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Research Article

M/M/1 Retrial Queue with Working Vacation Interruption and Feedback under N-Policy

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We consider an M/M/1 retrial queue with working vacations, vacation interruption, Bernoulli feedback, and N-policy simultaneously. During the working vacation period, customers can be served at a lower rate. Using the matrix-analytic method, we get the necessary and sufficient condition for the system to be stable. Furthermore, the stationary probability distribution and some performance measures are also derived. Moreover, we prove the conditional stochastic decomposition for the queue length in the orbit. Finally, we present some numerical examples and use the parabolic method to search the optimum value of service rate in working vacation period.

1. Introduction

In the queueing theory, vacation queues and retrial queues have been intensive research topics; we can find general models in Tian and Zhang [1] and Artalejo and Gómez-Corral [2]. In 2002, Servi and Finn [3] first introduced working vacation policy and studied an M/M/1/WV queue. Their work is motivated and illustrated by the analysis of a WDM optical access network using multiple wavelengths which can be reconfigured. The study of queueing system with working vacations can also provide the theory and analysis method to design the optimal lower speed period. Wu and Takagi [4] extended the M/M/1/WV queue to an M/G/1/WV queue. Using the matrix-analytic method, Baba [5] considered a GI/M/1 queue with working vacations. Krishnamoorthy and Sreenivasan [6, 7] analyzed an M/M/2 queue with working vacations.

Furthermore, during the working vacation period, if there are customers at a service completion instant, the server can stop the vacation and come back to the normal working level. This policy is called vacation interruption. In some practical situations, the server can take service in the vacation period and must come back to work at times. For example, when the number of customers exceeds the special value and if the server continues to take the vacation, the costs of waiting customers and providing service in the vacation period will be large. In 2007, Li and Tian [8] first introduced vacation interruption policy and studied an M/M/1 queue. Next, Li et al. [9] analyzed the GI/M/1 queue. Using the method of a supplementary variable, Zhang and Hou [10] considered an M/G/1 queue with working vacations and vacation interruption. Sreenivasan et al. [11] studied an MAP/PH/1 queue with working vacations, vacation interruption, and N-policy.

Retrial queueing systems are described by the feature that the arriving customers who find the server busy join the retrial orbit to try their requests again. Retrial queues are widely and successfully used as mathematical models of several computer systems and telecommunication networks. For example, peripherals in computer systems may make retrials to receive service from a central processor. Choi et al. [12] analyzed an M/M/1 retrial queue with general retrial times. Martin and Gómez-Corral [13] considered an M/G/1 retrial queue with liner control policy. Lillo [14] investigated a G/M/1 retrial queue. Sherman and Kharoufeh [15] studied an M/M/1 retrial queue with unreliable server.

One additional feature which has been widely discussed in retrial queueing systems is the Bernoulli feedback of customers. The phenomena of feedback in retrial queueing systems occurred in many practical situations. For example, the retrial queue with feedback can be used to model the Automative Repeat Request protocol in a high frequency communication network. Falin [16] studied an M/M/1 retrial queue with feedback. Kumar et al. [17] investigated an M/G/1 retrial queue with feedback and starting failures. Ke and Chang [18] considered a modified vacation policy for the M/G/1 retrial queue with balking and feedback. Kumar et al. [19] discussed an M/M/1 retrial queue with feedback and collisions. Kumar et al. [20] analyzed an M/G/1 retrial queue with feedback and negative customers.

Do [21] first studied an M/M/1 retrial queue with working vacations. Zhang and Xu [22] considered an M/M/1/WV queue with N-policy. On the basis of the model in [21, 22], we study an M/M/1 retrial queue with working vacations, vacation interruption, Bernoulli feedback, and N-policy simultaneously. Compared with the model in [23], we will see that the infinitesimal generator $\widetilde{Q}$ is different, and in our model, $\pi_{k0} = 0$, $k \geq N+1$.

This paper is organized as follows. In Section 2, we introduce the model and obtain the infinitesimal generator. In Section 3, we derive the stability condition and the stationary probability distribution. Some important and interesting performance measures are also given. In Section 4, we prove the conditional stochastic decomposition for the queue length given that the server is busy and there are at least N customers in the orbit. Some numerical results are presented in Section 5. Using the parabolic method, a cost minimization problem is also analyzed. Finally, Section 6 concludes this paper.

2. Quasi Birth and Death (QBD) Process Model

In this paper, we consider an M/M/1 retrial queue with working vacation interruption and feedback under N-policy. The detailed description of this model is given as follows.

(1) The interarrival times of customers are exponentially distributed with parameter λ. Upon the arrival of customers, if the server is busy, customers are forced to wait in the orbit of infinite size. If the server is not occupied, arriving customers get service immediately.

(2) Request retrials from the orbit follow a Poisson process with rate α. Upon the arrival of requests, if the server is busy, the retrial customers come back to the orbit. If the server is free, on the other hand, requests get service immediately.

(3) The service time S_b in regular busy period is governed by an exponential distribution with parameter μ. The service time S_v in working vacation period follows an exponential distribution with parameter η.

(4) The server begins a working vacation each time when the system becomes empty, and the vacation time follows an exponential distribution with parameter θ. When a vacation ends, if there are at least N customers in the orbit, the server switches to the normal working level. Otherwise, the server begins another vacation.

(5) In the working vacation period, if there are at least N customers in the orbit at a service completion instant, the server will stop the vacation and come back to the normal busy period, which means that vacation interruption happens. If the number of customers in the orbit is less than N, the server will continue the vacation.

(6) When a customer completes his/her service, s/he may leave the system with probability p $(0 < p \leq 1)$ or join the retrial group for another service with probability $\overline{p}$ $(\overline{p} = 1 - p)$.

We assume that interarrival times, interretrial times, service times, and vacation times are mutually independent. Let $Q(t)$ be the number of customers in the orbit at time t, and let $J(t)$ be the state of server at time t. There are four possible states of the server as follows:

$$J(t) = \begin{cases} 0, & \begin{array}{c}\text{the server is in a working vacation period}\\ \text{at time } t \text{ and the server is free,}\end{array} \\ 1, & \begin{array}{c}\text{the server is in a working vacation period}\\ \text{at time } t \text{ and the server is busy,}\end{array} \\ 2, & \begin{array}{c}\text{the server is during a normal service}\\ \text{period at time } t \text{ and the server is free,}\end{array} \\ 3, & \begin{array}{c}\text{the server is during a normal service}\\ \text{period at time } t \text{ and the server is busy.}\end{array} \end{cases} \tag{1}$$

Clearly, $\{Q(t), J(t)\}$ is a Markov process with state space

$$\Omega = \{(k, j), k \geq 0,\ j = 0, 1, 2, 3\}. \tag{2}$$

Using the lexicographical sequence for the states, the infinitesimal generator can be written as

$$\widetilde{Q} = \begin{array}{c} 0 \\ 1 \\ \vdots \\ N-1 \\ N \\ N+1 \\ \vdots \end{array} \begin{pmatrix} A_0 & C_1 & & & & & \\ B & A_1 & C_1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & B & A_1 & C_1 & & \\ & & & B & A & C & \\ & & & & B & A & C \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{3}$$

where

$$A_0 = \begin{pmatrix} -\lambda & \lambda & 0 & 0 \\ p\eta & -\lambda-\eta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p\mu & 0 & 0 & -\lambda-\mu \end{pmatrix};$$

$$C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \overline{p}\eta & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{p}\mu & \lambda \end{pmatrix};$$

$$B = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$A_1 = \begin{pmatrix} -\lambda-\alpha & \lambda & 0 & 0 \\ p\eta & -\lambda-\eta & 0 & 0 \\ 0 & 0 & -\lambda-\alpha & \lambda \\ 0 & 0 & p\mu & -\lambda-\mu \end{pmatrix}; \tag{4}$$

$$A = \begin{pmatrix} -\lambda-\alpha-\theta & \lambda & \theta & 0 \\ 0 & -\lambda-\eta-\theta & p\eta & \theta \\ 0 & 0 & -\lambda-\alpha & \lambda \\ 0 & 0 & p\mu & -\lambda-\mu \end{pmatrix};$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda & \overline{p}\eta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{p}\mu & \lambda \end{pmatrix}.$$

Due to the block structure of matrix $\widetilde{Q}$, $\{Q(t), J(t)\}$ is called a QBD process.

3. Stability Condition and Stationary Distribution

Theorem 1. *The QBD process $\{Q(t), J(t)\}$ is positive recurrent if and only if $(p\mu - \lambda)\alpha > \lambda(\overline{p}\mu + \lambda)$.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.1 in [23]; we omit it here. □

Theorem 2. *If $(p\mu - \lambda)\alpha > \lambda(\overline{p}\mu + \lambda)$, the matrix equation $R^2B + RA + C = \mathbf{0}$ has the minimal nonnegative solution*

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & r_1 & r_2 & r_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r_4 & r_5 \end{pmatrix}, \tag{5}$$

where

$$r_1 = \frac{\lambda}{\lambda+\eta+\theta}, \qquad r_2 = \frac{\lambda+\overline{p}\eta}{\alpha},$$
$$r_3 = \frac{\lambda(\lambda+\alpha+\eta+\theta)(\lambda+\overline{p}\eta)+\lambda\alpha\theta}{p\mu\alpha(\lambda+\eta+\theta)}, \tag{6}$$
$$r_4 = \frac{\lambda+\overline{p}\mu}{\alpha}, \qquad r_5 = \frac{\lambda(\lambda+\alpha+\overline{p}\mu)}{p\mu\alpha}.$$

Proof. The proof of this theorem is similar to the proof of Theorem 3.3 in [23]; we omit it here.

Under the stability condition, let (Q, J) be the stationary limit of the process $\{Q(t), J(t)\}$ and denote

$$\begin{aligned} \pi_k &= (\pi_{k0}, \pi_{k1}, \pi_{k2}, \pi_{k3}), \quad k \geq 0; \\ \pi_{kj} &= P\{Q = k, J = j\} \\ &= \lim_{t\to\infty} P\{Q(t) = k, J(t) = j\}, \quad (k, j) \in \Omega. \end{aligned} \tag{7}$$

□

Note that if there is no customer in the orbit, the probability that the server is free in the normal service period is zero. Thus, $\pi_{02} = 0$.

Theorem 3. *If $(p\mu-\lambda)\alpha > \lambda(\overline{p}\mu+\lambda)$, the stationary probability distribution of (Q, J) is given by*

$$\begin{aligned} \pi_{k0} &= 0, \\ \pi_{k1} &= \pi_{N1} r_1^{k-N}, \\ \pi_{k2} &= \pi_{N1}\left(r_2 r_1^{k-N-1} + \frac{r_3 r_4}{r_5 - r_1}\left(r_5^{k-N-1} - r_1^{k-N-1}\right)\right) \\ &\quad + \pi_{N3} r_4 r_5^{k-N-1}, \\ \pi_{k3} &= \pi_{N1}\frac{r_3}{r_5 - r_1}\left(r_5^{k-N} - r_1^{k-N}\right) + \pi_{N3} r_5^{k-N}, \end{aligned} \tag{8}$$

for $k \geq N + 1$, and

$$\begin{aligned} \pi_{k0} &= \frac{\eta}{\lambda+\alpha}\pi_{01} + \frac{p\eta}{\lambda+\alpha}(\pi_{11} - \pi_{01})\frac{1-q_1^k}{1-q_1} \\ &\quad + \frac{\overline{p}\eta}{\lambda+\alpha}(\pi_{11} - \pi_{01})\frac{1-q_1^{k-1}}{1-q_1}, \quad 1 \leq k \leq N-1, \end{aligned} \tag{9}$$

$$\pi_{k1} = \pi_{01} + (\pi_{11} - \pi_{01})\frac{1-q_1^k}{1-q_1}, \quad 0 \leq k \leq N-1, \tag{10}$$

$$\begin{aligned} \pi_{k2} &= \frac{\mu}{\lambda+\alpha}\pi_{03} + \frac{p\mu}{\lambda+\alpha}(\pi_{13} - \pi_{03})\frac{1-q_2^k}{1-q_2} \\ &\quad + \frac{\overline{p}\mu}{\lambda+\alpha}(\pi_{13} - \pi_{03})\frac{1-q_2^{k-1}}{1-q_2}, \quad 1 \leq k \leq N-1, \\ \pi_{k3} &= \pi_{03} + (\pi_{13} - \pi_{03})\frac{1-q_2^k}{1-q_2}, \quad 0 \leq k \leq N-1, \\ \pi_{N0} &= \frac{\overline{p}\eta}{\lambda+\alpha+\theta}\pi_{N-1,1}, \\ \pi_{N1} &= \frac{\lambda}{\lambda+\eta+\theta}\pi_{N0} + \frac{\lambda}{\lambda+\eta+\theta}\pi_{N-1,1}, \\ \pi_{N2} &= \frac{(\lambda+\overline{p}\mu)\pi_{N-1,3} + \theta\pi_{N0} + (\lambda+\eta+\theta)\pi_{N1}}{\alpha}, \\ \pi_{N3} &= \frac{\lambda\pi_{N-1,3} + (\lambda+\overline{p}\eta+\theta)\pi_{N1} + \lambda\pi_{N2}}{p\mu}, \end{aligned} \tag{11}$$

where

$$q_1 = \frac{(\lambda+\alpha)(\lambda+\overline{p}\eta)-\overline{p}\eta\alpha}{p\eta\alpha},$$
$$q_2 = \frac{(\lambda+\alpha)(\lambda+\overline{p}\mu)-\overline{p}\mu\alpha}{p\mu\alpha}, \quad (12)$$

$$\pi_{11} = -K^{-1}\left[\frac{\overline{p}\eta\alpha}{\lambda+\alpha+\theta}-\frac{\eta\alpha}{\lambda+\alpha}-K\right]\pi_{01}, \quad (13)$$

$$\pi_{00} = \frac{(\lambda+\alpha)(\lambda+\eta)-\overline{p}\eta\alpha}{\lambda(\lambda+\alpha)}\pi_{01}-\frac{p\eta\alpha}{\lambda(\lambda+\alpha)}\pi_{11},$$

$$\pi_{03} = \frac{\lambda}{p\mu}\pi_{00}-\frac{\eta}{\mu}\pi_{01},$$

$$\pi_{13} = \frac{(\lambda+\alpha)(\lambda+\mu)-\overline{p}\mu\alpha}{p\mu\alpha}\pi_{03}, \quad (14)$$

where $K = (\lambda+(\overline{p}\eta\lambda/(\lambda+\alpha)))((1-q_1^{N-2})/(1-q_1))+((\overline{p}\eta\alpha/(\lambda+\alpha+\theta))-\lambda-\eta+(p\eta\lambda/(\lambda+\alpha)))((1-q_1^{N-1})/(1-q_1))$. *Finally,* π_{01} *can be obtained by the normalization condition.*

Proof. Using the matrix-geometric solution method (see [24]), we have

$$\pi_k = (\pi_{k0},\pi_{k1},\pi_{k2},\pi_{k3}) = \pi_N R^{k-N} = (\pi_{N0},\pi_{N1},\pi_{N2},\pi_{N3})R^{k-N}, \quad k\ge N+1. \quad (15)$$

For $k \ge N+1$,

$$R^{k-N} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & r_1^{k-N} & r_2 r_1^{k-N-1}+\dfrac{r_3 r_4}{r_5-r_1}\left(r_5^{k-N-1}-r_1^{k-N-1}\right) & \dfrac{r_3}{r_5-r_1}\left(r_5^{k-N}-r_1^{k-N}\right) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r_4 r_5^{k-N-1} & r_5^{k-N} \end{pmatrix}. \quad (16)$$

□

Subsisting R^{k-N} into the above equation, we get (8). On the other hand, $\pi_0,\pi_1,\ldots,\pi_N$ satisfies the equation

$$(\pi_0,\pi_1,\ldots,\pi_N)B[R] = \mathbf{0}, \quad (17)$$

where

$$B[R] = \begin{matrix} 0 \\ 1 \\ \vdots \\ N-1 \\ N \end{matrix}\begin{pmatrix} A_0 & C_1 & & & \\ B & A_1 & C_1 & & \\ & \ddots & \ddots & \ddots & \\ & & B & A_1 & C_1 \\ & & & B & RB+A \end{pmatrix}. \quad (18)$$

Thus, we obtain

$$-\lambda\pi_{00}+p\eta\pi_{01}+p\mu\pi_{03} = 0, \quad (19)$$

$$\overline{p}\eta\pi_{k-1,1}-(\lambda+\alpha)\pi_{k0}+p\eta\pi_{k1} = 0, \quad 1\le k\le N-1, \quad (20)$$

$$\overline{p}\eta\pi_{N-1,1}-(\lambda+\alpha+\theta)\pi_{N0} = 0, \quad (21)$$

$$\lambda\pi_{00}-(\lambda+\eta)\pi_{01}+\alpha\pi_{10} = 0, \quad (22)$$

$$\lambda\pi_{k-1,1}+\lambda\pi_{k0}-(\lambda+\eta)\pi_{k1}+\alpha\pi_{k+1,0} = 0, \quad 1\le k\le N-1, \quad (23)$$

$$\lambda\pi_{N-1,1}+\lambda\pi_{N0}-(\lambda+\eta+\theta)\pi_{N1} = 0, \quad (24)$$

$$\overline{p}\mu\pi_{k-1,3}-(\lambda+\alpha)\pi_{k2}+p\mu\pi_{k3} = 0, \quad 1\le k\le N-1, \quad (25)$$

$$\overline{p}\mu\pi_{N-1,3}+\theta\pi_{N0}+p\eta\pi_{N1}-(\lambda+\alpha)\pi_{N2}+p\mu\pi_{N3} = 0, \quad (26)$$

$$-(\lambda+\mu)\pi_{03}+\alpha\pi_{12} = 0, \quad (27)$$

$$\lambda\pi_{k-1,3}+\lambda\pi_{k2}-(\lambda+\mu)\pi_{k3}+\alpha\pi_{k+1,2} = 0, \quad 1\le k\le N-1, \quad (28)$$

$$\lambda\pi_{N-1,3}+(r_2\alpha+\theta)\pi_{N1}+\lambda\pi_{N2}+(r_4\alpha-\lambda-\mu)\pi_{N3} = 0. \quad (29)$$

From (20) and (23), we get (10) by some computation. Taking (10) into (20), we get (9). In a similar way, (11) can be obtained from (25) and (28). Taking r_2 and r_4 into (29), together with (21), (24), and (26), we can derive π_{N0}, π_{N1}, π_{N2}, and π_{N3}. Then, π_{00}, π_{03}, and π_{13} can be obtained from (19), (22), and (27). Let k take $N-1$ in (23); using the expressions of $\pi_{N-2,1}$, $\pi_{N-1,0}$, $\pi_{N-1,1}$, and π_{N0}, (13) can be derived. Finally, from the normalization condition $\sum_{j=0}^{3}\sum_{k=0}^{\infty}\pi_{kj} = 1$, we can obtain π_{01}.

Remark 4. From Theorem 3, we can see that, in our model, $\pi_{k0} = 0$, $(k\ge N+1)$, which is different from the result in [23]. However, if we use the same technique to analyze the M/M/1 retrial queue with working vacations and feedback under N-policy but without vacation interruption, π_{k0} $(k \ge N+1)$ cannot be 0.

From Theorem 3, the probability that the server is busy is given by

$$P_b = \sum_{k=0}^{\infty}\pi_{k1}+\sum_{k=0}^{\infty}\pi_{k3} = N\left(\frac{\pi_{11}}{1-q_1}-\frac{q_1\pi_{01}}{1-q_1}\right)$$

$$-\frac{\pi_{11}-\pi_{01}}{(1-q_1)^2}\left(1-q_1^N\right)$$
$$+N\left(\frac{\pi_{13}}{1-q_2}-\frac{q_2\pi_{03}}{1-q_2}\right)-\frac{\pi_{13}-\pi_{03}}{(1-q_2)^2}\left(1-q_2^N\right)$$
$$+\frac{1-r_5+r_3}{(1-r_1)(1-r_5)}\pi_{N1}+\frac{1}{1-r_5}\pi_{N3}. \quad (30)$$

The probability that the server is free is

$$P_f=\sum_{k=0}^{N}\pi_{k0}+\sum_{k=1}^{\infty}\pi_{k2}=1-P_b. \quad (31)$$

The mean number of customers in the orbit is

$$\begin{aligned}
E[L] &= \sum_{k=1}^{N}k\pi_{k0}+\sum_{k=1}^{\infty}k(\pi_{k1}+\pi_{k2}+\pi_{k3}) \\
&= \sum_{k=1}^{N}k(\pi_{k0}+\pi_{k2})+\sum_{k=1}^{N-1}k(\pi_{k1}+\pi_{k3}) \\
&\quad + N\pi_{N1}\frac{(1+r_2)(1-r_5)+r_3(1+r_4)}{(1-r_1)(1-r_5)} \\
&\quad + N\pi_{N3}\frac{1+r_4}{1-r_5}+\pi_{N1} \\
&\quad \times \frac{(r_1+r_2)(1-r_5)^2+r_3r_4(2-r_1-r_5)+r_3(1-r_1r_5)}{(1-r_1)^2(1-r_5)^2} \\
&\quad + \pi_{N3}\frac{r_4+r_5}{(1-r_5)^2}.
\end{aligned} \quad (32)$$

The mean number of customers in the system is given by

$$\begin{aligned}
E[\widetilde{L}] &= \sum_{k=1}^{N}k\pi_{k0}+\sum_{k=1}^{\infty}k\pi_{k2}+\sum_{k=0}^{\infty}(k+1)(\pi_{k1}+\pi_{k3}) \\
&= E[L]+P_b.
\end{aligned} \quad (33)$$

Let W be the waiting time of a customer in the orbit, using Little's formula, $E[W]=E[L]/\lambda$. The expected sojourn time of a customer in the system is $E[\widetilde{W}]=E[\widetilde{L}]/\lambda$.

The system busy period T is defined as the period that starts at an epoch when an arriving customer finds an empty system and ends at the departure epoch at which the system is empty. Using the theory of regenerative process,

$$\pi_{00}=\frac{E[T_{00}]}{1/\lambda+E[T]}, \quad (34)$$

where $E[T_{00}]$ is the amount of time in the state $(0,0)$ during a regenerative cycle. Obviously, $E[T_{00}]=1/\lambda$. Thus, $E[T]=\lambda^{-1}(\pi_{00}^{-1}-1)$.

4. Conditional Stochastic Decomposition

Lemma 5. *If $(p\mu-\lambda)\alpha>\lambda(\overline{p}\mu+\lambda)$, let Q_0 be the conditional queue length of an $M/M/1$ retrial queue with feedback in the orbit given that the server is busy; then Q_0 has probability generating function*

$$G_{Q_0}(z)=\frac{1-r_5}{1-r_5z}. \quad (35)$$

Proof. Consider an M/M/1 retrial queue with feedback; let $Q^*(t)$ be the number of customers in the orbit at time t, and

$$J^*(t)=\begin{cases}0, & \text{the server is free at time } t,\\ 1, & \text{the server is busy at time } t;\end{cases} \quad (36)$$

then $\{Q^*(t),J^*(t)\}$ is a Markov process with state space $\{(k,j),k\geq 0,j=0,1\}$. And the infinitesimal generator is given by

$$\widetilde{Q}^*=\begin{pmatrix}A_0 & C & & & \\ B & A & C & & \\ & B & A & C & \\ & & \ddots & \ddots & \ddots\end{pmatrix}, \quad (37)$$

where

$$\begin{aligned}
A_0&=\begin{pmatrix}-\lambda & \lambda\\ p\mu & -\lambda-\mu\end{pmatrix}; \qquad B=\begin{pmatrix}0 & \alpha\\ 0 & 0\end{pmatrix};\\
A&=\begin{pmatrix}-\lambda-\alpha & \lambda\\ p\mu & -\lambda-\mu\end{pmatrix}; \qquad C=\begin{pmatrix}0 & 0\\ \overline{p}\mu & \lambda\end{pmatrix}.
\end{aligned} \quad (38)$$

Following the steps we used before, the QBD process $\{Q^*(t),J^*(t)\}$ is positive recurrent if and only if $(p\mu-\lambda)\alpha>\lambda(\overline{p}\mu+\lambda)$, and the stationary probability distribution is

$$\begin{aligned}
\widetilde{\pi}_{k0}&=\widetilde{\pi}_{01}r_4r_5^{k-1}, \quad k\geq 1,\\
\widetilde{\pi}_{k1}&=\widetilde{\pi}_{01}r_5^{k}, \quad k\geq 0,
\end{aligned} \quad (39)$$

where

$$\widetilde{\pi}_{00}=\left(1+\frac{1+r_4}{1-r_5}\frac{\lambda}{p\mu}\right)^{-1}, \qquad \widetilde{\pi}_{01}=\frac{\lambda}{p\mu}\widetilde{\pi}_{00}. \quad (40)$$

Thus,

$$G_{Q_0}(z)=\sum_{k=0}^{\infty}P\{Q_0=k\}z^k=\frac{\sum_{k=0}^{\infty}\widetilde{\pi}_{01}r_5^kz^k}{\sum_{k=0}^{\infty}\widetilde{\pi}_{01}r_5^k}=\frac{1-r_5}{1-r_5z}. \quad (41)$$

□

For the model considered in this paper, we introduce a random variable $Q^N=\{Q-N\mid Q\geq N,J=1\text{ or }3\}$. And Q^N is a conditional queue length given that the server is busy and there are at least N customers in the orbit. Let P_b^* be the

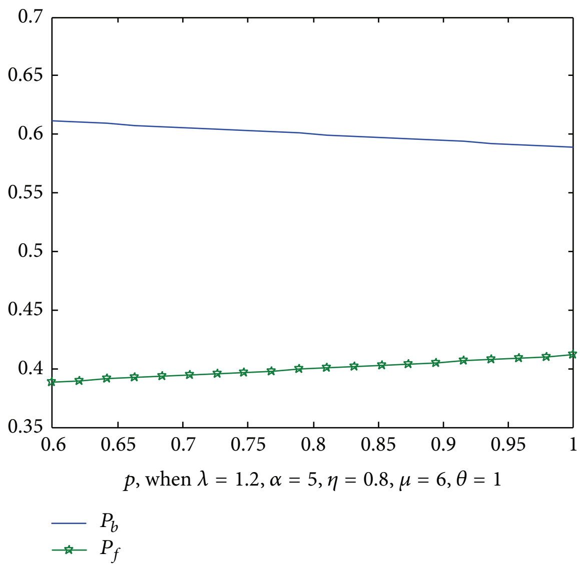

Figure 1: P_b and P_f with the change of p.

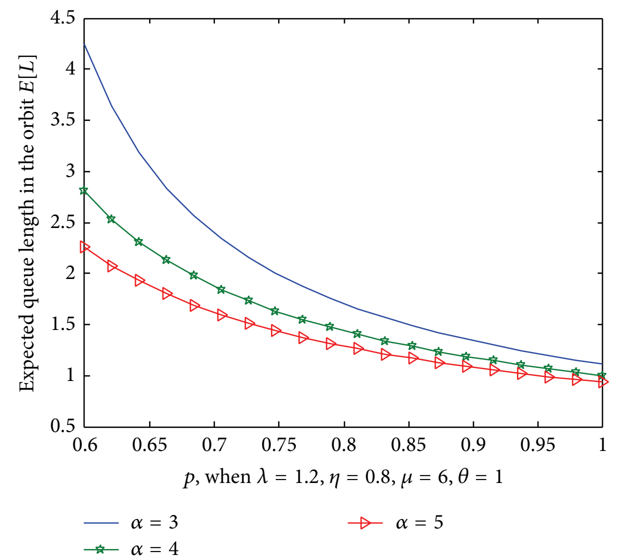

Figure 2: $E[L]$ with the change of p.

probability that the server is busy and there are at least N customers in the orbit. Obviously,

$$\begin{aligned} P_b^* &= P\{Q \geq N, J = 1 \text{ or } 3\} = \sum_{k=N}^{\infty} \pi_{k1} + \sum_{k=N}^{\infty} \pi_{k3} \\ &= \sum_{k=N}^{\infty} \pi_{N1} r_1^{k-N} + \sum_{k=N}^{\infty} \pi_{N1} \frac{r_3}{r_5 - r_1} \left(r_5^{k-N} - r_1^{k-N} \right) \\ &\quad + \sum_{k=N}^{\infty} \pi_{N3} r_5^{k-N} \\ &= \frac{1 + r_3 - r_5}{(1 - r_1)(1 - r_5)} \pi_{N1} + \frac{1}{1 - r_5} \pi_{N3}. \end{aligned} \tag{42}$$

Theorem 6. *If $(p\mu - \lambda)\alpha > \lambda(\overline{p}\mu + \lambda)$, the conditional queue length Q^N can be decomposed into the sum of two independent random variables: $Q^N = Q_0 + Q_c$, where Q_0 is defined in Lemma 5 and follows a geometric distribution with parameter $1 - r_5$. Additional queue length Q_c has a distribution*

$$\begin{aligned} P\{Q_c = 0\} &= \frac{1}{P_b^*} \frac{\pi_{N1} + \pi_{N3}}{1 - r_5}, \\ P\{Q_c = k\} &= \frac{\pi_{N1}}{P_b^*} \frac{r_1 + r_3 - r_5}{1 - r_5} r_1^{k-1}, \quad k \geq 1. \end{aligned} \tag{43}$$

Proof. The proof of this theorem is similar to the proof of Theorem 5.2 in [23]; we omit it here. □

5. Numerical Results

5.1. Sensitivity Analysis. In Figure 1, with the change of probability p, the curves of probability P_b (the server is busy) and P_f (the server is free) are provided. Figure 2 illustrates the expected queue length $E[L]$ with the change of probability p at different retrial rate α. In Figure 1, we find that P_f increases as p increases while P_b decreases as p increases. From Figure 2, we can see that $E[L]$ decreases evidently with increasing value of p. We can easily imagine that $E[L]$ will increase dramatically with p decreasing, as long as the stability condition in Theorem 1 holds.

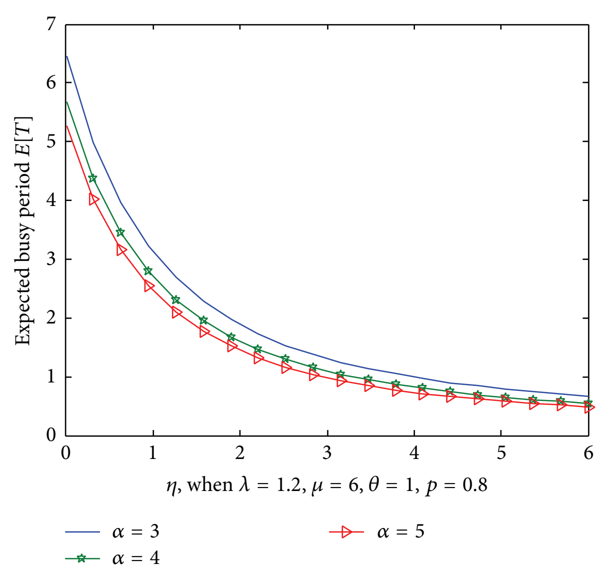

Figure 3: $E[T]$ with the change of η.

From Figures 3 and 4, it is obvious that expected busy period $E[T]$ and expected queue length $E[L]$ both decrease evidently with service rate η increasing. Thus, compared with ordinary vacation policy, working vacation policy can utilize the server and decrease the waiting jobs effectively. And it is easy to see that, if the other conditions are the same, the larger α is, the smaller $E[T]$ and $E[L]$ become.

Under the stability condition, we vary the retrial rate α from 3 to 5. Figures 5 and 6 illustrate the effect of α on $E[T]$

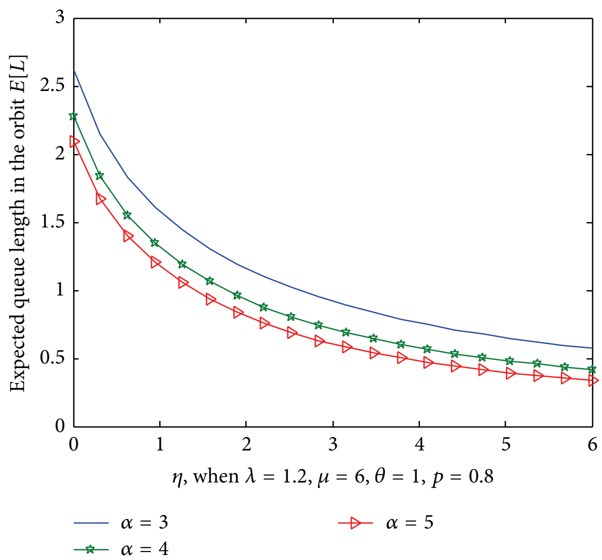

FIGURE 4: $E[L]$ with the change of η.

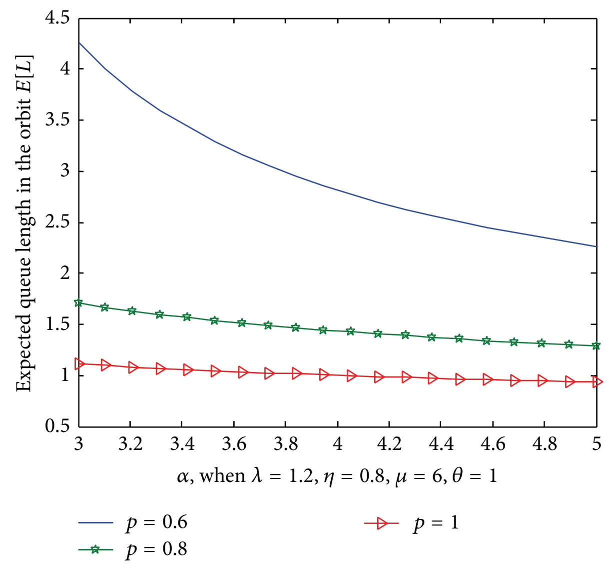

FIGURE 6: $E[L]$ with the change of α.

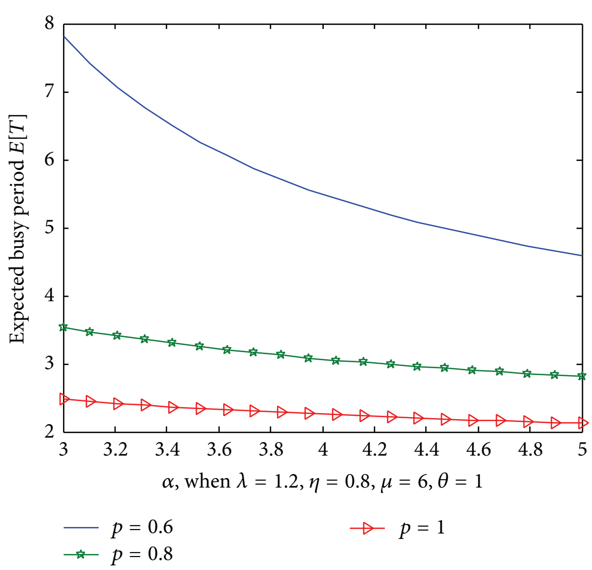

FIGURE 5: $E[T]$ with the change of α.

and $E[L]$, respectively. We can see that $E[T]$ and $E[L]$ both decrease with the rate α increasing; this is due to the fact that the interretrial time becomes shorter. When the probability p is small, $E[T]$ and $E[L]$ are sensitive to retrial rate α; this is because customers may join the retrial group for another service with probability $1-p$.

5.2. Cost Analysis. Queueing managers are always interested in minimizing operating cost of unit time. In this section, we establish a cost function to search for the optimal service rate η.

Define the following cost elements:

C_L = cost per unit time for each customer present in the orbit;

C_μ = cost per unit time for service during a normal service period;

C_η = cost per unit time for service in a working vacation period;

C_θ = fixed cost per unit time during a working vacation period.

We establish an expected operating cost function per unit time as

$$\min_{\eta} : f(\eta) = C_L E[L] + C_\mu \mu + C_\eta \eta + C_\theta \theta. \tag{44}$$

Assume that $C_L = 6$, $C_\mu = 15$, $C_\eta = 10$, and $C_\theta = 4$; Figure 7 illustrates the curve of cost function with the change of η. We can see that there is an optimal service rate η to make the cost minimize. In order to solve the optimization problem (44), we can use the parabolic method in [25] to find the optimum value of η, say η^*. As is known to us, the unique optimum of a quadratic function agreeing with $f(x)$ at 3-point pattern $\{x_0, x_1, x_2\}$ occurs at

$$\bar{x} = \frac{1}{2}\frac{f(x_0)(x_1^2-x_2^2)+f(x_1)(x_2^2-x_0^2)+f(x_2)(x_0^2-x_1^2)}{f(x_0)(x_1-x_2)+f(x_1)(x_2-x_0)+f(x_2)(x_0-x_1)}. \tag{45}$$

Assume the stopping tolerance $\varepsilon = 10^{-4}$ and with the information of Figure 7, we select the initial 3-point pattern $\eta_0 = 0.4$, $\eta_1 = 0.6$, and $\eta_2 = 0.8$. After six iterations, Table 1 shows that the minimum expected operating cost per unit time converges to the solution $\eta^* = 0.566601$ with a value of 71.614654.

6. Conclusion

In this paper, we analyze an M/M/1 retrial queue with working vacation, interruption, and feedback under N-policy.

Table 1: The parabolic method in searching for the optimum solution.

Number of iterations	η_0	η_1	η_2	$f(\eta_0)$	$f(\eta_1)$	$f(\eta_2)$	$\bar{\eta}$	$f(\bar{\eta})$	Tolerance
0	0.400000	0.600000	0.800000	71.882122	71.623400	71.973607	0.700000	71.741964	0.100000
1	0.400000	0.600000	0.700000	71.882122	71.623400	71.741964	0.578266	71.615748	0.021734
2	0.400000	0.578266	0.600000	71.882122	71.615748	71.623400	0.570064	71.614752	0.008202
3	0.400000	0.570064	0.578266	71.882122	71.614752	71.615748	0.567778	71.614666	0.002287
4	0.400000	0.567778	0.570064	71.882122	71.614666	71.614752	0.566941	71.614655	8.367041×10^{-4}
5	0.400000	0.566941	0.567778	71.882122	71.614655	71.614666	0.566689	71.614654	2.521251×10^{-4}
6	0.400000	0.566689	0.566941	71.882122	71.614654	71.614655	0.566601	71.614654	8.826668×10^{-5}

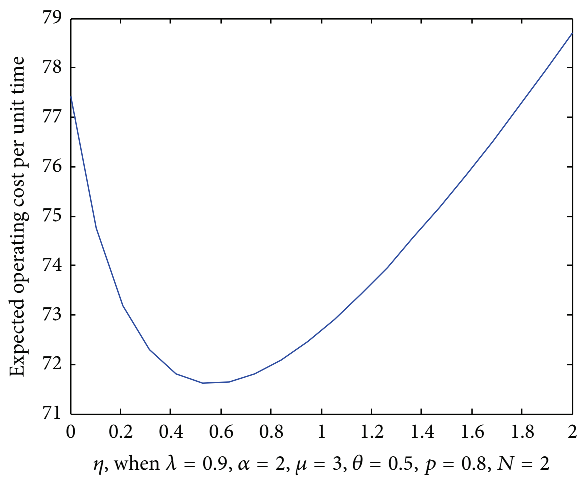

Figure 7: Effect of η on the expected operating cost per unit time.

Using the matrix-analytic method, the stationary probability distribution and some performance measures are obtained. The conditional stochastic decomposition is also given. We present several numerical examples to study the effect of some parameters. Finally, a cost optimization problem is studied.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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