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Research Article

The Optimization on Ranks and Inertias of a Quadratic Hermitian Matrix Function and Its Applications

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We solve optimization problems on the ranks and inertias of the quadratic Hermitian matrix function $Q - XPX^*$ subject to a consistent system of matrix equations $AX = C$ and $XB = D$. As applications, we derive necessary and sufficient conditions for the solvability to the systems of matrix equations and matrix inequalities $AX = C$, $XB = D$, and $XPX^* = (>, <, \geq, \leq)Q$ in the Löwner partial ordering to be feasible, respectively. The findings of this paper widely extend the known results in the literature.

1. Introduction

Throughout this paper, we denote the complex number field by $\mathbb{C}$. The notations $\mathbb{C}^{m\times n}$ and $\mathbb{C}_h^{m\times m}$ stand for the sets of all $m \times n$ complex matrices and all $m \times m$ complex Hermitian matrices, respectively. The identity matrix with an appropriate size is denoted by I. For a complex matrix A, the symbols A^* and $r(A)$ stand for the conjugate transpose and the rank of A, respectively. The Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is defined to be the unique solution X to the following four matrix equations

$$\begin{aligned} &(1)\ AXA = A, \qquad (2)\ XAX = X, \\ &(3)\ (AX)^* = AX, \qquad (4)\ (XA)^* = XA. \end{aligned} \tag{1}$$

Furthermore, L_A and R_A stand for the two projectors $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$ induced by A, respectively. It is known that $L_A = L_A^*$ and $R_A = R_A^*$. For $A \in \mathbb{C}_h^{m\times m}$, its inertia

$$\mathbb{I}_n(A) = \left(i_+(A), i_-(A), i_0(A)\right) \tag{2}$$

is the triple consisting of the numbers of the positive, negative, and zero eigenvalues of A, counted with multiplicities, respectively. It is easy to see that $i_+(A) + i_-(A) = r(A)$. For two Hermitian matrices A and B of the same sizes, we say $A > B$ ($A \geq B$) in the Löwner partial ordering if $A - B$ is positive (nonnegative) definite.

The investigation on maximal and minimal ranks and inertias of linear and quadratic matrix function is active in recent years (see, e.g., [1–24]). Tian [21] considered the maximal and minimal ranks and inertias of the Hermitian quadratic matrix function

$$h(X) = AXBX^*A^* + AXC + C^*X^*A^* + D, \tag{3}$$

where B and D are Hermitian matrices. Moreover, Tian [22] investigated the maximal and minimal ranks and inertias of the quadratic Hermitian matrix function

$$f(X) = Q - XPX^* \tag{4}$$

such that $AX = C$.

The goal of this paper is to give the maximal and minimal ranks and inertias of the matrix function (4) subject to the consistent system of matrix equations

$$AX = C, \qquad XB = D, \tag{5}$$

where $Q \in \mathbb{C}_h^{n\times n}$, $P \in \mathbb{C}_h^{p\times p}$ are given complex matrices. As applications, we consider the necessary and sufficient

conditions for the solvability to the systems of matrix equations and inequality

$$\begin{aligned} &AX = C, \quad XB = D, \quad XPX^* = Q, \\ &AX = C, \quad XB = D, \quad XPX^* > Q, \\ &AX = C, \quad XB = D, \quad XPX^* < Q, \\ &AX = C, \quad XB = D, \quad XPX^* \geq Q, \\ &AX = C, \quad XB = D, \quad XPX^* \leq Q, \end{aligned} \tag{6}$$

in the Löwner partial ordering to be feasible, respectively.

2. The Optimization on Ranks and Inertias of (4) Subject to (5)

In this section, we consider the maximal and minimal ranks and inertias of the quadratic Hermitian matrix function (4) subject to (5). We begin with the following lemmas.

Lemma 1 (see [3]). *Let $A \in \mathbb{C}_h^{m\times m}$, $B \in \mathbb{C}^{m\times p}$, and $C \in \mathbb{C}^{q\times m}$ be given and denote*

$$\begin{aligned} P_1 &= \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix}, \\ P_3 &= \begin{bmatrix} A & B & C^* \\ B^* & 0 & 0 \end{bmatrix}, \qquad P_4 = \begin{bmatrix} A & B & C^* \\ C & 0 & 0 \end{bmatrix}. \end{aligned} \tag{7}$$

Then

$$\begin{aligned} &\max_{Y\in\mathbb{C}^{p\times q}} r\left[A - BYC - (BYC)^*\right] \\ &\quad = \min\left\{r\left[A \;\; B \;\; C^*\right], r\left(P_1\right), r\left(P_2\right)\right\}, \\ &\min_{Y\in\mathbb{C}^{p\times q}} r\left[A - BYC - (BYC)^*\right] \\ &\quad = 2r\left[A \;\; B \;\; C^*\right] \\ &\qquad + \max\left\{w_+ + w_-, g_+ + g_-, w_+ + g_-, w_- + g_+\right\}, \\ &\max_{Y\in\mathbb{C}^{p\times q}} i_\pm\left[A - BYC - (BYC)^*\right] = \min\left\{i_\pm\left(P_1\right), i_\pm\left(P_2\right)\right\}, \\ &\min_{Y\in\mathbb{C}^{p\times q}} i_\pm\left[A - BYC - (BYC)^*\right] \\ &\quad = r\left[A \;\; B \;\; C^*\right] + \max\left\{i_\pm\left(P_1\right) - r\left(P_3\right), i_\pm\left(P_2\right) - r\left(P_4\right)\right\}, \end{aligned} \tag{8}$$

where

$$w_\pm = i_\pm\left(P_1\right) - r\left(P_3\right), \qquad g_\pm = i_\pm\left(P_2\right) - r\left(P_4\right). \tag{9}$$

Lemma 2 (see [4]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, $D \in \mathbb{C}^{m\times p}$, $E \in \mathbb{C}^{q\times n}$, $Q \in \mathbb{C}^{m_1\times k}$, and $P \in \mathbb{C}^{l\times n_1}$ be given. Then*

$$\begin{aligned} &(1)\ r(A) + r\left(R_A B\right) = r(B) + r\left(R_B A\right) = r\left[A \;\; B\right], \\ &(2)\ r(A) + r\left(CL_A\right) = r(C) + r\left(AL_C\right) = r\begin{bmatrix} A \\ C \end{bmatrix}, \\ &(3)\ r(B) + r(C) + r\left(R_B A L_C\right) = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \\ &(4)\ r(P) + r(Q) + r\begin{bmatrix} A & BL_Q \\ R_P C & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & P \\ 0 & Q & 0 \end{bmatrix}, \\ &(5)\ r\begin{bmatrix} R_B A L_C & R_B D \\ EL_C & 0 \end{bmatrix} + r(B) + r(C) = r\begin{bmatrix} A & D & B \\ E & 0 & 0 \\ C & 0 & 0 \end{bmatrix}. \end{aligned} \tag{10}$$

Lemma 3 (see [23]). *Let $A \in \mathbb{C}_h^{m\times m}$, $B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}_h^{n\times n}$, $Q \in \mathbb{C}^{m\times n}$, and $P \in \mathbb{C}^{p\times n}$ be given, and, $T \in \mathbb{C}^{m\times m}$ be nonsingular. Then*

$$\begin{aligned} &(1)\ i_\pm\left(TAT^*\right) = i_\pm(A), \\ &(2)\ i_\pm\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} = i_\pm(A) + i_\pm(C), \\ &(3)\ i_\pm\begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix} = r(Q), \\ &(4)\ i_\pm\begin{bmatrix} A & BL_P \\ L_P B^* & 0 \end{bmatrix} + r(P) = i_\pm\begin{bmatrix} A & B & 0 \\ B^* & 0 & P^* \\ 0 & P & 0 \end{bmatrix}. \end{aligned} \tag{11}$$

Lemma 4. *Let A, C, B, and D be given. Then the following statements are equivalent.*

(1) *System* (5) *is consistent.*

(2) *Let*

$$r\left[A \;\; C\right] = r(A), \quad \begin{bmatrix} D \\ B \end{bmatrix} = r(B), \quad AD = CB. \tag{12}$$

In this case, the general solution can be written as

$$X = A^\dagger C + L_A D B^\dagger + L_A V R_B, \tag{13}$$

where V is an arbitrary matrix over $\mathbb{C}$ with appropriate size.

Now we give the fundamental theorem of this paper.

Theorem 5. *Let $f(X)$ be as given in* (4) *and assume that $AX = C$ and $XB = D$ in* (5) *is consistent. Then*

$$\begin{aligned}
&\max_{AX=C,XB=D} r\left(Q-XPX^*\right)\\
&\quad=\min\left\{n+r\begin{bmatrix}0 & P & P\\ AQ & CP & 0\\ -D^* & 0 & B^*\end{bmatrix}-r(A)-r(B)\right.\\
&\qquad -r(P), 2n+r\left(AQA^*-CPC^*\right)\\
&\qquad \left.-2r(A), r\begin{bmatrix}Q & 0 & D\\ 0 & -P & B\\ D^* & B^* & 0\end{bmatrix}-2r(B)\right\},\\
&\min_{AX=C,XB=D} r\left(Q-XPX^*\right)\\
&\quad=2n+2r\begin{bmatrix}0 & P & P\\ AQ & CP & 0\\ -D^* & 0 & B^*\end{bmatrix}-2r(A)-2r(B)-r(P)\\
&\qquad+\max\left\{s_+ + s_-, t_+ + t_-, s_+ + t_-, s_- + t_+\right\},\\
&\max_{AX=C,XB=D} i_\pm\left(Q-XPX^*\right)\\
&\quad=\min\left\{n+i_\pm\left(AQA^*-CPC^*\right)\right.\\
&\qquad \left.-r(A), i_\pm\begin{bmatrix}Q & 0 & D\\ 0 & -P & B\\ D^* & B^* & 0\end{bmatrix}-r(B)\right\},
\end{aligned}\tag{14}$$

$$\begin{aligned}
&\min_{AX=C,XB=D} i_\pm\left(Q-XPX^*\right)\\
&\quad=n+r\begin{bmatrix}0 & P & P\\ AQ & CP & 0\\ -D^* & 0 & B^*\end{bmatrix}-r(A)-r(B)\\
&\qquad-i_\pm(P)+\max\left\{s_\pm, t_\pm\right\},
\end{aligned}\tag{15}$$

where

$$\begin{aligned}
s_\pm &= -n+r(A)-i_\mp(P)+i_\pm\left(AQA^*-CPC^*\right)-r\begin{bmatrix}CP & AQA^*\\ B^* & D^*A^*\end{bmatrix},\\
t_\pm &= -n+r(A)-i_\mp(P)+i_\pm\begin{bmatrix}Q & 0 & D\\ 0 & -P & B\\ D^* & B^* & 0\end{bmatrix}-\begin{bmatrix}0 & P & B\\ AQ & CP & 0\\ D^* & B^* & 0\end{bmatrix}.
\end{aligned}\tag{16}$$

Proof. It follows from Lemma 4 that the general solution of (4) can be expressed as

$$X = X_0 + L_A V R_B, \tag{17}$$

where V is an arbitrary matrix over $\mathbb{C}$ and X_0 is a special solution of (5). Then

$$Q - XPX^* = Q - \left(X_0 + L_A V R_B\right) P \left(X_0 + L_A V R_B\right)^*. \tag{18}$$

Note that

$$\begin{aligned}
&r\left[Q-\left(X_0+L_AVR_B\right)P\left(X_0+L_AVR_B\right)^*\right]\\
&\quad=r\begin{bmatrix}Q & \left(X_0+L_AVR_B\right)P\\ P\left(X_0+L_AVR_B\right)^* & P\end{bmatrix}-r(P)\\
&\quad=r\left[\begin{bmatrix}Q & X_0P\\ PX_0^* & P\end{bmatrix}+\begin{bmatrix}L_A\\ 0\end{bmatrix}V\begin{bmatrix}0 & R_BP\end{bmatrix}\right.\\
&\qquad\left.+\left(\begin{bmatrix}L_A\\ 0\end{bmatrix}V\begin{bmatrix}0 & R_BP\end{bmatrix}\right)^*\right]-r(P),
\end{aligned}\tag{19}$$

$$\begin{aligned}
&i_\pm\left[Q-\left(X_0+L_AVR_B\right)P\left(X_0+L_AVR_B\right)^*\right]\\
&\quad=i_\pm\begin{bmatrix}Q & \left(X_0+L_AVR_B\right)P\\ P\left(X_0+L_AVR_B\right)^* & P\end{bmatrix}-i_\pm(P)\\
&\quad=i_\pm\left[\begin{bmatrix}Q & X_0P\\ PX_0^* & P\end{bmatrix}+\begin{bmatrix}L_A\\ 0\end{bmatrix}V\begin{bmatrix}0 & R_BP\end{bmatrix}\right.\\
&\qquad\left.+\left(\begin{bmatrix}L_A\\ 0\end{bmatrix}V\begin{bmatrix}0 & R_BP\end{bmatrix}\right)^*\right]-i_\pm(P).
\end{aligned}\tag{20}$$

Let

$$\begin{aligned}
q(V) &= \begin{bmatrix}Q & X_0P\\ PX_0^* & P\end{bmatrix}+\begin{bmatrix}L_A\\ 0\end{bmatrix}V\begin{bmatrix}0 & R_BP\end{bmatrix}\\
&\quad+\left(\begin{bmatrix}L_A\\ 0\end{bmatrix}V\begin{bmatrix}0 & R_BP\end{bmatrix}\right)^*.
\end{aligned}\tag{21}$$

Applying Lemma 1 to (19) and (20) yields

$$\begin{aligned}
&\max_V r\left[q(V)\right]=\min\left\{r(M), r(M_1), r(M_2)\right\},\\
&\min_V r\left[q(V)\right]\\
&\quad=2r(M)+\max\left\{s_+ + s_-, t_+ + t_-, s_+ + t_-, s_- + t_+\right\},\\
&\max_V i_\pm\left[q(V)\right]=\min\left\{i_\pm(M_1), i_\pm(M_2)\right\},\\
&\min_V i_\pm\left[q(V)\right]=r(M)+\max\left\{s_\pm, t_\pm\right\},
\end{aligned}\tag{22}$$

where

$$\begin{aligned}
&M=\begin{bmatrix}Q & X_0P & L_A & 0\\ PX_0^* & P & 0 & PR_B\end{bmatrix},\quad M_1=\begin{bmatrix}Q & X_0P & L_A\\ PX_0^* & P & 0\\ L_A & 0 & 0\end{bmatrix},\\
&M_2=\begin{bmatrix}Q & X_0P & 0\\ PX_0^* & P & PR_B\\ 0 & R_BP & 0\end{bmatrix},\quad M_3=\begin{bmatrix}Q & X_0P & L_A & 0\\ PX_0^* & P & 0 & PR_B\\ L_A & 0 & 0 & 0\end{bmatrix},\\
&M_4=\begin{bmatrix}Q & X_0P & L_A & 0\\ PX_0^* & P & 0 & PR_B\\ 0 & R_BP & 0 & 0\end{bmatrix},\\
&s_\pm=i_\pm(M_1)-r(M_3),\qquad t_\pm=i_\pm(M_2)-r(M_4).
\end{aligned}\tag{23}$$

Applying Lemmas 2 and 3, elementary matrix operations and congruence matrix operations, we obtain

$$\begin{aligned}
r(M) &= n + r\begin{bmatrix} 0 & P & P \\ AQ & CP & 0 \\ -D^* & 0 & B^* \end{bmatrix} - r(A) - r(B), \\
r(M_1) &= 2n + r(AQA^* - CPC^*) - 2r(A) + r(P), \\
i_\pm(M_1) &= n + i_\pm(AQA^* - CPC^*) - r(A) + i_\pm(P), \\
r(M_2) &= r\begin{bmatrix} Q & 0 & D \\ 0 & -P & B \\ D^* & B^* & 0 \end{bmatrix} - 2r(B) + r(P), \\
i_\pm(M_2) &= i_\pm\begin{bmatrix} Q & 0 & D \\ 0 & -P & B \\ D^* & B^* & 0 \end{bmatrix} - r(B) + i_\pm(P), \\
r(M_3) &= 2n + r(P) - 2r(A) - r(B) + r\begin{bmatrix} CP & AQA^* \\ B^* & D^*A^* \end{bmatrix}, \\
r(M_4) &= n + r(P) + r\begin{bmatrix} 0 & P & B \\ AQ & CP & 0 \\ D^* & B^* & 0 \end{bmatrix} - 2r(B) - r(A).
\end{aligned} \tag{24}$$

Substituting (24) into (22), we obtain the results. □

Using immediately Theorem 5, we can easily get the following.

Theorem 6. *Let $f(X)$ be as given in (4), $s_\pm$ and let $t_\pm$ be as given in Theorem 5 and assume that $AX = C$ and $XB = D$ in (5) are consistent. Then we have the following.*

(a) *$AX = C$ and $XB = D$ have a common solution such that $Q - XPX^* \geq 0$ if and only if*

$$\begin{aligned}
n + r\begin{bmatrix} 0 & P & P \\ AQ & CP & 0 \\ -D^* & 0 & B^* \end{bmatrix} - r(A) - r(B) - i_-(P) + s_- &\leq 0, \\
n + r\begin{bmatrix} 0 & P & P \\ AQ & CP & 0 \\ -D^* & 0 & B^* \end{bmatrix} - r(A) - r(B) - i_-(P) + t_- &\leq 0.
\end{aligned} \tag{25}$$

(b) *$AX = C$ and $XB = D$ have a common solution such that $Q - XPX^* \leq 0$ if and only if*

$$\begin{aligned}
n + r\begin{bmatrix} 0 & P & P \\ AQ & CP & 0 \\ -D^* & 0 & B^* \end{bmatrix} - r(A) - r(B) - i_+(P) + s_+ &\leq 0, \\
n + r\begin{bmatrix} 0 & P & P \\ AQ & CP & 0 \\ -D^* & 0 & B^* \end{bmatrix} - r(A) - r(B) - i_+(P) + t_+ &\leq 0.
\end{aligned} \tag{26}$$

(c) *$AX = C$ and $XB = D$ have a common solution such that $Q - XPX^* > 0$ if and only if*

$$\begin{aligned}
i_+(AQA^* - CPC^*) - r(A) &\geq 0, \\
i_+\begin{bmatrix} Q & 0 & D \\ 0 & -P & B \\ D^* & B^* & 0 \end{bmatrix} - r(B) &\geq n.
\end{aligned} \tag{27}$$

(d) *$AX = C$ and $XB = D$ have a common solution such that $Q - XPX^* < 0$ if and only if*

$$\begin{aligned}
i_-(AQA^* - CPC^*) - r(A) &\geq 0, \\
i_-\begin{bmatrix} Q & 0 & D \\ 0 & -P & B \\ D^* & B^* & 0 \end{bmatrix} - r(B) &\geq n.
\end{aligned} \tag{28}$$

(e) *All common solutions of $AX = C$ and $XB = D$ satisfy $Q - XPX^* \geq 0$ if and only if*

$$\begin{aligned}
n + i_-(AQA^* - CPC^*) - r(A) &= 0, \\
\text{or, } i_-\begin{bmatrix} Q & 0 & D \\ 0 & -P & B \\ D^* & B^* & 0 \end{bmatrix} - r(B) &= 0.
\end{aligned} \tag{29}$$

(f) *All common solutions of $AX = C$ and $XB = D$ satisfy $Q - XPX^* \leq 0$ if and only if*

$$\begin{aligned}
n + i_+(AQA^* - CPC^*) - r(A) &= 0, \\
\text{or, } i_+\begin{bmatrix} Q & 0 & D \\ 0 & -P & B \\ D^* & B^* & 0 \end{bmatrix} - r(B) &= 0.
\end{aligned} \tag{30}$$

(g) *All common solutions of $AX = C$ and $XB = D$ satisfy $Q - XPX^* > 0$ if and only if*

$$n + r\begin{bmatrix} 0 & P & P \\ AQ & CP & 0 \\ -D^* & 0 & B^* \end{bmatrix} - r(A) - r(B) - i_+(P) + s_+ = n, \tag{31}$$

or

$$n + r\begin{bmatrix} 0 & P & P \\ AQ & CP & 0 \\ -D^* & 0 & B^* \end{bmatrix} - r(A) - r(B) - i_+(P) + t_+ = n. \tag{32}$$

(h) *All common solutions of $AX = C$ and $XB = D$ satisfy $Q - XPX^* < 0$ if and only if*

$$n + r\begin{bmatrix} 0 & P & P \\ AQ & CP & 0 \\ -D^* & 0 & B^* \end{bmatrix} - r(A) - r(B) - i_-(P) + s_- = n, \tag{33}$$

or

$$n + r\begin{bmatrix} 0 & P & P \\ AQ & CP & 0 \\ -D^* & 0 & B^* \end{bmatrix} - r(A) - r(B) - i_-(P) + t_- = n. \tag{34}$$

(i) *$AX = C$, $XB = D$, and $Q = XPX^*$ have a common solution if and only if*

$$2n+2r\begin{bmatrix} 0 & P & P \\ AQ & CP & 0 \\ -D^* & 0 & B^* \end{bmatrix}-2r(A)-2r(B)-r(P)+s_+ + s_- \le 0,$$

$$2n+2r\begin{bmatrix} 0 & P & P \\ AQ & CP & 0 \\ -D^* & 0 & B^* \end{bmatrix}-2r(A)-2r(B)-r(P)+t_+ + t_- \le 0,$$

$$2n+2r\begin{bmatrix} 0 & P & P \\ AQ & CP & 0 \\ -D^* & 0 & B^* \end{bmatrix}-2r(A)-2r(B)-r(P)+s_+ + t_- \le 0,$$

$$2n+2r\begin{bmatrix} 0 & P & P \\ AQ & CP & 0 \\ -D^* & 0 & B^* \end{bmatrix}-2r(A)-2r(B)-r(P)+s_- + t_+ \le 0. \tag{35}$$

Let $P = I$ in Theorem 5, we get the following corollary.

Corollary 7. *Let $Q \in \mathbb{C}^{n\times n}$, A, B, C, and D be given. Assume that* (5) *is consistent. Denote*

$$T_1 = \begin{bmatrix} C & AQ \\ B^* & D^* \end{bmatrix}, \qquad T_2 = AQA^* - CC^*,$$
$$T_3 = \begin{bmatrix} Q & D \\ D^* & B^*B \end{bmatrix}, \qquad T_4 = \begin{bmatrix} C & AQA^* \\ B^* & D^*A^* \end{bmatrix}, \tag{36}$$
$$T_5 = \begin{bmatrix} CB & AQ \\ B^*B & D^* \end{bmatrix}.$$

Then,

$$\begin{aligned}
&\max_{AX=C,XB=D} r(Q-XX^*) \\
&\quad = \min\{n + r(T_1) - r(A) - r(B), 2n + r(T_2) \\
&\qquad -2r(A), n + r(T_3) - 2r(B)\}, \\
&\min_{AX=C,XB=D} r(Q-XX^*) \\
&\quad = 2r(T_1) + \max\{r(T_2) - 2r(T_4), -n + r(T_3) \\
&\qquad - 2r(T_5), i_+(T_2) + i_-(T_3) \\
&\qquad - r(T_4) - r(T_5), -n + i_-(T_2) \\
&\qquad + i_+(T_3) - r(T_4) - r(T_5)\} \\
&\max_{AX=C,XB=D} i_+(Q-XX^*) \\
&\quad = \min\{n + i_+(T_2) - r(A), i_+(T_3) - r(B)\}, \\
&\max_{AX=C,XB=D} i_-(Q-XX^*) \\
&\quad = \min\{n + i_-(T_2) - r(A), n + i_-(T_3) - r(B)\}, \\
&\min_{AX=C,XB=D} i_+(Q-XX^*) \\
&\quad = r(T_1) + \max\{i_+(T_2) - r(T_4), i_+(T_3) - n - r(T_5)\}, \\
&\min_{AX=C,XB=D} i_-(Q-XX^*) \\
&\quad = r(T_1) + \max\{i_-(T_2) - r(T_4), i_-(T_3) - r(T_5)\}.
\end{aligned} \tag{37}$$

Remark 8. Corollary 7 is one of the results in [24].

Let B and D vanish in Theorem 5, then we can obtain the maximal and minimal ranks and inertias of (4) subject to $AX = C$.

Corollary 9. *Let $f(X)$ be as given in* (4) *and assume that $AX = C$ is consistent. Then*

$$\begin{aligned}
&\max_{AX=C} r(Q-XPX^*) \\
&\quad = \min\{n + r[AQ \;\; CP] - r(A) - r(B), \\
&\qquad 2n + r(AQA^* - CPC^*) - 2r(A), r(Q) + r(P)\} \\
&\min_{AX=C} r(Q-XPX^*) \\
&\quad = 2n + 2r[AQ \;\; CP] - 2r(A) \\
&\qquad + \max\{s_+ + s_-, t_+ + t_-, s_+ + t_-, s_- + t_+\}, \\
&\max_{AX=C} i_\pm(Q-XPX^*) \\
&\quad = \min\{n + i_\pm(AQA^* - CPC^*) - r(A), i_\pm(Q) + i_\mp(P)\}, \\
&\min_{AX=C} i_\pm(Q-XPX^*) \\
&\quad = n + r[AQ \;\; CP] - r(A) + i_\mp(P) + \max\{s_\pm, t_\pm\},
\end{aligned} \tag{38}$$

where

$$\begin{aligned}
s_\pm &= -n + r(A) - i_\mp(P) \\
&\quad + i_\pm(AQA^* - CPC^*) - r[CP \;\; AQA^*], \\
t_\pm &= -n + r(A) + i_\pm(Q) - r(P) - [AQ \;\; CP].
\end{aligned} \tag{39}$$

Remark 10. Corollary 9 is one of the results in [22].

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