Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 971357, 7 pages
http://dx.doi.org/10.1155/2013/971357

Research Article

Oscillation Criteria for Nonlinear Fractional Differential Equations

Run Xu

Department of Mathematics, Qufu Normal University, Qufu, Shandong 273165, China

Correspondence should be addressed to Run Xu; xurun_2005@163.com

Received 5 October 2013; Accepted 3 December 2013

Academic Editor: Nazim Idrisoglu Mahmudov

Several oscillation criteria are established for nonlinear fractional differential equations of the form $\{a(t)[(r(t)D_-^\alpha x(t))']^\eta\}' - F(t, \int_t^\infty (v-t)^{-\alpha}x(v)dv) = 0$, where $D_-^\alpha x$ is the Liouville right-side fractional derivative of order $\alpha \in (0,1)$ of x and η is a quotient of two odd positive integers. We also give some examples to illustrate the main results. To the best of our knowledge, the results are initiation for the oscillatory behavior of the equations.

1. Introduction

In this paper, we are dealing with the oscillation problem of nonlinear fractional differential equations of the form

$$\left\{a(t)\left[\left(r(t)D_-^\alpha x(t)\right)'\right]^\eta\right\}' - F\left(t, \int_t^\infty (v-t)^{-\alpha}x(v)\,dv\right) = 0, \quad t \geq t_0 > 0, \tag{1}$$

where $\alpha \in (0,1)$ is a constant, η is a ratio of two odd positive integers. $D_-^\alpha x(t)$ is the Liouville right-side fractional derivative of order α of $x(t)$ defined by

$$D_-^\alpha x(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt} \times \int_t^\infty (v-t)^{-\alpha}x(v)\,dv, \quad t \in R_+ = (0,\infty). \tag{2}$$

Throughout this paper, we will suppose that the following conditions hold:

(A_1) $a(t) \in C^1([t_0,\infty);R_+)$, $r(t) \in C^2([t_0,\infty);R_+)$, $\int_{t_0}^\infty (ds/a^{1/\eta}(s)) = \infty$, $\int_{t_0}^\infty (ds/r(s)) = \infty$;

(A_2) $F(t,G) \in C([t_0,\infty)\times R;R)$, there exists a function $q(t) \in C([t_0,\infty);R_+)$ such that $F(t,G)/G^\eta \geq q(t)$ for $G \neq 0$ and $x \neq 0$, $t \geq t_0$.

By a solution of (1), we mean a function $x(t) \in C(R_+;R)$ such that $\int_t^\infty (v-t)^{-\alpha}x(v)dv \in C^1(R_+;R)$, $r(t)D_-^\alpha x(t) \in C^2(R_+;R)$ and satisfies (1) on $[T_x,\infty)$. A nontrivial solution of (1) is called oscillatory if it has arbitrary large zero. Otherwise, it is called nonoscillatory. Equation (1) is called oscillatory if all of its solutions are oscillatory.

The theory of fractional derivatives goes back to Leibniz's note in [1], and this led to the appearance of the theory of derivatives and integrals of arbitrary order. The theory had taken a more or less finished form due primarily to Liouville, Grünwald, Letnikov, and Riemann by the end of nineteenth century. We can see some of the books such as [2, 3] on the subject of fractional derivatives and fractional integrals.

Fractional differential equations are generalizations of classical differential equations of integer order. In the last few decades, many researchers found that fractional derivatives and fractional integrals were applied in widespread fields of science and engineering, especially in mathematical modeling and simulation of systems and processes, instead of simply being applied in pure theoretical fields of mathematics. Nowadays, many articles have investigated some aspects of fractional differential equations, such as the existence, the uniqueness and stability of solutions, the methods for explicit and numerical solutions, and the stability of solutions (we refer the reader to see [4–9] and the references cited in there). In very recent days, the research on oscillation of various

fractional differential equations is being a hot topic; see [10–16].

In [10], Grace et al. discussed the oscillation of the following question:

$$D_a^q x + f_1(t,x) = v(t) + f_2(t,x), \qquad \lim_{t\to a^+} J_a^{1-q} x(t) = b_1, \tag{3}$$

where D_a^q denotes the Riemann-Liouville differential operator of order q with $0 < q \le 1$, and the functions f_1, f_2, and v are continuous functions.

In [11], Chen et al. established several oscillation criteria for (3) with some additional initial conditions and $m-1 \le q \le m$, $m \ge 1$ is an integer. They improved and extended some results of [10].

In [12], Chen considered the oscillation of the fractional differential equation

$$\begin{aligned}&\left[r(t)\left(D_-^\alpha y\right)^\eta(t)\right]' \\ &\quad - q(t) f\left(\int_t^\infty (v-t)^{-\alpha} y(v)\,dv\right) = 0, \quad t > 0\end{aligned} \tag{4}$$

with $0 < \alpha \le 1$.

In [13], Han et al. have established some oscillation criteria for the equation

$$\begin{aligned}&\left[r(t)\, g\left(\left(D_-^\alpha y\right)(t)\right)\right]' \\ &\quad - p(t) f\left(\int_t^\infty (s-t)^{-\alpha} y(s)\,ds\right) = 0, \quad t > 0.\end{aligned} \tag{5}$$

In [14], Qi and Cheng studied the following equation:

$$\begin{aligned}&\left(a(t)\left[r(t) D_-^\alpha x(t)\right]'\right)' + p(t)\left[r(t) D_-^\alpha x(t)\right]' \\ &\quad - q(t)\int_t^\infty (\xi - t)^{-\alpha} x(\xi)\,d\xi = 0, \quad t > t_0\end{aligned} \tag{6}$$

with $0 < \alpha < 1$ and established some new interval oscillation criteria by using a generalized Riccati transformation and inequality technique.

In [15], Feng and Meng paid attention to the oscillation of the fractional differential equation

$$\begin{aligned}&D_t^\alpha\left[r(t)\psi(x(t)) D_t^\alpha x(t)\right] \\ &\quad + q(t) f(x(t)) = e(t), \quad t > t_0 > 0,\ 0 < \alpha < 1.\end{aligned} \tag{7}$$

In [16], Chen considered the oscillation of the fractional differential equation

$$\begin{aligned}&\left[r(t)\left(D_-^\alpha y\right)^\eta\right]' \\ &\quad - q(t) f\left(\int_t^\infty (s-t)^{-\alpha} x(s)\,ds\right) = 0, \quad t > 0,\ 0 < \alpha < 1.\end{aligned} \tag{8}$$

The purpose of this paper is to establish some oscillation criteria for (1) by generalized Riccati function and present some applications for our results.

In order to prove our theories, we use the general weighted functions from the class X. We say that a function $H = H(t,s)$ belongs to the function class X, if $H \in C(D; R_+)$, where $D = \{(t,s) : t_0 \le s \le t < \infty\}$, which satisfies $H(t,t) = 0$, $H(t,s) > 0$ for $t > s$, and has nonpositive continuous partial derivative $H_s'(t,s)$ on D.

2. Main Results

First, we set

$$G(t) = \int_t^\infty (v-t)^{-\alpha} x(v)\,dv, \tag{9}$$

then it follows that

$$G'(t) = -\Gamma(1-\alpha)\left(D_-^\alpha x\right)(t). \tag{10}$$

We give the following lemmas for our results.

Lemma 1 (see [17]). *Let X and Y be nonnegative; then*

$$mXY^{m-1} - X^m \le (m-1)Y^m \quad \text{for } m > 1. \tag{11}$$

Lemma 2. *Assume that $x(t)$ is an eventually positive solution of* (1), *and*

$$\int_{t_0}^\infty \frac{1}{r(\xi)}\int_\xi^\infty \left[\frac{1}{a(\tau)}\int_\tau^\infty q(s)ds\right]^{1/\eta} d\tau\, d\xi = \infty; \tag{12}$$

then there exists a sufficiently large T such that $(r(t)D_-^\alpha x(t))' < 0$ on $[T,\infty)$, and one of the following two conditions hold:

(i) $D_-^\alpha x(t) < 0$ *on* $[T,\infty)$,

(ii) $D_-^\alpha x(t) > 0$ *on* $[T,\infty)$ *and* $\lim_{t\to\infty} G(t) = 0$.

Proof. From the hypothesis, there exists a t_1 such that $x(t) > 0$ on $[t_1,\infty)$, so that $G(t) > 0$ on $[t_1,\infty)$, and we have

$$\begin{aligned}&\left\{a(t)\left[\left(r(t) D_-^\alpha x(t)\right)'\right]^\eta\right\}' \\ &\quad = F\left(t, \int_t^\infty (v-t)^{-\alpha} x(v)\,dv\right) \\ &\quad \ge q(t) G^\eta(t) > 0, \quad t > t_1.\end{aligned} \tag{13}$$

Then $a(t)[(r(t)D_-^\alpha x(t))']^\eta$ is strictly increasing on $[t_1,\infty)$, and we can conclude that $(r(t)D_-^\alpha x(t))'$ is eventually of one sign. We claim that $(r(t)D_-^\alpha x(t))' < 0$ on $[t_2,\infty)$, where $t_2 > t_1$ is sufficiently large. Otherwise, if there exists a $t_3 > t_2$ such that $(r(t)D_-^\alpha x(t))'|_{t_3} > 0$, then we can get $(r(t)D_-^\alpha x(t))' > 0$ on $[t_3,\infty)$; from those conditions we get

$$\begin{aligned}&a(t)\left[\left(r(t) D_-^\alpha x(t)\right)'\right]^\eta \\ &\quad \ge a(t)\left[\left(r(t) D_-^\alpha x(t)\right)'\right]^\eta\Big|_{t_3} \doteq \delta > 0, \quad t \ge t_3,\end{aligned} \tag{14}$$

that is,

$$\left(r(t) D_-^\alpha x(t)\right)' \ge \frac{\delta}{(a(t))^{1/\eta}} > 0. \tag{15}$$

Integrating two sides of the previous inequality from t_3 to t leads to

$$r(t)D_-^\alpha x(t) - r(t_3)D_-^\alpha x(t_3) \geq \delta \int_{t_3}^{t} \frac{ds}{(a(s))^{1/\eta}}. \quad (16)$$

Then from (A_1) we have $\lim_{t\to\infty} r(t)D_-^\alpha x(t) = +\infty$, which implies that for a certain constant $t_4 > t_3$, $D_-^\alpha x(t) > 0$, $t \in [t_4, \infty)$, then

$$\begin{aligned} G(t) - G(t_4) &= \int_{t_4}^{t} G'(s)\,ds \\ &= -\Gamma(1-\alpha)\int_{t_4}^{t} \frac{r(s)D_-^\alpha x(s)}{r(s)}ds \\ &\leq -\Gamma(1-\alpha)\, r(t_4)D_-^\alpha x(t_4)\int_{t_4}^{t}\frac{ds}{r(s)}; \end{aligned} \quad (17)$$

by (A_1) we obtain $\lim_{t\to\infty} G(t) = -\infty$, which contradicts to $G(t) > 0$ on $[t_1, \infty)$.

So we have $(r(t)D_-^\alpha x(t))' < 0$ on $[t_2, \infty)$, and $D_-^\alpha x(t)$ is eventually of one sign. There are two possibilities: (i) $D_-^\alpha x(t) < 0$ on $[T, \infty)$, (ii) $D_-^\alpha x(t) > 0$ on $[T, \infty)$, where T is sufficiently large.

Assume that $D_-^\alpha x(t) > 0$, $t \in [t_5, \infty)$ for certain sufficiently large constant $t_5 > t_4$; then $G'(t) < 0$, $t \in [t_5, \infty)$ and we have $\lim_{t\to\infty} G(t) = \beta \geq 0$. We claim that $\beta = 0$. Otherwise, let $\beta > 0$; then $G(t) \geq \beta$ on $[t_5, \infty)$, and by (13), we have

$$\left\{a(t)\left[\left(r(t)D_-^\alpha x(t)\right)'\right]^\eta\right\}' \geq q(t)G^\eta(t) \geq q(t)\beta^\eta, \quad t \in [t_5, \infty). \quad (18)$$

Integrating two sides of above inequality from t to ∞ leads to

$$\begin{aligned} &-a(t)\left[\left(r(t)D_-^\alpha x(t)\right)'\right]^\eta \\ &\geq \int_t^\infty q(s)\beta^\eta ds - \lim_{t\to\infty} a(t)\left[\left(r(t)D_-^\alpha x(t)\right)'\right]^\eta \\ &\geq \beta^\eta \int_t^\infty q(s)\,ds, \end{aligned} \quad (19)$$

which means

$$\begin{aligned} &\left(r(t)D_-^\alpha x(t)\right)' \\ &\quad \leq -\beta\left[\frac{1}{a(t)}\int_t^\infty q(s)\,ds\right]^{1/\eta}, \quad t \in [t_5, \infty). \end{aligned} \quad (20)$$

Integrating two sides of (20) from t to ∞ yields

$$\begin{aligned} &-r(t)D_-^\alpha x(t) \\ &\leq -\lim_{t\to\infty} r(t)D_-^\alpha x(t) - \beta\int_t^\infty\left[\frac{1}{a(\tau)}\int_\tau^\infty q(s)\,ds\right]^{1/\eta}d\tau \\ &\leq -\beta\int_t^\infty\left[\frac{1}{a(\tau)}\int_\tau^\infty q(s)\,ds\right]^{1/\eta}d\tau, \quad t \in [t_5, \infty). \end{aligned} \quad (21)$$

That is,

$$G'(t) \leq -\beta\Gamma(1-\alpha)\frac{1}{r(t)}\int_t^\infty\left[\frac{1}{a(\tau)}\int_\tau^\infty q(s)ds\right]^{1/\eta}d\tau. \quad (22)$$

Integrating two sides of (22) from t_5 to t, we have

$$\begin{aligned} &G(t) - G(t_5) \\ &\quad \leq -\beta\Gamma(1-\alpha) \\ &\quad \times \int_{t_5}^{t}\frac{1}{r(\xi)} \\ &\quad \times \left\{\int_\xi^\infty\left[\frac{1}{a(\tau)}\int_\tau^\infty q(s)\,ds\right]^{1/\eta}d\tau\right\}d\xi. \end{aligned} \quad (23)$$

By (12), we have $\lim_{t\to\infty} G(t) = -\infty$, which contradicts to the fact that $G(t) > 0$. Then we get that $\beta = 0$, which is $\lim_{t\to\infty} G(t) = 0$. The proof is complete. □

Lemma 3. *Assume that $x(t)$ is an eventually positive solution of (1) such that $(r(t)D_-^\alpha x(t))' < 0$, $D_-^\alpha x(t) < 0$ on $[t_1, \infty)$, where t_1 is sufficiently large and $t_1 > t_0$. Then one has*

$$G'(t) \geq -\Gamma(1-\alpha)\frac{1}{r(t)}A(t_1, t)\,a^{1/\eta}(t)\left(r(t)D_-^\alpha x(t)\right)', \quad (24)$$

where $A(t_1, t) = \int_{t_1}^{t} a^{-(1/\eta)}(s)ds$.

Proof. From (13), we get that $a(t)[(r(t)D_-^\alpha x(t))']^\eta$ is strictly increasing on $[t_1, \infty)$, so we get

$$\begin{aligned} &r(t)D_-^\alpha x(t) \\ &\quad \leq r(t)D_-^\alpha x(t) - r(t_1)D_-^\alpha x(t_1) \\ &\quad = \int_{t_1}^{t}\frac{a^{1/\eta}(s)\left(r(s)D_-^\alpha x(s)\right)'}{a^{1/\eta}(s)}ds \\ &\quad \leq a^{1/\eta}(t)\left(r(t)D_-^\alpha x(t)\right)'\int_{t_1}^{t}\frac{ds}{a^{1/\eta}(s)} \\ &\quad = A(t)\,a^{1/\eta}(t)\left(r(t)D_-^\alpha x(t)\right)'. \end{aligned} \quad (25)$$

Then we get

$$G'(t) \geq -\Gamma(1-\alpha)\frac{1}{r(t)}A(t)\,a^{1/\eta}(t)\left(r(t)D_-^\alpha x(t)\right)'. \quad (26)$$

□

Theorem 4. *Assume that (A_1), (A_2), and (12) hold; if there exists a function $\rho \in C^1([t_0, \infty); R_+)$ such that*

$$\begin{aligned} &\int_T^\infty\left[\rho(s)q(s) - \frac{1}{(\eta+1)^{\eta+1}}\left(\frac{\rho_+'(s)}{\rho(s)}\right)^{\eta+1}\right. \\ &\qquad \left.\times\frac{\rho(s)r^\eta(s)}{\left[\Gamma(1-\alpha)A(t_2, s)\right]^\eta}\right]ds = \infty \end{aligned} \quad (27)$$

for all sufficiently large constants t_2 and T, where $\rho'_+(s) = \max\{\rho'(s), 0\}$, $A(t_2, t)$ is defined in Lemma 3, $t \geq T \geq t_2 \geq t_0 > 0$, then every solution of (1) *is oscillatory or satisfies* $\lim_{t\to\infty} G(t) = 0$.

Proof. Suppose to the contrary that (1) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)$; without loss of generality, we assume that $x(t) > 0$ on $[t_1, \infty)$, where t_1 is sufficiently large. By Lemma 2 we have $(r(t)D^{\alpha}_{-}x(t))' < 0$ on $[t_2, \infty)$, where t_2 is sufficiently large and $t_2 > t_1$, and either $D^{\alpha}_{-}x(t) < 0$ on $[t_2, \infty)$ or $\lim_{t\to\infty} G(t) = 0$. (In the case $x(t)$ is eventually negative, it can be proved similarly.)

If $D^{\alpha}_{-}x(t) < 0$ on $[t_2, \infty)$, then the conclusion of Lemma 3 holds. We define

$$w(t) = -\rho(t)\frac{a(t)\left[\left(r(t)D^{\alpha}_{-}x(t)\right)'\right]^{\eta}}{G^{\eta}(t)}, \tag{28}$$

then $w(t) > 0$ on $[t_2, \infty)$, and from (1), (A_2), and Lemma 3, we have

$$\begin{aligned}
w'(t) &= \frac{\rho'(t)}{\rho(t)}w(t) \\
&\quad - \rho(t)\left(\left(\left\{a(t)\left[\left(r(t)D^{\alpha}_{-}x(t)\right)'\right]^{\eta}\right\}' G^{\eta}(t)\right.\right. \\
&\qquad \left.\left. -a(t)\left[\left(r(t)D^{\alpha}_{-}x(t)\right)'\right]^{\eta}\eta G^{\eta-1}(t)G'(t)\right) \\
&\qquad \times\left(G^{2\eta}(t)\right)^{-1}\Bigg) \\
&\leq \frac{\rho'_+(t)}{\rho(t)}w(t) - \rho(t)\frac{F(t, G(t))}{G^{\eta}(t)} \\
&\quad + \eta\rho(t)\left(\left(a(t)\left[\left(r(t)D^{\alpha}_{-}x(t)\right)'\right]^{\eta}\right.\right. \\
&\qquad \times\left[-\Gamma(1-\alpha)\frac{A(t_2, t)}{r(t)}a^{1/\eta}(t)\right. \\
&\qquad \left.\left. \times\left(r(t)D^{\alpha}_{-}x(t)\right)'\right]\right)\times\left(G^{\eta+1}(t)\right)^{-1}\Bigg) \\
&\leq \frac{\rho'_+(t)}{\rho(t)}w(t) - \rho(t)q(t) - \eta\Gamma(1-\alpha)\rho(t) \\
&\quad \times\frac{A(t_2, t)}{r(t)}\frac{a^{(\eta+1)/\eta}(t)\left[\left(r(t)D^{\alpha}_{-}x(t)\right)'\right]^{\eta+1}}{G^{\eta+1}(t)} \\
&= \frac{\rho'_+(t)}{\rho(t)}w(t) - \rho(t)q(t) \\
&\quad - \eta\Gamma(1-\alpha)\frac{A(t_2, t)}{r(t)}\frac{1}{\rho^{1/\eta}(t)}w^{(\eta+1)/\eta}(t).
\end{aligned} \tag{29}$$

Let $m = 1 + (1/\eta)$, $X = [\eta\Gamma(1-\alpha)A(t_2, t)/\rho^{1/\eta}(t)r(t)]^{1/m} w(t)$, $Y^{m-1} = (\eta/(\eta+1))(\rho'_+(t)/\rho(t))\rho^{1/\eta}(t)r(t)/\eta\Gamma(1-\alpha)A(t)]^{\eta/(\eta+1)}$; apply Lemma 1 to (29), we get

$$\begin{aligned}
w'(t) \leq & -\rho(t)q(t) + \frac{1}{(\eta+1)^{\eta+1}} \\
& \times\left(\frac{\rho'_+(t)}{\rho(t)}\right)^{\eta+1}\frac{\rho(t)r^{\eta}(t)}{\left[\Gamma(1-\alpha)A(t_2, t)\right]^{\eta}}.
\end{aligned} \tag{30}$$

Integrating two sides of (30) from t_2 to t, we have

$$\begin{aligned}
&\int_{t_2}^{t}\left[\rho(s)q(s) - \frac{1}{(\eta+1)^{\eta+1}}\left(\frac{\rho'_+(s)}{\rho(s)}\right)^{\eta+1}\right. \\
&\qquad \left.\times\frac{\rho(s)r^{\eta}(s)}{\left[\Gamma(1-\alpha)A(t_2, s)\right]^{\eta}}\right] \\
&\qquad \leq w(t_2) - w(t) \leq w(t_2),
\end{aligned} \tag{31}$$

which contradicts to (27).

If $D^{\alpha}_{-}x(t) > 0$ on $[t_2, \infty)$, then from Lemma 2, we get that $\lim_{t\to\infty} G(t) = 0$. This completes the proof. □

Corollary 5. *Assume that* (A_1), (A_2), *and* (12) *hold; if there exists a function* $\rho \in C^1([t_0, \infty); R_+)$ *such that*

$$\begin{aligned}
&\int_{T}^{\infty}\rho(s)q(s)\,ds = \infty, \\
&\int_{T}^{\infty}\left(\frac{\rho'_+(s)}{\rho(s)}\right)^{\eta+1}\frac{\rho(s)r^{\eta}(s)}{A^{\eta}(t_2, s)}ds < \infty
\end{aligned} \tag{32}$$

for a sufficiently large constant T, where $\rho'_+(s) = \max\{\rho'(s), 0\}$, $A(t_2, t)$ is defined in Lemma 3, then every solution of (1) *is oscillatory or satisfies* $\lim_{t\to\infty} G(t) = 0$.

Theorem 6. *Assume that* (A_1), (A_2), *and* (12) *hold; if there exist functions* $\rho \in C^1([t_0, \infty); R_+)$ *and* $H(t, s) \in X$ *such that*

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H(t, t_0)} \\
&\quad \times\int_{t_2}^{t}\rho(s)\left\{H(t, s)q(s)\right. \\
&\qquad \left. -\frac{r^{\eta}(s)h_+^{\eta+1}(t, s)}{(\eta+1)^{\eta+1}\left[\Gamma(1-\alpha)A(t_2, s)H(t, s)\right]^{\eta}}\right\}ds \\
&= \infty,
\end{aligned} \tag{33}$$

where t_2 is sufficiently large, $t \geq t_2 \geq t_0$, $A(t_2, t)$ is defined in Lemma 3, $h(t, s) = H(t, s)(\rho'_+(s)/\rho(s)) + H'_s(t, s)$, $h_+(t, s) = \max\{0, h(t, s)\}$, then every solution of (1) *is oscillatory or satisfies* $\lim_{t\to\infty} G(t) = 0$.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1); without loss of generality, we let $x(t)$ be an eventually positive solution of (1). According to the proof of Theorem 4, if $D_-^\alpha x(t) < 0$ on $[t_2, \infty)$, then (29) holds. Multiplying two sides of (29) by $H(t,s)$ and integrating it from t_2 to t, we get

$$\begin{aligned}
&\int_{t_2}^{t} \rho(s) q(s) H(t,s)\, ds \\
&\quad \le -\int_{t_2}^{t} H(t,s) w'(s)\, ds + \int_{t_2}^{t} H(t,s) \frac{\rho_+'(s)}{\rho(s)} w(s)\, ds \\
&\qquad - \eta\Gamma(1-\alpha) \int_{t_2}^{t} H(t,s) \frac{A(t_2,s)}{\rho^{1/\eta}(s) r(s)} w^{(\eta+1)/\eta}(s)\, ds \\
&\quad \le H(t,t_2) w(t_2) \\
&\qquad + \int_{t_2}^{t} \left[\left(H(t,s) \frac{\rho_+'(s)}{\rho(s)} + H_s'(t,s) \right) w(s) \right. \\
&\qquad\qquad \left. - \frac{\eta\Gamma(1-\alpha) A(t_2,s) H(t,s)}{\rho^{1/\eta}(s) r(s)} w^{(\eta+1)/\eta}(s) \right] ds \\
&\quad \le H(t,t_2) w(t_2) \\
&\qquad + \int_{t_2}^{t} \left[h_+(t,s) w(s) \right. \\
&\qquad\qquad \left. - \frac{\eta\Gamma(1-\alpha) A(t_2,s) H(t,s)}{\rho^{1/\eta}(s) r(s)} w^{(\eta+1)/\eta}(s) \right] ds.
\end{aligned} \tag{34}$$

Taking

$$\begin{aligned}
m &= 1 + \frac{1}{\eta}, \\
X &= \left[\frac{\eta\Gamma(1-\alpha) A(t_2,s) H(t,s)}{\rho^{1/\eta}(s) r(s)} \right]^{\eta/(\eta+1)} w(s), \\
Y^{m-1} &= \frac{\eta}{\eta+1} \\
&\quad \times \frac{\rho(s) r^{\eta/(\eta+1)}(s)}{[\eta\Gamma(1-\alpha) A(t_2,s) H(t,s)]^{\eta/(\eta+1)}} h_+(t,s),
\end{aligned} \tag{35}$$

applying Lemma 1 to (34), we obtain that for $t \ge t_2$,

$$\begin{aligned}
&\int_{t_2}^{t} \rho(s) q(s) H(t,s)\, ds \\
&\quad \le H(t,t_2) w(t_2) + \frac{1}{(\eta+1)^{\eta+1} \Gamma^{\eta}(1-\alpha)} \\
&\qquad \times \int_{t_2}^{t} h_+^{\eta+1}(t,s) \frac{\rho(s) r^{\eta}(s)}{(A(t_2,s) H(t,s))^{\eta}} ds.
\end{aligned} \tag{36}$$

Then we have

$$\begin{aligned}
&\int_{t_2}^{t} \left[\rho(s) q(s) H(t,s) \right. \\
&\qquad \left. - \frac{\rho(s) r^{\eta}(s) h_+^{\eta+1}(t,s)}{(\eta+1)^{\eta+1} [\Gamma(1-\alpha) A(s) H(t,s)]^{\eta}} \right] ds \\
&\quad \le H(t,t_2) w(t_1) \le H(t,t_0) w(t_2).
\end{aligned} \tag{37}$$

From above inequality we have

$$\begin{aligned}
&\frac{1}{H(t,t_0)} \\
&\quad \times \int_{t_2}^{t} \left[\rho(s) q(s) H(t,s) \right. \\
&\qquad \left. - \frac{r^{\eta}(s) h_+^{\eta+1}(t,s)}{(\eta+1)^{\eta+1} [\Gamma(1-\alpha) A(t_2,s) H(t,s)]^{\eta}} \right] ds \\
&\quad \le w(t_2).
\end{aligned} \tag{38}$$

Letting $t \to \infty$ we have

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \\
&\quad \times \int_{t_2}^{t} \left\{ \rho(s) q(s) H(t,s) \right. \\
&\qquad \left. - \frac{r^{\eta}(s) h_+^{\eta+1}(t,s)}{(\eta+1)^{\eta+1} [\Gamma(1-\alpha) A(t_2,s) H(t,s)]^{\eta}} \right\} ds \\
&\quad < \infty,
\end{aligned} \tag{39}$$

which contradicts to (33). If $D_-^\alpha x(t) > 0$ on $[t_2, \infty)$, from Lemma 2, we have $\lim_{t\to\infty} G(t) = 0$. So the proof is complete. □

Corollary 7. *Assume that* (A_1), (A_2), *and* (12) *hold; if there exist functions* $\rho \in C^1([t_0,\infty); R_+)$ *and* $H(t,s) \in X$ *such that*

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_2}^{t} \rho(s) q(s) H(t,s)\, ds = \infty, \\
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_2}^{t} \frac{\rho(s) r^{\eta}(s) h_+^{\eta+1}(t,s)}{[A(t_2,s) H(t,s)]^{\eta}} ds < \infty,
\end{aligned} \tag{40}$$

where t_2 *is sufficiently large,* $t \ge t_2 \ge t_0$, $A(t_2,t)$ *is defined in Lemma 3, then every solution of* (1) *is oscillatory or satisfies* $\lim_{t\to\infty} G(t) = 0$.

With an appropriate choice of the functions H and ρ, one can derive from Theorem 6 a number of oscillation criteria for (1). For example, let $H(t,s) = \ln(t/s)$, $(t,s) \in D$, and $\rho(t) = t^{\mu}$; then $h(t,s) = (1/s)(\mu \ln(t/s) - 1)$, $h_+(t,s) = \max\{h(t,s), 0\}$. Based on the above results we obtain the following corollary.

Corollary 8. *Assume that* (A_1), (A_2), *and* (12) *hold; if*

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{\ln t}\\&\quad\times\int_{t_2}^{t}s^{\mu}\left[q(s)\ln\frac{t}{s}\right.\\&\qquad\left.-\frac{r^{\eta}(s)\,h_{+}(t,s)^{\eta+1}}{(\eta+1)^{\eta+1}\Gamma^{\eta}(1-\alpha)\left(A(t_2,s)\ln(t/s)\right)^{\eta}}\right]ds\\&=\infty,\end{aligned}\tag{41}$$

where t_2 *is sufficiently large,* $t\geq t_2\geq t_0$, $A(t_2,t)$ *is defined in Lemma 3, then every solution of* (1) *is oscillatory or satisfies* $\lim_{t\to\infty}G(t)=0$.

Corollary 9. *Assume that* (A_1), (A_2), *and* (12) *hold; if*

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{\ln t}\int_{t_2}^{t}s^{\mu}q(s)\ln\frac{t}{s}ds=\infty,\\&\limsup_{t\to\infty}\frac{1}{\ln t}\int_{t_2}^{t}\frac{s^{\mu}r^{\eta}(s)\left[(1/s)\left(\mu\ln(t/s)-1\right)\right]^{\eta+1}}{\left(A(t_2,s)\ln(t/s)\right)^{\eta}}ds<\infty,\end{aligned}\tag{42}$$

where t_2 *is sufficiently large,* $t\geq t_2\geq t_0$, $A(t_2,t)$ *is defined in Lemma 3, then every solution of* (1) *is oscillatory or satisfies* $\lim_{t\to\infty}G(t)=0$.

3. Examples

Example 1. Consider the fractional differential equation

$$\begin{aligned}&\left\{t^{\eta/2}\left[\left(D_{-}^{\alpha}x(t)\right)'\right]^{\eta}\right\}'\\&\quad-F\left(t,\int_{t}^{\infty}(v-t)^{-\alpha}x(v)\,dv\right)=0,\quad t>t_0,\end{aligned}\tag{43}$$

where $\alpha\in(0,1)$ and η is a quotient of two odd positive integers.

Here, $a(t)=t^{\eta/2}$, $r(t)=1$, $F(t,G)/G^{\eta}\geq q(t)$ for $G\neq 0$ and $x\neq 0$, $t\geq t_0$, $q(t)\geq t^{-(\eta+1)}$.

We set $\rho(t)=t^{\mu}$; if t is sufficiently large, then $A(t_1,t)=\int_{t_1}^{t}(ds/s^{1/2})=2(\sqrt{t}-\sqrt{t_1})\geq\sqrt{t}$, and $\int_{t_0}^{\infty}(ds/a^{1/\eta}(s))=\int_{t_0}^{\infty}(ds/s^{1/2})=\infty$, $\int_{t_0}^{\infty}(ds/r(s))=\infty$. Consider

$$\begin{aligned}&\int_{t_0}^{\infty}\frac{1}{r(\xi)}\int_{\xi}^{\infty}\left[\frac{1}{a(\tau)}\int_{\tau}^{\infty}q(s)\,ds\right]^{1/\eta}d\tau\,d\xi\\&\quad\geq\int_{t_0}^{\infty}\int_{\xi}^{\infty}\left[\frac{1}{\tau^{\eta/2}}\int_{\tau}^{\infty}s^{-(\eta+1)}ds\right]^{1/\eta}d\tau\,d\xi=\infty,\end{aligned}\tag{44}$$

so (A_1), (A_2), and (12) hold.

Furthermore, for a sufficiently large constant T and $\eta<\mu<(3/2)\eta$, we have

$$\begin{aligned}&\int_{T}^{\infty}\rho(s)\,q(s)\geq\int_{T}^{\infty}s^{\mu}s^{-(\eta+1)}ds=\infty,\\&\int_{T}^{\infty}\left(\frac{\rho_{+}'(s)}{\rho(s)}\right)^{\eta+1}\frac{\rho(s)\,r^{\eta}(s)}{A(t_1,s)^{\eta}}ds\\&\quad\leq\int_{T}^{\infty}\left(\frac{\mu}{s}\right)^{\eta+1}\frac{s^{\mu}}{s^{\eta/2}}ds=\mu^{\eta+1}\frac{s^{\mu-(3/2)\eta}}{\mu-(3/2)\,\eta}\bigg|_{T}^{\infty}<\infty.\end{aligned}\tag{45}$$

So the conditions of Corollary 5 hold, and we deduce that every solution of (43) is oscillatory or satisfies $\lim_{t\to\infty}G(t)=0$.

Example 2. Consider the fractional differential equation

$$\begin{aligned}&\left\{t^{\eta}\left[\left(\frac{\ln t}{t}D_{-}^{\alpha}x(t)\right)'\right]^{\eta}\right\}'\\&\quad-F\left(t,\int_{t}^{\infty}(v-t)^{-\alpha}x(v)\,dv\right)=0,\quad t>t_0,\end{aligned}\tag{46}$$

where $\alpha\in(0,1)$ and η is a quotient of two odd positive integers.

Here, $a(t)=t^{\eta}$, $r(t)=\ln t/t$, $F(t,G)/G^{\eta}\geq q(t)$ for $G\neq 0$ and $x\neq 0$, $t\geq t_0$, $q(t)\geq t^{-(\eta+1)}$.

We set $\rho(t)=1$, $H(t,s)=\ln(t/s)$; then $h(t,s)=(1/s)(\mu\ln(t/s)-1)$, and $\int_{t_0}^{\infty}(ds/a^{1/\eta}(s))=\int_{t_0}^{\infty}(ds/s)=\infty$, $\int_{t_0}^{\infty}(ds/r(s))=\int_{t_0}^{\infty}(s\,ds/\ln s)=\infty$.

May as well we assume that $t_2<\sqrt{t}$; then we have $A(t_2,t)=\ln(t/t_2)>(1/2)\ln t$. By calculation we have

$$\begin{aligned}&\int_{t_0}^{\infty}\frac{1}{r(\xi)}\int_{\xi}^{\infty}\left[\frac{1}{a(\tau)}\int_{\tau}^{\infty}q(s)ds\right]^{1/\eta}d\tau\,d\xi\\&\quad\geq\int_{t_0}^{\infty}\frac{\xi}{\ln\xi}\int_{\xi}^{\infty}\left[\frac{1}{\tau^{\eta}}\int_{\tau}^{\infty}s^{-(\eta+1)}ds\right]^{1/\eta}d\tau\,d\xi=\infty,\end{aligned}\tag{47}$$

so (A_1), (A_2), and (12) hold.

Furthermore,

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{\ln t}\int_{t_0}^{t}s^{\mu}\ln\frac{t}{s}q(s)\,ds\\&\quad\geq\limsup_{t\to\infty}\frac{1}{\ln t}\int_{t_0}^{t}s^{\mu-\eta-1}\ln\frac{t}{s}ds=\infty,\end{aligned}\tag{48}$$

and for $\eta<\mu<2\eta$,

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{\ln t}\\&\quad\times\int_{t_2}^{t}\frac{s^{\mu}r^{\eta}(s)\left[(1/s)\left(\mu\ln(t/s)-1\right)\right]^{(\eta+1)}}{\left(A(t_2,s)\ln(t/s)\right)^{\eta}}ds\end{aligned}$$

$$
\begin{aligned}
&\leq \limsup_{t\to\infty}\frac{1}{\ln t}\\
&\quad\times\int_{t_2}^{t}\frac{s^{\mu}\ln^{\eta}s\left[(1/s)\left(\mu\ln(t/s)-1\right)\right]^{(\eta+1)}}{s^{\eta}((\ln s/2)\ln(t/s))^{\eta}}ds\\
&\leq 2^{(1/\eta)}\limsup_{t\to\infty}\frac{1}{\ln t}\int_{t_0}^{t}\frac{s^{\mu-2\eta-1}\left[\mu\ln(t/s)\right]^{\eta+1}}{(\ln(t/s))^{\eta}}ds\\
&= 2^{(1/\eta)}\limsup_{t\to\infty}\frac{\mu^{\eta+1}}{\ln t}\int_{t_0}^{t}s^{\mu-2\eta-1}\ln\frac{t}{s}ds\\
&= 2^{(1/\eta)}\limsup_{t\to\infty}\frac{\mu^{\eta+1}}{\ln t}\frac{1}{\mu-2\eta}\\
&\quad\times\left(-\ln\frac{t}{t_0}t_0^{\mu-2\eta}+\frac{1}{\mu-2\eta}\left(t^{\mu-2\eta}-t_0^{\mu-2\eta}\right)\right)<\infty,
\end{aligned}
\tag{49}
$$

the conditions of Corollary 9 hold, and we get that every solution of (46) is oscillatory or satisfies $\lim_{t\to\infty}G(t)=0$.

Acknowledgment

This research is supported by National Science Foundation of China (11171178 and 11271225). The authors sincerely thank the reviewers for their valuable suggestions and useful comments.

References

[1] G. W. Leibniz, *Mathmatische Schiften*, Grorg Olms Verlagsbuchhandlung, Hildesheim, Germany, 1962.

[2] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, Calif, USA, 1999.

[3] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science BV, Amsterdam, The Netherlands, 2006.

[4] A. Saadatmandi and M. Dehghan, "A new operational matrix for solving fractional-order differential equations," *Computers and Mathematics with Applications*, vol. 59, no. 3, pp. 1326–1336, 2010.

[5] Y. Zhou, F. Jiao, and J. Li, "Existence and uniqueness for P-type fractional neutral differential equations," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 71, no. 7-8, pp. 2724–2733, 2009.

[6] L. Galeone and R. Garrappa, "Explicit methods for fractional differential equations and their stability properties," *Journal of Computational and Applied Mathematics*, vol. 228, no. 2, pp. 548–560, 2009.

[7] W. Deng, "Smoothness and stability of the solutions for nonlinear fractional differential equations," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 72, no. 3-4, pp. 1768–1777, 2010.

[8] J. T. Edwards, N. J. Ford, and C. A. Simpson, "The numerical solution of linear multi-term fractional differential equations: systems of equations," *Journal of Computational and Applied Mathematics*, vol. 148, no. 2, pp. 401–418, 2002.

[9] M. Muslim, "Existence and approximation of solutions to fractional differential equations," *Mathematical and Computer Modelling*, vol. 49, no. 5-6, pp. 1164–1172, 2009.

[10] S. R. Grace, R. P. Agarwal, P. J. Y. Wong, and A. Zafer, "On the oscillation of fractional differential equations," *Fractional Calculus and Applied Analysis*, vol. 15, no. 2, pp. 222–231, 2012.

[11] D. X. Chen, P. X. Qu, and Y.-H. Lan, "Forced oscillation of certain fractional differential equations," *Advances in Difference Equations*, vol. 2013, p. 125, 2013.

[12] D. X. Chen, "Oscillation criteria of fractional differential equations," *Advances in Difference Equations*, vol. 33, pp. 1–18, 2012.

[13] Z. Han, Y. Zhao, Y. Sun, and C. Zhang, "Oscillation for a class of fractional differential equation," *Discrete Dynamics in Nature and Society*, vol. 2013, Article ID 390282, 6 pages, 2013.

[14] C. Qi and J. Cheng, "Interval oscillation criteria for a class of fractional differential equations with damping term," *Mathematical Problems in Engineering*, vol. 2013, Article ID 301085, 8 pages, 2013.

[15] Q. Feng and F. Meng, "Oscillation of solutions to nonlinear forced fractional differential equation," *Electronic Journal of Differential Equations*, vol. 2013, no. 169, pp. 1–10, 2013.

[16] D. Chen, "Oscillatory behavior of a class of fractional differential equations with damping," *Universitatea Politehnica din Bucuresti Scientific Bulletin A*, vol. 75, no. 1, pp. 107–118, 2013.

[17] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, vol. 2, Cambridge University Press, Cambridge, UK, 1988.