z - 1\right)^{-1}\frac{dz}{z^{n+1}}. \quad (65)$$

An asymptotic formula for r-Stirling numbers of the second kind was first considered by Corcino et al. in [14]. However, the formula only holds for integral arguments. In this section, we are going to establish an asymptotic formula for r-Stirling numbers of the second kind that will hold for real arguments.

Consider the integral representation in (59) where $|\Im z| < 2\pi$. To see the analysis of Moser-Wyman applies to r-Stirling numbers with real arguments x and y, we deform the path $\mathscr{H}$ into the following contour: $C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5$ where

(1) C_1 is the line $\Im z = -2\pi + \delta, \delta > 0$ and $\Re z \le \epsilon$, ϵ is a small positive number;

(2) C_2 is the line segment $\Re z = \epsilon$, going from $\epsilon + i(\delta - 2\pi)$ to the circle $|z| = R$;

(3) C_5 and C_4 are the reflections in the real axis of C_1 and C_2, respectively; and

(4) C_3 is the portion of the circle $|z| = R$, meeting C_2 and C_4.

The new contour is $C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5$ in the counterclockwise sense. This idea of deforming the contour $\mathscr{H}$ is also done in [11].

The integrals along C_1, C_2, C_4, and C_5 are seen to be

$$O\left(\frac{(2+\epsilon)^y}{(2\pi - \delta)^x}\right). \quad (66)$$

It will also be shown that these integrals go to 0 as $x \to \infty$ provided that $x - y \ge x^\alpha$ where $0 < \alpha < 1$. To see this, we consider C_1. For the other contours, the estimate can be seen similarly.

Note that for z in C_1, we can choose δ so that

$$\left[1 - \left(e^{\Re z}\cos(\Im z)\right)^{-1}\right]^2 < 1. \quad (67)$$

From this and the assumptions that $\Re z < \epsilon$ and $\epsilon < 1$, it follows that

$$\left|e^z - 1\right| \le e^{\Re z} < \epsilon < \frac{2+\epsilon}{2-\epsilon} < 2 + \epsilon. \quad (68)$$

Thus,

$$\left|e^z - 1\right|^y < (2+\epsilon)^y. \quad (69)$$

Consequently,

$$\left|\int_{C_1}\frac{\left(e^z - 1\right)^y}{z^{x+1}}dz\right| \le \int_{C_1}\frac{(2+\epsilon)^y}{(2\pi - \delta)^{x+1}}|dz|, \quad (70)$$

where $\int_{C_1}|dz|$ is the length of C_1. With C_1 the horizontal line $\Im z = -2\pi + \delta$, the length of C_1 is a linear function of the real part of z, given by

$$l(C_1) = \lim_{t\to\infty}(\epsilon - t). \quad (71)$$

Hence, we have

$$\left|\int_{C_1}\frac{\left(e^z - 1\right)^y}{z^{x+1}}dz\right| \le \frac{(2+\epsilon)^y}{(2\pi - \delta)^{x+1}}\lim_{t\to\infty}(\epsilon - t) < \frac{\lim_{t\to\infty}(\epsilon - t)}{(2\pi - \delta)^{x-y}}. \quad (72)$$

The last inequality follows from the fact that $2+\epsilon < 2\pi - \delta$. With the condition that $x - y \ge x^\alpha$, $0 < \alpha < 1$, and the fact that $2\pi - \delta > e$, it follows that the integral along C_1 goes to 0 as $x \to \infty$. Thus, we have

$$\left\{ {x+r \atop y+r} \right\}_r \sim \frac{x!}{y!}\frac{1}{2\pi i}\int_{C_3}\frac{e^{rz}\left(e^z - 1\right)^y}{z^{x+1}}dz. \quad (73)$$

Using the method of Moser and Wyman [3], we obtain the following asymptotic formula.

Theorem 9. *The r-Stirling numbers of the second kind with real arguments x and y have the following asymptotic formula:*

$$\left\{ {x+r \atop y+r} \right\}_r = \frac{x!}{y!}\frac{e^{rR}\left(e^R - 1\right)^y}{2R^x\sqrt{\pi y R H}}\left[1 + O\left(\frac{1}{y}\right)\right] \quad (74)$$

valid for $x - y \to \infty$ as $x \to \infty$, provided that $x - y \ge x^\alpha$ with $0 < \alpha < 1$, where

$$H = \frac{r}{2y} + \frac{e^R\left(e^R - R - 1\right)}{2\left(e^R - 1\right)^2}, \quad (75)$$

and R is the unique positive solution to the equation

$$w\left[r - y\left(1 - e^{-w}\right)^{-1}\right] - x = 0, \quad (76)$$

as a function of w.

Chelluri et al. [11] has made a modification of Moser and Wyman formula and analysis. Using this analysis of Chelluri, we can restate Theorem 9 as follows.

Theorem 10. *The r-Stirling numbers of the second kind with real arguments x and y have the following asymptotic formula:*

$$\left\{ {x+r \atop y+r} \right\}_r = \frac{x!}{y!}\frac{e^{rR}\left(e^R - 1\right)^y}{2R^x\sqrt{\pi y R H}}\left[1 + O\left(\frac{1}{x}\right)\right] \quad (77)$$

valid for $x - y \to \infty$ as $x \to \infty$, provided that $x - y \ge x^{1/3}$, where

$$H = \frac{r}{2y} + \frac{e^R\left(e^R - R - 1\right)}{2\left(e^R - 1\right)^2}, \quad (78)$$

and R is the unique positive solution to the equation

$$w\left[r - y\left(1 - e^{-w}\right)^{-1}\right] - x = 0, \tag{79}$$

as a function of w.

As noted in (58), the (r, β)-Stirling numbers can be obtained using the r-Stirling numbers of the second kind for complex arguments x, y, r, and β. This implies, using Theorems 9 and 10, the following asymptotic formulas which agree with the formula in (55) for the integral values of x and y.

Corollary 11. *The (r, β)-Stirling numbers with real arguments x and y have the following asymptotic formulas:*

$$S_{\beta,r}(x, y) = \frac{\beta^{x-y} x!}{y!} \frac{e^{rR/\beta}\left(e^{R} - 1\right)^{y}}{2R^{x}\sqrt{\pi y R H}} \left[1 + O\left(\frac{1}{y}\right)\right] \tag{80}$$

provided that $x - y \geq x^{\alpha}$ with $0 < \alpha < 1$, and

$$S_{\beta,r}(x, y) = \frac{\beta^{x-y} x!}{y!} \frac{e^{rR/\beta}\left(e^{R} - 1\right)^{y}}{2R^{x}\sqrt{\pi y R H}} \left[1 + O\left(\frac{1}{x}\right)\right] \tag{81}$$

provided that $x - y \geq x^{1/3}$, where

$$H = \frac{r}{2\beta y} + \frac{e^{R}\left(e^{R} - R - 1\right)}{2\left(e^{R} - 1\right)^{2}} \tag{82}$$

and R is the unique positive solution to the equation

$$w\left[\frac{r}{\beta} - y\left(1 - e^{-w}\right)^{-1}\right] - x = 0 \tag{83}$$

as a function of w.

Acknowledgment

The authors would like to thank the referee for reading and evaluating the paper.

References

[1] N. M. Temme, "Asymptotic estimates of Stirling numbers," *Studies in Applied Mathematics*, vol. 89, no. 3, pp. 233–243, 1993.

[2] L. Moser and M. Wyman, "Asymptotic development of the Stirling numbers of the first kind," *Journal of the London Mathematical Society*, vol. 33, pp. 133–146, 1958.

[3] L. Moser and M. Wyman, "Stirling numbers of the second kind," *Duke Mathematical Journal*, vol. 25, pp. 29–43, 1957.

[4] R. B. Corcino, C. B. Corcino, and R. Aldema, "Asymptotic normality of the (r, β)-Stirling numbers," *Ars Combinatoria*, vol. 81, pp. 81–96, 2006.

[5] I. Mező, "A new formula for the Bernoulli polynomials," *Results in Mathematics*, vol. 58, no. 3-4, pp. 329–335, 2010.

[6] A. Ruciński and B. Voigt, "A local limit theorem for generalized Stirling numbers," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 35, no. 2, pp. 161–172, 1990.

[7] L. C. Hsu and P. J. S. Shuie, "A unified approach to generalized stirling numbers," *Advances in Applied Mathematics*, vol. 20, no. 3, pp. 366–384, 1998.

[8] A. Z. Broder, "The r-stirling numbers," *Discrete Mathematics*, vol. 49, no. 3, pp. 241–259, 1984.

[9] R. B. Corcino and R. Aldema, "Some combinatorial and statistical applications of (r, β)-Stirling numbers," *Matimyás Matematika*, vol. 25, no. 1, pp. 19–29, 2002.

[10] R. B. Corcino and C. B. Corcino, "On the maximum of generalized Stirling numbers," *Utilitas Mathematica*, vol. 86, pp. 241–256, 2011.

[11] R. Chelluri, L. B. Richmond, and N. M. Temme, "Asymptotic estimates for generalized Stirling numbers," *Analysis*, vol. 20, no. 1, pp. 1–13, 2000.

[12] M. A. R. P. Vega and C. B. Corcino, "An asymptotic formula for the generalized Stirling numbers of the first kind," *Utilitas Mathematica*, vol. 73, pp. 129–141, 2007.

[13] M. A. R. P. Vega and C. B. Corcino, "More asymptotic formulas for the generalized Stirling numbers of the first kind," *Utilitas Mathematica*, vol. 75, pp. 259–272, 2008.

[14] C. B. Corcino, L. C. Hsu, and E. L. Tan, "Asymptotic approximations of r-stirling numbers," *Approximation Theory and Its Applications*, vol. 15, no. 3, pp. 13–25, 1999.

[15] R. B. Corcino, C. B. Corcino, and M. B. Montero, "On generalized Bell numbers for complex argument," *Utilitas Mathematica*, vol. 88, pp. 267–279, 2012.

[16] F. W. J. Olver, D. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of Mathematical Functions*, Cambridge University Press, Washington, DC, USA, 2010.

[17] K. N. Boyadzhiev, "Evaluation of series with Hurwitz and Lerch zeta function coefficients by using Hankel contour integrals," *Applied Mathematics and Computation*, vol. 186, no. 2, pp. 1559–1571, 2007.