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Research Article

Iterative Reproducing Kernel Method for Solving Second-Order Integrodifferential Equations of Fredholm Type

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We present an efficient iterative method for solving a class of nonlinear second-order Fredholm integrodifferential equations associated with different boundary conditions. A simple algorithm is given to obtain the approximate solutions for this type of equations based on the reproducing kernel space method. The solution obtained by the method takes form of a convergent series with easily computable components. Furthermore, the error of the approximate solution is monotone decreasing with the increasing of nodal points. The reliability and efficiency of the proposed algorithm are demonstrated by some numerical experiments.

1. Introduction

The topic of integrodifferential equations (IDEs) which has attracted growing interest for some time has been recently developed in many applied fields, so a wide variety of problems in the physical sciences and engineering can be reduced to IDEs, in particular in relation to mathematical modeling of biological phenomena [1–3], aeroelasticity phenomena [4], population dynamics [5], neural networks [6], electrocardiology [7], electromagnetic [8], electrodynamics [9], and so on. Thus, it is important to study boundary value problems (BVPs) for especially the nonlinear IDEs, which can be classified into two types: Fredholm and Volterra IDEs, where the upper bound of the integral part of Fredholm type is a fixed number whilst it is a variable for Volterra type [10]. In this point, these types of IDEs arise in the theories of singular integral equations with degenerate symbol and BVPs for mixed type partial differential equations. Therefore, the investigations in this area are of great interest; see [11] and the references therein for an overview of the current state of the art in their numerical methods; also it is well known that it is extremely difficult to analytically solve nonlinear IDEs. Unfortunately, few of these equations can be solved explicitly. Thus, it is required to obtain an efficient approximation method in order to solve these types of IDEs. So far, several numerical methods are currently improved in this regard.

The existence and uniqueness of the solutions for the BVPs of higher-order IDEs have been investigated by Agarwal [12]; but no numerical method was presented. Particularly, the analytical approximate solutions for first-order up to higher-order IDEs have been obtained by the numerical integration techniques such as Runge-Kutta methods [13], Euler-Chebyshev methods [14], Wavelet-Galerkin method [15], and Taylor polynomials method [16] and by semianalytical-numerical techniques such as Adomian decomposition method [17], reproducing kernel Hilbert space (RKHS) method [18–22], homotopy analysis method [23], and variational iteration method [24].

In this paper, we apply the RKHS technique to develop a novel numerical method in the space $W_2^3[a,b]$ for solving second-order Fredholm IDEs of the following form:

$$-\frac{d^2u(x)}{dx^2}+\int_a^b k(x,s)\,u(s)\,ds=f(x,u(x)),\quad a\le x\le b, \tag{1}$$

where a and b are real finite constants, $u(x)$ is an unknown function to be determined, and the forcing function $f(x,u)$

can be linear or nonlinear function of $u(x)$. Subject to the typical boundary conditions,

$$u(a) = u_a, \qquad u(b) = u_b, \tag{2}$$

where u_a and u_b are already known boundary values.

Consequently here, we assume that IDEs (1) and (2) satisfy the following two assumptions. Firstly, the forcing function $f(x,u)$ and all its partial derivatives are continuous, and $\partial f(x,u)/\partial u$ is nonpositive for $a \le x \le b$. Secondly, the kernel function $K(x,y)$ satisfies the positive definite property:

$$\int\int_a^b k(x,s)\,\xi(x)\,\xi(s)\,ds\,dx > 0, \tag{3}$$

where $\xi(x)$ is any continuous nonzero function and holds

$$\int\int_a^b |k(x,s)|^2 ds\,dx < +\infty. \tag{4}$$

For a comprehensive introduction about the existence and uniqueness theory of solution of such problems, we refer to [25, 26]. Additionally, we assume that IDEs (1) and (2) have a unique solution $u(x)$ under the above two assumptions on the given interval.

In this paper, the attention is given to obtain the approximate solution of second-order Fredholm IDEs with different boundary conditions using the RKHS method. The present method can approximate the solutions and their derivatives at every point of the range of integration; also it has several advantages such that the conditions for determining solutions can be imposed on the reproducing kernel space, the conditions about the nonlinearity of the forcing function f are simple and may include u, u', or any others operator of u, and the iterative sequence $u_n(x)$ of approximate solutions converges in C to the solution $u(x)$.

This paper is comprised of five sections including the introduction. The next section is devoted to several reproducing kernel spaces and essential theorems. An associated linear operator and solution representation in $W_2^3[a,b]$ are obtained in Section 3. Meanwhile, an iterative method is developed for the existence of solutions for IDEs (1) and (2) based on reproducing kernel space. The applications of the proposed numerical scheme are illustrated in Section 4. Conclusions are presented in Section 5.

2. Analysis of Reproducing Kernel Hilbert Space (RKHS)

In functional analysis, the RKHS is a Hilbert space of functions in which pointwise evaluation is a continuous linear functional. Equivalently, they are spaces that can be defined by reproducing kernels. In this section, we utilize the reproducing kernel concept to construct two reproducing kernel Hilbert spaces and to find out their representation of reproducing functions for solving the IDEs (1) and (2) via RKHS technique.

Definition 1 (see [27]). Hilbert spaces $\mathscr{H}$ of functions on a nonempty abstract set E are called a reproducing kernel Hilbert spaces if there exists a reproducing kernel K of $\mathscr{H}$.

It is worth mentioning that the reproducing kernel K of a Hilbert space $\mathscr{H}$ is unique, and the existence of K is due to the Riesz representation theorem, where K completely determines the space $\mathscr{H}$. Moreover, every sequence of functions $f_1, f_2, \ldots, f_n, \ldots$ which converges strongly to a function f in $\mathscr{H}$, converges also in the pointwise sense. This convergence is uniform on every subset on E on which $x \to K(x,x)$ is bounded. In this occasion, these spaces have wide applications including complex analysis, harmonic analysis, quantum mechanics, statistics, and machine learning. Subsequently, the space $W_2^3[a,b]$ is constructed in which every function satisfies the boundary conditions (2) and then utilized the space $W_2^1[a,b]$. For the theoretical background of reproducing kernel Hilbert space theory and its applications, we refer the reader to [27–31].

Definition 2. The inner product space $W_2^3[a,b]$ is defined as $W_2^3[a,b] = \{u(x) : u''(x)$ is an absolutely continuous function in $[a,b]$, $u'''(x) \in L^2[a,b]$, $u(a) = u(b) = 0$, $x \in [a,b]\}$ and is equipped with inner product and norm, respectively,

$$\langle u, v\rangle_{W_2^3} = \sum_{i=0}^{2} u^{(i)}(a)\,v^{(i)}(a) + \int_a^b u'''(s)\,v'''(s)\,ds, \tag{5}$$

$\|u\|^2_{W_2^3} = \langle u,u\rangle_{W_2^3}$, where $u, v \in W_2^3[a,b]$ and $L^2[a,b] = \{u(x) : \int_a^b u^2(x) < \infty\}$.

The Hilbert space $W_2^3[a,b]$ is called a reproducing kernel if, for each $x \in [a,b]$, there exist $R(x,y) \in W_2^3[a,b]$, simply $R_x(y)$, such that $\langle u(\xi), R_x(\xi)\rangle_{W_2^3} = u(x)$ for any $u(\xi) \in W_2^3[a,b]$ and $\xi \in [a,b]$.

Theorem 3. *The Hilbert space $W_2^3[a,b]$ is a complete reproducing kernel and its reproducing kernel function can be written as*

$$R_x(y) = \begin{cases} \sum_{i=1}^{6} p_i(x)\,y^{i-1}, & y \le x, \\ \sum_{i=1}^{6} q_i(x)\,y^{i-1}, & y > x, \end{cases} \tag{6}$$

where the functions $p_i(x)$ and $q_i(x)$ can be determined by $(\partial^i R_x(y)/\partial y^i)|_{y=x+0} = (\partial^i R_x(y)/\partial y^i)|_{y=x-0}$, $i = 0,1,2,3,4$, and $-(\partial^5 R_x(y)/\partial y^5)|_{y=x+0} + (\partial^5 R_x(y)/\partial y^5)|_{y=x-0} = 1$, in which $R_x(a) = 0$, $R'_x(a) + R_x^{(4)}(a) = 0$, $R''_x(a) - R'''_x(a) = 0$ and $R_x(b) = 0$, $R'''_x(b) = 0$, $R_x^{(4)}(b) = 0$.

By using the Mathematica 7.0 software package, the coefficients of the reproducing kernel $R_x(y)$ are given in the appendix, whilst, the proof of completeness and the process of obtaining the coefficients of the reproducing kernel $R_x(y)$ are similar to the proof of Theorem 2.1 in [20].

Remark 4. The reproducing kernel function $R_x(y)$ is symmetric and unique, and $R_x(y) \geq 0$ for any fixed $x \in [a,b]$.

Definition 5. The inner product space $W_2^1[a,b]$ is defined as $W_2^1[a,b] = \{u(x) : u(x)$ is an absolutely continuous function in $[a,b]$, $u'(x) \in L^2[a,b]\}$.

On the other hand, the inner product and norm in $W_2^1[a,b]$ are defined, respectively, by $\langle u,v\rangle_{W_2^1} = \int_a^b (u(s)v(s) + u'(s)v'(s))ds$ and $\|u\|_{W_2^1}^2 = \langle u,u\rangle_{W_2^1}$, where $u,v \in W_2^1[a,b]$.

In [29], the authors had proved that $W_2^1[a,b]$ is complete reproducing kernel space and its reproducing kernel is

$$K_x(y) = \frac{1}{2\sinh(b-a)}\left[\cosh(x+y-b-a) + \cosh(|x-y|-b+a)\right]. \tag{7}$$

Lemma 6. *The space $W_2^3[a,b]$ is imbedded to space $C[a,b]$.*

Theorem 7. *An arbitrary bounded set of $W_2^3[a,b]$ is a compact set of $C[a,b]$.*

Proof. Let $\{u_n(x)\}_{n=1}^{\infty}$ be a bounded set of $W_2^3[a,b]$. Assume that $\|u_n(x)\| \leq M$, where M is positive constant. From the representation of $R_x(y)$, we get

$$\begin{aligned} \left|u_n^{(i)}(x)\right| &= \left|\left\langle u(x), R_x^{(i)}(y)\right\rangle_{W_2^3}\right| \\ &\leq \left\|R_x^{(i)}(y)\right\|_{W_2^3}\|u(x)\|_{W_2^3}, \quad i=1,2,3. \end{aligned} \tag{8}$$

Since $R_x^{(i)}(y)$, $i=1,2,3$, is uniformly bounded about x and y, so we have $|u_n^{(i)}(x)| \leq M_i\|u(x)\|_{W_2^3}$. Hence, $\|u_n(x)\|_C \leq M$. Now, we need to prove that $\{u_n(x)\}_{n=1}^{\infty}$ is a compact set of $C[a,b]$; that is, $\{u_n(x)\}_{n=1}^{\infty}$ are equicontinuous functions. Again, from the property of $R_x(y)$, it follows that

$$\begin{aligned} |u_n(x_2) - u_n(x_1)| &= \left|\left\langle u_n(x), R_{x_2}(y) - R_{x_1}(y)\right\rangle_{W_2^3}\right| \\ &\leq \left\|R_{x_2}(y) - R_{x_1}(y)\right\|_{W_2^3}\|u_n(x)\|_{W_2^3} \\ &\leq M\left\|R_{x_2}(y) - R_{x_1}(y)\right\|_{W_2^3}. \end{aligned} \tag{9}$$

By the symmetry of $R_x(y)$ and the mean value theorem of differentials, it follows that

$$\begin{aligned} \left|R_{x_2}(y) - R_{x_1}(y)\right| &= \left|R_y(x_2) - R_y(x_1)\right| \\ &= \left[\frac{d}{dx}R_y(x)\right]\bigg|_{x=\eta}|x_2 - x_1| \\ &\leq M_1|x_2 - x_1|. \end{aligned} \tag{10}$$

Thus, if $\delta \leq |x_2 - x_1| \leq \epsilon/M_1M$, we can get $|u_n(x_2) - u_n(x_1)| < \epsilon$. □

3. The Exact and Approximate Solution

In this section, formulation of a differential linear operator and implementation method are presented in the spaces $W_2^3[a,b]$ and $W_2^1[a,b]$. Meanwhile, we construct an associated orthogonal function system based on Gram-Schmidt orthogonalization process in order to obtain the exact and approximate solutions of IDEs (1) and (2). In order to apply the RKHS method, as in [30–33], we firstly define a differential linear operator $L : W_2^3[a,b] \to W_2^1[a,b]$ such that $Lu(x) = d^2u(x)/dx^2$. After homogenization of the boundary conditions of Equation (2), the IDEs (1) and (2) can be transformed into the equivalent operator equation as

$$\begin{aligned} Lu(x) &= F(x, u(x), Tu(x)), \quad a \leq x \leq b, \\ u(a) &= 0, \qquad u(b) = 0, \end{aligned} \tag{11}$$

where $Tu(x) = \int_a^b k(x,s)u(s)ds$, $u(x) \in W_2^3[a,b]$ and $F(x,y,z) \in W_2^1[a,b]$ for $y = y(x)$, $z = z(x) \in W_2^3[a,b]$, $-\infty < y$, $z < +\infty$ and $x \in [a,b]$. It is clear that L is a bounded linear operator and L^{-1} exists.

Lemma 8. *Let $\zeta_0 \in W_2^1[a,b]$ and $\zeta_i(x) = K_{x_i}(y)$, $i = 1,2,\ldots$, where $K_{x_i}(y)$ is the reproducing kernel of $W_2^1[a,b]$. If $\{x_1, x_2, \ldots, x_n, \ldots\}$ is a dense subset on $[a,b]$, then $\{\zeta_0(x), \zeta_1(x), \ldots, \zeta_n(x), \ldots\}$ is the complete system of the space $W_2^1[a,b]$.*

Proof. For all $v(x) \in W_2^1[a,b]$, let $\langle v(x), \zeta_i(x)\rangle_{W_2^1} = 0$, $i = 1,2,\ldots$; then $\langle v(x), \zeta_i(x)\rangle_{W_2^1} = \langle v(x), K_{x_i}(y)\rangle_{W_2^1} = v(x_i) = 0$. Thus, by the density of $\{x_1, x_2, \ldots, x_n, \ldots\}$ and the continuity of $v(x)$, we have $v(x) = 0$. Thus, $\{\zeta_1(x), \zeta_2(x), \ldots, \zeta_n(x), \ldots\}$ is the complete system of $W_2^1[a,b]$. Therefore, $\{\zeta_0(x), \zeta_1(x), \ldots, \zeta_n(x), \ldots\}$ is also the complete system of $W_2^1[a,b]$.

The normal orthogonal system $\{\overline{\zeta}_i(x)\}_{i=0}^{\infty}$ of the space $W_2^1[a,b]$ of the sequence $\{\zeta_i(x)\}_{i=0}^{\infty}$ can be constructed as $\overline{\zeta}_i(x) = \sum_{k=0}^{i}\sigma_{ik}\zeta_k(x)$, $i = 0,1,\ldots$, where σ_{ik}, $i = 0,1,\ldots$, are the coefficients of orthogonalization. □

Lemma 9. *Let $\eta_i(x) = L^*\zeta_i(x)$, $i = 1,2,\ldots$, where L^* is the adjoint operator of L. If $\{x_i(x)\}_{i=1}^{\infty}$ is dense in $[a,b]$, then $\eta_i(x) = L_yR_x(y)|_{y=x_i}$, $i = 1,2,\ldots$.*

Proof. We note that $\eta_i(x) = L^*\zeta_i(x) = \langle L^*\zeta_i(y), R_x(y)\rangle_{W_2^3} = \langle \zeta_i(y), L_yR_x(y)\rangle_{W_2^1} = L_yR_x(y)|_{y=x_i}$. The subscript y by the operator L indicates that the operator L applies to the function of y. Clearly $\eta_i(x) \in W_2^3[a,b]$. □

Corollary 10. *For (11), if $\{x_i(x)\}_{i=1}^{\infty}$ is dense in $[a,b]$, then $\{\eta_i(x)\}_{i=1}^{\infty}$ is the complete system of $W_2^3[a,b]$.*

The normal orthogonal system $\{\overline{\eta_i}(x)\}_{i=1}^{\infty}$ of the space $W_2^3[a,b]$ can be derived from Gram-Schmidt orthogonalization process of the sequence $\{\eta_i(x)\}_{i=1}^{\infty}$ as follows:

$$\overline{\eta_i}(x) = \sum_{k=1}^{i} \alpha_{ik}\eta_k(x), \quad i = 1, 2, \ldots, \tag{12}$$

where the orthogonalization coefficients α_{ik} are given by $\alpha_{ii} > 0$, $\alpha_{ij} = 1/\|\eta_i(x)\|$, for $i = j = 1$, $\alpha_{ij} = 1/\beta_{ik}$, $i \neq j \neq 1$ and $\alpha_{ij} = (-1/\beta_{ik})\sum_{k=j}^{i-1} c_{ik}\alpha_{kj}$, and $i > j$ such that $\beta_{ik} = \sqrt{\|\eta_i(x)\|^2 - \sum_{k=1}^{i-1}(c_{ik})^2}$, $c_{ik} = \langle \eta_i(x), \overline{\eta_k}(x)\rangle_{W_2^3}$.

Theorem 11. *For all $u(x) \in W_2^3[a,b]$, the series $\sum_{i=1}^{\infty}\langle u(x), \overline{\eta_i}(x)\rangle\overline{\eta_i}(x)$ is convergent in the sense of the norm $\|\cdot\|_{W_2^3}$. On the other hand, if $u(x)$ is the exact solution of* (11)*, then*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \alpha_{ik} F(x_k, u(x_k), Tu(x_k))\overline{\eta_i}(x). \tag{13}$$

Proof. Since $u(x) \in W_2^3[a,b]$ can be expanded in the form of Fourier series about normal orthogonal system $\{\eta_i(x)\}_{i=1}^{\infty}$ as

$$u(x) = \sum_{i=1}^{\infty}\langle u(x), \overline{\eta_i}(x)\rangle \overline{\eta_i}(x), \tag{14}$$

and since the space $W_2^3[a,b]$ is Hilbert space, so the series $\sum_{i=1}^{\infty}\langle u(x), \overline{\eta_i}(x)\rangle\overline{\eta_i}(x)$ is convergent in the norm of $\|\cdot\|_{W_2^3}$. From (12) and (14), it can be written as $u(x) = \sum_{i=1}^{\infty}\langle u(x), \overline{\eta_i}(x)\rangle\overline{\eta_i}(x) = \sum_{i=1}^{\infty}\langle u(x), \sum_{k=1}^{i}\alpha_{ik}\eta_k(x)\rangle\overline{\eta_i}(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\alpha_{ik}\langle Lu(x), \zeta_k(x)\rangle\overline{\eta_i}(x)$.

If $u(x)$ is the exact solution of (11) and $Lu(x) = F(x, u(x), Tu(x))$, for $x \in [a,b]$, then $u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\alpha_{ik}F(x_k, u(x_k), Tu(x_k))\overline{\eta_i}(x)$.

As a result, the approximate solution can be obtained by the n-term intercept of the exact solution $u(x)$, and it is given by

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \alpha_{ik} F(x_k, u(x_k), Tu(x_k))\overline{\eta_i}(x). \tag{15}$$

□

Remark 12. If (11) is linear, then the approximate solution to (11) can be obtained directly from (15), while, if (11) is nonlinear, then the exact and approximate solutions can be obtained using the following iterative algorithm.

According to (14), we construct the iterative sequences $u_n(x)$:

$$\begin{gathered}\forall \text{ fixed } u_0(x) \in W_2^3[a,b], \\ u_n(x) = \sum_{i=1}^{n} A_i\overline{\eta_i}(x), \quad n = 0, 1, \ldots,\end{gathered} \tag{16}$$

where the coefficients A_i of $\overline{\eta_i}(x)$, $i = 1, 2, \ldots, n$, are given as

$$\begin{aligned} A_1 &= \alpha_{11}F(x_1, u_0(x_1), Tu_0(x_1)), \\ A_2 &= \sum_{k=1}^{2}\alpha_{1k}F(x_k, u_{k-1}(x_k), Tu_{k-1}(x_k)), \\ &\vdots \\ A_n &= \sum_{k=1}^{n}\alpha_{nk}F(x_k, u_{k-1}(x_k), Tu_{k-1}(x_k)). \end{aligned} \tag{17}$$

In the iteration process of (16), we can guarantee that the approximation solution $u_n(x)$ satisfies the boundary conditions of (2).

Corollary 13. *The sequence $\{u_n\}_{n=1}^{\infty}$ in* (16) *is monotone increasing in the sense of the norm $\|\cdot\|_{W_2^3}$.*

Proof. Since $\{\overline{\eta_i}(x)\}_{i=1}^{\infty}$ is the complete orthonormal system in $W_2^3[a,b]$, one can get $\|u_n\|_{W_2^3}^2 = \langle u_n(x), u_n(x)\rangle_{W_2^3} = \langle\sum_{i=1}^{n}A_i\overline{\eta_i}(x), \sum_{i=1}^{n}A_i\overline{\eta_i}(x)\rangle_{W_2^3} = \sum_{i=1}^{n}(A_i)^2$. Hence, $\|u_n\|_{W_2^3}$ is monotone increasing.

Since $W_2^3[a,b]$ is Hilbert space, then $\sum_{i=1}^{n}(A_i)^2 < \infty$ and thus u_n is convergent. On the other hand, the solution of IDEs (1) and (2) is considered the fixed point of the following functional under the suitable choice of the initial term $u_0(x)$:

$$u_n(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \alpha_{ik} F(x_k, u(x_k), Tu(x_k))\overline{\eta_i}(x). \tag{18}$$

□

Theorem 14 (see [28]). *Assume that X is a Banach space and $D: X \to X$ is a nonlinear mapping and suppose that $\|Du - Dv\| \le \alpha\|u - v\|$, $u, v \in X$, for some constants $\alpha < 1$. Then D has a unique fixed point. Furthermore, the sequence $u_n = Du_n$, with an arbitrary choice of $u_0 \in X$, converges to the fixed point of D.*

According to above theorem, for the nonlinear mapping,

$$\begin{aligned} Du(x) &= L^{-1}F(x, u(x), Tu(x)) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\alpha_{ik}F(x_k, u(x_k), Tu(x_k))\overline{\eta_i}(x). \end{aligned} \tag{19}$$

A sufficient condition for convergence of the present iteration method is strictly contraction of D. Furthermore, the sequence (16) converges to the fixed point of D which is also the solution of IDEs (1) and (2).

The approximate solution $u_{n,m}(x)$ can be obtained by taking finitely many terms in the series representation of $u_n(x)$, given by

$$u_{n,m}(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\alpha_{ik}F(x_k, u_{m-1}(x_k), Tu_{m-1}(x_k))\overline{\eta_i}(x). \tag{20}$$

4. Numerical Results and Discussions

To illustrate the accuracy and applicability of the RKHS method, three examples are given in this section. Results obtained are compared with the exact solution of each example and are found to be in good agreement with each other. In the process of computation, all the symbolic and numerical computations are performed by using Mathematica 7.0 software package.

Example 15 (see [34]). Consider the following Fredholm IDE:

$$u''(x) + xu'(x) - xu(x) = e^{x} - 2\sin(x) + \int_{-1}^{1} \sin(x) e^{-s} u(s)\, ds, \quad -1 \le x \le 1, \tag{21}$$

subject to the boundary conditions $u(-1) = e^{-1}$, $u(1) = e$ with the exact solution $u(x) = e^{x}$.

We employ the RKHS method to solve this example. Taking $x_i = (i-1)/(n-1)$, $i = 1, 2, \ldots, n$, $n = 64$, the approximate solutions $u_{64}(x)$ are obtained using (15) and the reproducing kernel function $R_x(y)$ on $[-1, 1]$ is given such that

$$R_x(y) = \frac{-1}{138240} \begin{cases} (x-1)(1+y) f_1(x,y), & x < y, \\ (1+x)(y-1) f_2(x,y), & x > y, \end{cases} \tag{22}$$

where $f_1(x, y)$ and $f_2(x, y)$ are given in the appendix.

In Table 1, there is a comparison of the numerical result against the Taylor polynomial solution, used in [35], and Tau-Chebyshev and Legendre method, used in [36], at some selected grid points on $[-1, 1]$. It is worth noting that the RKHS results become very highly accurate only with a few iterations and become very close to the exact solution.

As it is evident from the comparison results, it was found that our method in comparison with the mentioned methods is better with a view to accuracy and utilization.

Example 16 (see [37]). Consider the following nonlinear Fredholm IDE:

$$u''(x) = 6x + \int_{-1}^{1} xs\left(u'(s)\right)^{2} (u(s))^{2} ds, \quad -1 \le x \le 1, \tag{23}$$

subject to the boundary conditions $u(-1) = u(1) = 0$ with the exact solution $u(x) = x^{3} - x$.

Again, we employ the RKHS method to solve this example, taking $x_i = (i-1)/(n-1)$, $i = 1, 2, \ldots, n$, $n = 64$. The approximate solutions $u_{64}(x)$ at some selected gird points on $[-1, 1]$ are obtained using (16). The results and errors are reported in Table 2.

The result from numerical analysis is an approximation, in general, which can be made as accurate as desired. Because a computer has a finite word length, only a fixed number of digits are stored and used during computations. From Table 2, it can be seen that, with the few iterations, the RKHS approximate solutions with high accuracy are achievable.

Example 17 (see [37]). Consider the following nonlinear Fredholm IDE:

$$u''(x) - \int_{0}^{1} (x-s)^{2} e^{u(s)} ds + \left(u'(x)\right)^{2} + \left(\frac{3}{2}x^{2} - \frac{5}{3}x + \frac{7}{12}\right) = 0, \quad 0 \le x \le 1, \tag{24}$$

subject to the boundary conditions $u(0) = 0$, $u(1) = \ln(2)$ with the exact solution $u(x) = \ln(x + 1)$.

The influence of the number of nodes n on the absolute error function $|u(x) - u_n(x)|$ of the RKHS method is explored in Figure 1. By taking $x_i = (i-1)/(n-1)$, $i = 1, 2, \ldots, n$, $(n = 16, 32, 64)$, the approximate solutions $u_{16}(x)$, $u_{32}(x)$, and $u_{64}(x)$ are obtained using (16), where the reproducing kernel function $R_x(y)$ on $[0, 1]$ is given such that

$$R_x(y) = \frac{-1}{18720} \begin{cases} y(x-1) f_3(x,y), & x < y, \\ x(y-1) f_4(x,y), & x > y, \end{cases} \tag{25}$$

where $f_3(x, y)$ and $f_4(x, y)$ are given in the appendix. The absolute error charts are shown in Figure 1.

It is observed that the increase in the number of nodes results in a reduction in the absolute error and correspondingly an improvement in the accuracy of the obtained solution. This goes in agreement with the known fact: the error is monotone decreasing, where more accurate solutions are achieved using an increase in the number of nodes.

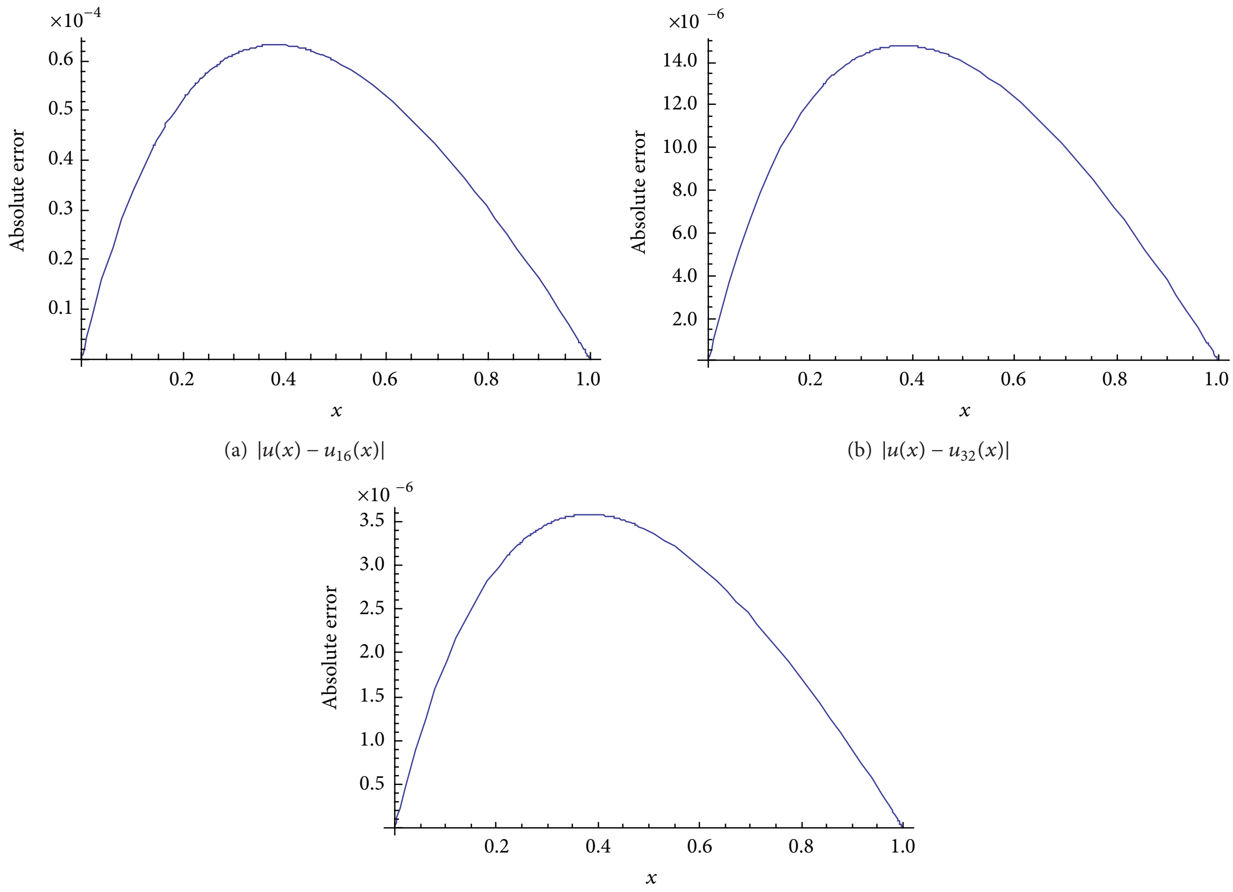
5. Conclusion

In this paper, the RKHS method was employed successfully for solving a class of second-order Fredholm IDEs by constructing a reproducing kernel space in which each function satisfies the given boundary conditions of the considered problems. Moreover, the exact solution $u(x)$ and the approximate solution $u_n(x)$ are represented in the form of series in the space $W_2^3[a, b]$, and it can be obvious that they are in better agreement with each other. Many of the results obtained in this paper can be extended to significantly more general classes of linear and nonlinear Fredholm-Volterra IDEs, which show that the present method is an accurate and reliable analytical technique for the solutions of various kinds of these IDEs.

Appendices

A. The Coefficients of the Reproducing Kernel Function $R_x(y)$

We denote the reproducing kernel function of the space $W_2^3[a, b]$ by $R_x(y)$, where $x, y \in [a, b]$. Next, we give the



(a) $|u(x) - u_{16}(x)|$

(b) $|u(x) - u_{32}(x)|$

(c) $|u(x) - u_{64}(x)|$

Figure 1: Figures of absolute errors $|u(x) - u_{16}(x)|$, $|u(x) - u_{32}(x)|$, $|u(x) - u_{64}(x)|$ for Example 17.

Table 1: Numerical result of Example 15 compared with the Taylor polynomial solution and Tau-Chebyshev and Legendre method.

x_i	Exact solution	RKHS method	Taylor polynomial	Tau-Chebyshev	Tau-Legendre
−1.0	0.36787944	0.36787943	0.36787911	0.36787942	0.36787942
−0.8	0.44932896	0.44932895	0.44932892	0.44932896	0.44932896
−0.6	0.54881164	0.54881164	0.54881163	0.54881164	0.54881164
−0.4	0.67032005	0.67032005	0.67032005	0.67032004	0.67032004
−0.2	0.81873075	0.81873075	0.81873076	0.81873075	0.81873075
0.00	1	1	1	1	1
0.20	1.22140276	1.22140276	1.22140277	1.22140276	1.22140276
0.40	1.49182470	1.49182472	1.49182482	1.49182469	1.49182469
0.60	1.82211880	1.82211881	1.82211942	1.82211880	1.82211880
0.80	2.22554093	2.22554093	2.22554335	2.22554092	2.22554092
1.00	2.71828183	2.71828182	2.71828972	2.71828180	2.71828181

Table 2: Numerical results and errors for Example 16.

x_i	Exact solution	Approximate solution	Absolute error	Relative error
−0.96	0.075264	0.07526450	5.02448×10^{-7}	6.67580×10^{-6}
−0.80	0.288000	0.28800103	1.03221×10^{-6}	3.58406×10^{-6}
−0.64	0.377856	0.37785481	1.18971×10^{-6}	3.14858×10^{-6}
−0.48	0.369408	0.36940714	8.56297×10^{-7}	2.31803×10^{-6}
−0.32	0.287232	0.28722987	2.13009×10^{-6}	7.41592×10^{-6}
−0.16	0.155904	0.15590388	1.20843×10^{-7}	7.75112×10^{-7}
0.16	−0.155904	−0.15590313	8.65379×10^{-7}	5.55072×10^{-6}
0.32	−0.287232	−0.28723072	1.27927×10^{-6}	4.45379×10^{-6}
0.48	−0.369408	−0.36940667	1.32766×10^{-6}	3.59402×10^{-6}
0.64	−0.377856	−0.37785490	1.09653×10^{-6}	2.90199×10^{-6}
0.80	−0.288000	−0.28799933	6.71899×10^{-7}	2.33298×10^{-6}
0.96	−0.075264	−0.07526386	1.39755×10^{-7}	1.85687×10^{-6}

expression of the coefficients $p_i(x)$ and $q_i(x)$, $i = 1, \dots, 6$, of $R_x(y)$ as

$$
\begin{aligned}
p_1(x) &= \frac{1}{\Delta_1}\Big(a\big(-15a^7 b(b-x) \\
&\quad + 120bx\left(30b^3 + 6b^4 + 5bx^3 - x^4 - 10b^2 x(3+x)\right) \\
&\quad + 30a^6 b\left(2b^2 - b(-4+x) - x(4+x)\right) \\
&\quad - 10a^5 b\big(8b^3 - 21x^2 + b^2(42+5x) \\
&\qquad -bx(21+13x)\big) \\
&\quad - 30a\big(20bx^4 - 4x^5 + 6b^5\left(4+x^2\right) \\
&\qquad + 5b^3 x\left(24 - 24x + x^3\right) \\
&\qquad - 10b^4\left(-12 - 12x + x^3\right) \\
&\qquad - b^2x^2\left(240 + 40x + x^3\right)\big) \\
&\quad + 6a^4\big(6b^5 - 25b^3(-2+x)x - x^5 + 5b^4(16+5x) \\
&\qquad + 5bx\left(-64 + x^3\right) \\
&\qquad -10b^2\left(-32 + 13x^2 + x^3\right)\big) \\
&\quad + 10a^2 b\big(6b^4 x(6+x) - 10b^3\left(-36 + x^3\right) \\
&\qquad - 6x^2\left(60 + x^3\right) \\
&\qquad - bx\left(360 + 600x - 30x^3 + x^4\right) \\
&\qquad + 5b^2\left(144 + 48x - 12x^3 + x^4\right)\big) \\
&\quad - 15a^3\big(50b^4 x - 2x^5 + 6b^5(2+x) \\
&\qquad - 10b^3\left(-32 + 5x^2 + x^3\right) \\
&\qquad - bx\left(240 + 160x - 10x^3 + x^4\right) \\
&\qquad + 5b^2\left(48 - 32x - 4x^3 + x^4\right)\big)\big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
p_2(x) &= \frac{1}{\Delta_2}\Big(3a^8(b-x) \\
&\quad - 24bx\left(30b^3 + 6b^4 + 5bx^3 - x^4 - 10b^2 x(3+x)\right) \\
&\quad - 6a^7\left(4b + b^2 - x(4+x)\right) \\
&\quad - 2a^6\left(5b^3 + 21x^2 + 5bx^2 - b^2(21+10x)\right) \\
&\quad + 6a^5\big(10b^3 + 5b^4 + 64x - 5b^2 x(4+x) \\
&\qquad + 2b\left(-32 + 5x^2\right)\big) \\
&\quad + 12a\big(10bx^4 - 2x^5 + 5b^3x^2\left(-12 + x^2\right) \\
&\qquad + 6b^5\left(2 + x^2\right) - 10b^4\left(-6 - 6x + x^3\right) \\
&\qquad -b^2x^2\left(60 + 20x + x^3\right)\big) \\
&\quad - 3a^4\big(6b^5 - 20b^3x^2 + 10b^4(5+2x) \\
&\qquad - 10b^2\left(16 + 5x^2 + x^3\right) \\
&\qquad + x\left(240 + 160x - x^4\right) + 5b\left(-48 + x^4\right)\big) \\
&\quad - 6a^2 b\big(6b^4 x(4+x) - 4x^2\left(-30 + x^3\right) \\
&\qquad - 10b^3\left(-12 + x^3\right) \\
&\qquad - bx\left(240 + 120x - 20x^3 + x^4\right) \\
&\qquad + 5b^2\left(24 - 8x^3 + x^4\right)\big) \\
&\quad + 4a^3\big(60b^4 x + 6b^5(3+2x)
\end{aligned}
$$

$$- 3x^2 \left(-60 + x^3\right) - 20b^3 \left(-6 + 3x^2 + x^3\right)$$
$$+ 10b^2 \left(-18 - 24x - 3x^3 + x^4\right)$$
$$+ b\left(120x^2 + 15x^4 - 2x^5\right)\Big)\Big),$$

$$p_3(x) = \frac{1}{\Delta_3}\Big(-3a^7(b-x)$$
$$- 3a^5\Big(5b^3 - 16x^2 - 10bx(1+x) + b^2(26+5x)\Big)$$
$$+ a^6\left(13b^2 + b(21-5x) - x(21+8x)\right)$$
$$+ 15a^4\Big(-16x + b^2(5-2x)x + b^3(5+2x) - 2b\left(-8+5x^2\right)\Big)$$
$$+ 3b^2x\Big(-6b^3x + 10b^2x^2 + x\left(-120+x^3\right) - 5b\left(-24+x^3\right)\Big)$$
$$+ a^3\Big(6b^5 - 120b^3x - 10b^2\left(60 - 12x^2 + x^3\right) + x\left(360 + 360x - x^4\right) + 5b\left(-72 + 48x + x^4\right)\Big)$$
$$+ 3ab\Big(-10b^3x^3 + 6b^4x(2+x) - 2x^2\left(-120+x^3\right) - bx\left(120 - 120x - 10x^3 + x^4\right) + 5b^2\left(-24 - 24x - 4x^3 + x^4\right)\Big)$$
$$- 3a^2\Big(6b^5(1+x) - x^2\left(-120+x^3\right) - 10b^3\left(12+x^3\right) + bx\left(120 + 240x + 5x^3 - x^4\right) + 5b^2\left(-48 - 24x - 2x^3 + x^4\right)\Big)\Big),$$

$$p_4(x) = \frac{-1}{\Delta_3}(b-x)\Big(6a^5(b+x) + 60ab\left(2b + 4x + b^2x\right)$$
$$+ b^2x\left(-120 + 6b^2x - 4bx^2 + x^3\right)$$
$$- 15a^4\left(b^2 + 2x + 2b(1+x)\right)$$
$$- 30a^2\left(4b + 4x + 5b^2x + b^3(1+x)\right)$$
$$+ 10a^3b\left(b^2 + 12x + b(6+5x)\right)\Big),$$

$$p_5(x) = \frac{1}{\Delta_2}(b-x)\Big(6a^5 - 15a^4(2+b)$$
$$+ 10a^3b(6+b-x)$$
$$+ 30a^2\left(-4 + b^2(-1+x) + bx\right)$$
$$- 30ab\left(-4 + 2bx + b^2x\right)$$
$$+ bx\left(6b^3 - 4bx^2 + x^3 + 6b^2(5+x)\right)\Big),$$

$$p_6(x) = \frac{-1}{\Delta_1}(b-x)\Big(15a^4 + 6b^4 + x^4$$
$$+ 6b^3(5+x) + 6b^2x(5+x)$$
$$- 10a^3(6 + 5b + x) - 4b\left(-30 + x^3\right)$$
$$- 30a\left(4 + b^3 + 2bx + b^2(4+x)\right)$$
$$+ 30a^2\left(2b^2 + x + b(5+x)\right)\Big),$$

$$q_1(x) = \frac{1}{\Delta_1}\Big(b(a-x)$$
$$\times\Big(-15a^7(b-x)$$
$$+ 6b\left(20 + 5b^2 + b^3\right)x^4$$
$$+ 15a^6\left(4b^2 + b(8-3x) - x(8+x)\right)$$
$$- 5a^5\Big(16b^3 - 2b^2(-42+x) + 3(-6+x)x^2 - bx(66+17x)\Big)$$
$$+ a^4\Big(36b^4 - 20b^2x(6+7x) + 10b^3(48+7x) + 15x\left(-128 + 6x^2 + x^3\right) + 5b\left(384 - 90x^2 + 5x^3\right)\Big)$$
$$+ 6a\Big(-80bx^3 - 20x^4 + b^4\left(-120 + x^3\right) - 20b^2x\left(-30 - 10x + x^3\right) - 5b^3\left(120 - x^3 + x^4\right)\Big)$$
$$+ 6a^2\Big(80x^3 + b^4x(30+x) - 5b^3\left(-120 + 9x^2 + x^3\right) + 5bx\left(-240 - 56x + 5x^3\right) + 5b^2\left(240 - 80x + x^3 + 2x^4\right)\Big)$$

$$
\begin{aligned}
&- 2a^3 \Big(9b^4 (10 + 3x) - 5b^3 x (-27 + 7x) \\
&\qquad + 30x \left(-60 - 8x + x^3\right) \\
&\qquad - 5b^2 \left(-480 + 63x^2 + x^3\right) \\
&\qquad + 5b \left(360 - 432x + 15x^3 + 5x^4\right)\Big)\Big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
&q_2 (x) \\
&= \frac{1}{\Delta_2} \Big(3a^8 (b - x) \\
&\quad - 24bx \left(30b^3 + 6b^4 + 5bx^3 - x^4 - 10b^2 x (3 + x)\right) \\
&\quad - 6a^7 \left(4b + b^2 - x (4 + x)\right) \\
&\quad - 2a^6 \left(5b^3 + 21x^2 + 5bx^2 - b^2 (21 + 10x)\right) \\
&\quad + 6a^5 \Big(10b^3 + 5b^4 + 64x - 5b^2 x (4 + x) \\
&\qquad +2b \left(-32 + 5x^2\right)\Big) \\
&\quad + 12a \Big(10bx^4 - 2x^5 + 5b^3 x^2 \left(-12 + x^2\right) \\
&\qquad + 6b^5 \left(2 + x^2\right) - 10b^4 \left(-6 - 6x + x^3\right) \\
&\qquad - b^2 x^2 \left(60 + 20x + x^3\right)\Big) \\
&\quad - 3a^4 \Big(6b^5 - 20b^3 x^2 + 10b^4 (5 + 2x) \\
&\qquad - 10b^2 \left(16 + 5x^2 + x^3\right) \\
&\qquad + x \left(240 + 160x - x^4\right) + 5b \left(-48 + x^4\right)\Big) \\
&\quad - 6a^2 b \Big(6b^4 x (4 + x) - 4x^2 \left(-30 + x^3\right) \\
&\qquad - 10b^3 \left(-12 + x^3\right) \\
&\qquad - bx \left(240 + 120x - 20x^3 + x^4\right) \\
&\qquad + 5b^2 \left(24 - 8x^3 + x^4\right)\Big) \\
&\quad + 4a^3 \Big(60b^4 x + 6b^5 (3 + 2x) \\
&\qquad - 3x^2 \left(-60 + x^3\right) - 20b^3 \left(-6 + 3x^2 + x^3\right) \\
&\qquad + 10b^2 \left(-18 - 24x - 3x^3 + x^4\right) \\
&\qquad + b \left(120x^2 + 15x^4 - 2x^5\right)\Big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
&q_3 (x) \\
&= \frac{1}{\Delta_3} \Big((a - x) \\
&\quad \times \Big(-3a^6 (b - x) \\
&\qquad + a^5 \left(13b^2 + b (21 - 8x) - x (21 + 5x)\right) \\
&\qquad + a^4 \Big(-15b^3 + x^2 (27 + x) \\
&\qquad\quad - 2b^2 (39 + x) + bx (51 + 22x)\Big) \\
&\qquad + 3b^2 \Big(2b^3 x (3 + x) + x \left(120 + 40x - x^3\right) \\
&\qquad\quad + 5b \left(-24 + x^3\right)\Big) \\
&\qquad + a^3 \Big(15b^3 (5 + x) \\
&\qquad\quad - b^2 x (3 + 32x) + b \left(240 - 99x^2 - 8x^3\right) \\
&\qquad\quad + x \left(-240 - 3x^2 + x^3\right)\Big) \\
&\qquad - 3ab \Big(b^4 (6 + 4x) \\
&\qquad\quad - 2x \left(-120 - 40x + x^3\right) \\
&\qquad\quad + 5b^2 \left(-24 + 3x^2 + x^3\right) \\
&\qquad\quad - b \left(240 - 80x - 11x^3 + x^4\right)\Big) \\
&\qquad + 3a^2 \Big(2b^5 + 5b^3 (-3 + x) x \\
&\qquad\quad + x \left(120 + 40x - x^3\right) \\
&\qquad\quad + b^2 \left(-200 + 39x^2 + 6x^3\right) \\
&\qquad\quad - b \left(120 - 160x - 7x^3 + x^4\right)\Big)\Big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
&q_4 (x) \\
&= \frac{-1}{\Delta_3} b^2 (a - x) \\
&\quad \times \Big(6a^4 + x^4 + 10b^2 x (3 + x) \\
&\qquad - 3a^3 (10 + 5b + 3x) - 5b \left(-24 + x^3\right) \\
&\qquad - a \left(120 - x^3 + 5bx (12 + x) + 10b^2 (3 + 2x)\right) \\
&\qquad + a^2 \left(10b^2 + x (30 + x) + 5b (12 + 5x)\right)\Big),
\end{aligned}
$$

$$
\begin{aligned}
&q_5 (x) \\
&= \frac{1}{\Delta_2} b (a - x) \\
&\quad \times \Big(6a^4 + x^4 + 10b^2 x (3 + x) \\
&\qquad - 3a^3 (10 + 5b + 3x) - 5b \left(-24 + x^3\right) \\
&\qquad - a \left(120 - x^3 + 5bx (12 + x) + 10b^2 (3 + 2x)\right) \\
&\qquad + a^2 \left(10b^2 + x (30 + x) + 5b (12 + 5x)\right)\Big),
\end{aligned}
$$

$$
\begin{aligned}
&q_6 (x) \\
&= \frac{-1}{\Delta_1} (a - x)
\end{aligned}
$$

$$\times\Big(6a^4+x^4+10b^2x(3+x)-3a^3(10+5b+3x)-5b\big(-24+x^3\big)-a\big(120-x^3+5bx(12+x)+10b^2(3+2x)\big)+a^2\big(10b^2+x(30+x)+5b(12+5x)\big)\Big), \tag{A.1}$$

where

$$\begin{aligned}\Delta_1&=720(a-b)^2\times\big(-20+a^3-5b^2-b^3-a^2(5+3b)+ab(10+3b)\big),\\ \Delta_2&=144(a-b)^2\times\big(-20+a^3-5b^2-b^3-a^2(5+3b)+ab(10+3b)\big),\\ \Delta_3&=72(a-b)^2\times\big(-20+a^3-5b^2-b^3-a^2(5+3b)+ab(10+3b)\big).\end{aligned} \tag{A.2}$$

B. The Kernel Representations for (22) and (25)

Here, we give the expression form of the functions $f_1(x,y)$ and $f_2(x,y)$ for $R_x(y)$ in (22), where $x, y \in [-1,1]$, $f_3(x,y)$ and $f_4(x,y)$, for $R_x(y)$ in (25), where $x, y \in [0,1]$:

$$\begin{aligned}f_1(x,y)&=\Big(26831-23106y-5104y^2+1194y^3+761y^4+6x^2\times\big(391+174y+16y^2-6y^3+y^4\big)\\&\quad-4x^3\big(391+174y+16y^2-6y^3+y^4\big)+x^4\big(391+174y+16y^2-6y^3+y^4\big)\\&\quad+4x\big(7639-5874y-2096y^2-294y^3+49y^4\big)\Big),\\ f_2(x,y)&=\Big(26831+30556y+2346y^2-1564y^3+391y^4+16x^2\times\big(-319-524y+6y^2-4y^3+y^4\big)\\&\quad-6x^3\big(-199+196y+6y^2-4y^3+y^4\big)+x^4\big(761+196y+6y^2-4y^3+y^4\big)\\&\quad+6x\big(-3851-3916y+174y^2-116y^3+29y^4\big)\Big),\\ f_3(x,y)&=156y^4+6x^2\times\big(120+30y+10y^2-5y^3+y^4\big)-4x^3\big(120+30y+10y^2-5y^3+y^4\big)\\&\quad+x^4\big(120+30y+10y^2-5y^3+y^4\big)+12x\big(360-300y-100y^2-15y^3+3y^4\big),\\ f_4(x,y)&=30xy\big(-120+6y-4y^2+y^3\big)+10x^2y\big(-120+6y-4y^2+y^3\big)\\&\quad+120y\big(36+6y-4y^2+y^3\big)-5x^3y\big(36+6y-4y^2+y^3\big)\\&\quad+x^4\big(156+36y+6y^2-4y^3+y^4\big).\end{aligned} \tag{B.1}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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