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Research Article

Existence Results for a Coupled System of Nonlinear Fractional Differential Equations in Banach Spaces

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We investigate boundary value problems for a coupled system of nonlinear fractional differential equations involving Caputo derivative in Banach spaces. A generalized singular type coupled Gronwall inequality system is given to obtain an important a priori bound. Existence results are obtained by using fixed point theorems and an example is given to illustrate the results.

1. Introduction

Fractional differential equations involving the Riemann-Liouville fractional derivative or the Caputo fractional derivative have been shown to be very useful in the study of models of many phenomena in various fields of science and engineering, such as physics, chemistry, biology, signal and image processing, biophysics, blood flow phenomena, control theory, economics, aerodynamics and fitting of experimental data. For more details, see, for example, [1–6].

In recent years, many researchers paid much attention to the coupled system of fractional differential equations due to its applications in differential fields. The reader is referred to the papers [7–11] and the references cited therein.

Up to now, there are fewer results of fractional differential equations with boundary conditions in infinite dimensional spaces than in finite dimensional spaces. Recently, Wang et al. [12] investigated the existence and uniqueness of solutions for a fractional boundary value problem involving the Caputo derivative in Banach space as follows:

$$\begin{gathered} {}_0^C D_t^{\alpha} y(t) = f(t, y(t)), \quad 0 < \alpha < 1, \ t \in J = [0, T], \\ ax(0) + bx(T) = c, \end{gathered} \tag{1}$$

which extended the earlier work [13], where ${}_0^C D_t^{\alpha}$ is the Caputo fractional derivative of order α, $f : J \times X \to X$, where X is a Banach spaces and a, b, and c are real constants with $a + b \neq 0$.

To the best of our knowledge, there is no effort being made in the literature to study the existence of solutions for a coupled system of fractional boundary value problems involving the Caputo derivative in Banach space. Motivated by the above-mentioned works, in this paper, we study a coupled system of fractional differential equations with boundary conditions of the type

$$\begin{gathered} {}_0^C D_t^{\alpha} x(t) = f(t, x(t), y(t)), \quad 0 < \alpha < 1, \ t \in J = [0, T], \\ {}_0^C D_t^{\beta} y(t) = g(t, x(t), y(t)), \quad 0 < \beta < 1, \ t \in J = [0, T], \\ ax(0) + bx(T) = p, \\ cy(0) + dy(T) = q, \end{gathered} \tag{2}$$

where ${}_0^C D_t^{\alpha}$ and ${}_0^C D_t^{\beta}$ are the Caputo fractional derivatives of order α and β, respectively, $f, g : J \times X \times X \to X$, where X is a Banach spaces and a, b, c, d, p, and q are real constants with $a + b \neq 0$ and $c + d \neq 0$. We will apply Schaefer fixed point theorem, nonlinear alternative of Leray-Schauder type, and a new singular coupled Gronwall inequality system given by us to establish the existence of solutions for BVP (2).

This paper is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel, and we give a generalized singular type coupled Gronwall inequality system which can be used to obtain an important a priori bound. In Section 3, we give two existence results of the problem (2) which is based on two fixed point theorems, respectively. Finally, an example is given to illustrate the results in Section 4.

2. Preliminaries

For the convenience of the reader, we first briefly recall some definitions of fractional calculus; for more details, see [1, 2, 5], for example.

Definition 1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, \infty) \to R$ is given by

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds \tag{3}$$

provided that the right side is pointwise defined on $(0, \infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2. The Caputo fractional derivative of order $\gamma > 0$ of a function $u : (0, \infty) \to R$ can be written as

$$ {}_0^C D_t^{\gamma} u(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{u^{(n)}(s)}{(t-s)^{\gamma+1-n}} ds, \quad n-1 < \gamma < n. \tag{4}$$

Definition 3. The Mittag-Leffler function in two parameters is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \tag{5}$$

where $\alpha > 0$, $\beta > 0$, and $z \in \mathbb{C}$; $\mathbb{C}$ denotes the complex plane. In particular, for $\beta = 1$, one has

$$E_{\alpha,1}(z) = E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}. \tag{6}$$

The Laplace transform of Mittag-Leffler function is

$$\mathscr{L}\left\{t^{\beta-1}E_{\alpha,\beta}(-\lambda t^{\alpha})\right\} = \frac{s^{\alpha-\beta}}{s^{\alpha}+\lambda}, \quad \left(\mathscr{R}(s) > |\lambda|^{1/\alpha}\right), \tag{7}$$

where t and s are, respectively, the variables in the time domain and Laplace domain; $\mathscr{L}\{\cdot\}$ stands for the Laplace transform.

Throughout this paper, let $C(J, X)$ be the Banach space of all continuous functions from J into X with the norm $\|x\| := \sup\{\|x(t)\| : t \in J\}$. Let $E = C(J, X) \times C(J, X)$ be the Banach space endowed with the norm as follows:

$$\|(x, y)\| = \max\{\|x\|, \|y\|\}, \quad \forall (x, y) \in E. \tag{8}$$

Now, we give the definition of the solution for problem (2).

Definition 4. A $(u, v) \in E$ is said to be a solution of a coupled system of fractional BVP (2) if (u, v) satisfies the system ${}_0^C D_t^{\alpha} x(t) = f(t, x(t), y(t))$, ${}_0^C D_t^{\beta} y(t) = g(t, x(t), y(t))$ on J and the conditions $ax(0) + bx(T) = p$, $cy(0) + dy(T) = q$.

By Lemma 3.2 in [14], we have the following.

Lemma 5. *$(x, y) \in E$ is a solution of the fractional integral system*

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), y(s))\, ds - \frac{1}{a+b} \\ &\quad \times \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x(s), y(s))\, ds - p\right], \\ y(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x(s), y(s))\, ds - \frac{1}{c+d} \\ &\quad \times \left[\frac{d}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} g(s, x(s), y(s))\, ds - q\right]. \end{aligned} \tag{9}$$

Wang et al. in [15] gave a generalized Gronwall inequality as follows.

Lemma 6. *Let $y \in C(J, X)$ satisfy the following inequality:*

$$\begin{aligned} \|y(t)\| &\le a + b\int_0^t \|y(\theta)\|^{\lambda_1} d\theta + c\int_0^T \|y(\theta)\|^{\lambda_2} d\theta \\ &\quad + d\int_0^t \|y_{\theta}\|^{\lambda_3} d\theta + e\int_0^T \|y_{\theta}\|^{\lambda_4} d\theta, \quad t \in J, \end{aligned} \tag{10}$$

where $\lambda_1, \lambda_3 \in [0, 1]$, $\lambda_2, \lambda_4 \in (0, 1)$, $a, b, c, d, e \ge 0$ are constants, and $\|y_{\theta}\|_B = \sup_{0 \le s \le \theta} \|y(s)\|$. Then there exists a constant $M^* > 0$ such that

$$\|y(t)\| \le M^*. \tag{11}$$

By using the above generalized Gronwall inequality, we now give the following generalized singular type coupled Gronwall inequality system.

Lemma 7. *Let $x, y \in C(J, X)$ satisfy the following inequality system:*

$$\begin{aligned} \|x(t)\| &\le a_1 + b_1\int_0^t (t-s)^{\alpha-1} \|x(s)\|^{\lambda_1} ds \\ &\quad + c_1\int_0^t (t-s)^{\alpha-1} \|y(s)\|^{\lambda_2} ds \\ &\quad + d_1\int_0^T (T-s)^{\alpha-1} \|x(s)\|^{\lambda_1} ds \\ &\quad + e_1\int_0^T (T-s)^{\alpha-1} \|y(s)\|^{\lambda_2} ds, \\ \|y(t)\| &\le a_2 + b_2\int_0^t (t-s)^{\beta-1} \|x(s)\|^{\mu_1} ds \\ &\quad + b_2\int_0^t (t-s)^{\beta-1} \|y(s)\|^{\mu_2} ds \end{aligned}$$

$$+ d_2 \int_0^T (T-s)^{\beta-1} \|x(s)\|^{\mu_1} ds$$
$$+ e_2 \int_0^T (T-s)^{\beta-1} \|y(s)\|^{\mu_2} ds, \tag{12}$$

where α, $\beta \in (0,1)$, λ_i, $\mu_i \in [0,1/2)$, *and* $a_i, b_i, c_i, d_i, e_i \geq 0$ $(i = 1,2)$ *are constants. Then there exists a constant* $M^* > 0$ *such that*

$$\|x(t)\| \leq M^*, \qquad \|y(t)\| \leq M^*. \tag{13}$$

Proof. Let

$$u(t) = \begin{cases} 1, & \|x(t)\| \leq 1, \\ x(t), & \|x(t)\| > 1, \end{cases}$$
$$v(t) = \begin{cases} 1, & \|y(t)\| \leq 1, \\ y(t), & \|y(t)\| > 1. \end{cases} \tag{14}$$

By (12), we have

$$\begin{aligned} \|u(t)\| &\leq a_1 + 1 + b_1 \int_0^t (t-s)^{\alpha-1} \|u(s)\|^{\lambda_1} ds \\ &\quad + c_1 \int_0^t (t-s)^{\alpha-1} \|v(s)\|^{\lambda_2} ds \\ &\quad + d_1 \int_0^T (T-s)^{\alpha-1} \|u(s)\|^{\lambda_1} ds \\ &\quad + e_1 \int_0^T (T-s)^{\alpha-1} \|v(s)\|^{\lambda_2} ds \\ &\leq A_1 + B_1 \int_0^t (t-s)^{\alpha-1} \left[\|u(s)\|^{\lambda} + \|v(s)\|^{\lambda}\right] ds \\ &\quad + C_1 \int_0^T (T-s)^{\alpha-1} \left[\|u(s)\|^{\lambda} + \|v(s)\|^{\lambda}\right] ds, \end{aligned} \tag{15}$$

$$\begin{aligned} \|v(t)\| &\leq a_2 + 1 + b_2 \int_0^t (t-s)^{\beta-1} \|u(s)\|^{\mu_1} ds \\ &\quad + c_2 \int_0^t (t-s)^{\beta-1} \|v(s)\|^{\mu_2} ds \\ &\quad + d_2 \int_0^T (T-s)^{\beta-1} \|u(s)\|^{\mu_1} ds \\ &\quad + e_2 \int_0^T (T-s)^{\beta-1} \|v(s)\|^{\mu_2} ds \\ &\leq A_2 + B_2 \int_0^t (t-s)^{\beta-1} \left[\|u(s)\|^{\mu} + \|v(s)\|^{\mu}\right] ds \\ &\quad + C_2 \int_0^T (T-s)^{\beta-1} \left[\|u(s)\|^{\mu} + \|v(s)\|^{\mu}\right] ds, \end{aligned} \tag{16}$$

where $A_i = a_i + 1$, $B_i = \max\{b_i, c_i\}$, $C_i = \max\{d_i, e_i\}$ $(i = 1,2)$, $\lambda = \max\{\lambda_1, \lambda_2\}$, and $\mu = \max\{\mu_1, \mu_2\}$. It is easy to know that

$$a^{\theta} + b^{\theta} \leq 2(a+b)^{\theta}, \quad \theta \in (0,1), \ a, b \geq 0. \tag{17}$$

Adding (15) to (16), we get by Cauchy inequality and (17) that $(\theta = \max\{\lambda, \mu\})$

$$\begin{aligned} &\|u(t)\| + \|v(t)\| \\ &\leq A_1 + A_2 + \int_0^t \left[B_1 (t-s)^{\alpha-1} + B_2 (t-s)^{\beta-1}\right] \\ &\qquad \times \left(\|u(s)\|^{\theta} + \|v(s)\|^{\theta}\right) ds \\ &\quad + \int_0^T \left[C_1 (T-s)^{\alpha-1} + C_2 (T-s)^{\beta-1}\right] \\ &\qquad \times \left(\|u(s)\|^{\theta} + \|v(s)\|^{\theta}\right) ds \\ &\leq A_1 + A_2 + \left(\int_0^t \left[B_1 (t-s)^{\alpha-1} + B_2 (t-s)^{\beta-1}\right]^2 ds\right)^{1/2} \\ &\quad \cdot \left(\int_0^t \left(\|u(s)\|^{\theta} + \|v(s)\|^{\theta}\right)^2 ds\right)^{1/2} \\ &\quad + \left(\int_0^T \left[C_1 (T-s)^{\alpha-1} + C_2 (T-s)^{\beta-1}\right]^2 ds\right)^{1/2} \\ &\quad \cdot \left(\int_0^T \left(\|u(s)\|^{\theta} + \|v(s)\|^{\theta}\right)^2 ds\right)^{1/2} \\ &\leq A_1 + A_2 + \left(\int_0^t \left[B_1^2 (t-s)^{2(\alpha-1)} + 2B_1 B_2 (t-s)^{\alpha+\beta-2} \right.\right. \\ &\qquad \left.\left. + B_2^2 (t-s)^{2(\beta-1)}\right] ds\right)^{1/2} \\ &\quad \cdot 2 \int_0^t \left(\|u(s)\| + \|v(s)\|\right)^{2\theta} ds \\ &\quad + \left(\int_0^T \left[C_1^2 (T-s)^{2(\alpha-1)} + 2C_1 C_2 (T-s)^{\alpha+\beta-2} \right.\right. \\ &\qquad \left.\left. + C_2^2 (T-s)^{2(\beta-1)}\right] ds\right)^{1/2} \\ &\quad \cdot 2 \int_0^T \left(\|u(s)\| + \|v(s)\|\right)^{2\theta} ds \\ &\leq A_1 + A_2 + 2\left(\frac{B_1^2}{2\alpha-1} T^{2\alpha-1} + \frac{2B_1 B_2}{\alpha+\beta-1} T^{\alpha+\beta-1} \right. \\ &\qquad \left. + \frac{B_2^2}{2\beta-1} T^{2\beta-1}\right)^{1/2} \\ &\quad \times \int_0^t \left(\|u(s)\| + \|v(s)\|\right)^{2\theta} ds \\ &\quad + 2\left(\frac{C_1^2}{2\alpha-1} T^{2\alpha-1} + \frac{2C_1 C_2}{\alpha+\beta-1} T^{\alpha+\beta-1} \right. \\ &\qquad \left. + \frac{C_2^2}{2\beta-1} T^{2\beta-1}\right)^{1/2} \\ &\quad \times \int_0^T \left(\|u(s)\| + \|v(s)\|\right)^{2\theta} ds. \end{aligned} \tag{18}$$

From Lemma 6, we obtain that there exists $M^* > 0$ such that $\|u(t)\| + \|v(t)\| \le M^*$. Thus,

$$\begin{gathered} \|x(t)\| \le \|u(t)\| + \|v(t)\| \le M^*, \\ \|y(t)\| \le \|u(t)\| + \|v(t)\| \le M^*. \end{gathered} \tag{19}$$

□

Theorem 8 (Schaefer's fixed point theorem [16]). *Let $F : X \to X$ completely continuous operator. If the set*

$$E(F) = \{x \in X : x = \lambda Fx \text{ for some } \lambda \in [0,1]\} \tag{20}$$

is bounded, then F has fixed points.

Theorem 9 (Nonlinear alternative of Leray-Schauder type [17]). *Let X be a Banach space, D a closed, convex subset of X, U an open subset of D, and $0 \in D$. Assume that $\mathscr{A} : \overline{U} \to D$ is a continuous and compact map. Then either*

(i) *$\mathscr{A}$ has fixed points or*

(ii) *there exists $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda\mathscr{A}(u)$.*

3. Main Results

In order to obtain main result, we make the following assumptions.

(H1) The functions $f, g : J \times X \times X \to X$ are continuous.

(H2) There exist constants $\lambda_1, \lambda_2, \mu_1, \mu_2 \in [0, 1/2)$ such that

$$\begin{gathered} \|f(t,x,y)\| \le M_1 \left(1 + \|x\|^{\lambda_1} + |y|^{\lambda_2}\right), \\ \|g(t,x,y)\| \le M_2 \left(1 + \|x\|^{\mu_1} + |y|^{\mu_2}\right), \\ \forall t \in J, \ x, y \in X. \end{gathered} \tag{21}$$

(H3) For each $t \in J$, the sets

$$\begin{aligned} K_1 = \Big\{&(t-s)^{\alpha-1} f(s, x(s), y(s)) : \\ &x, y \in C(J,X), s \in [0,t]\Big\}, \\ K_2 = \Big\{&(t-s)^{\beta-1} g(s, x(s), y(s)) : \\ &x, y \in C(J,X), s \in [0,t]\Big\} \end{aligned} \tag{22}$$

are relatively compact.

Define the operator $F : E \to E$ as follows:

$$F(x,y)(t) := (F_1(x,y)(t), F_2(x,y)(t)), \tag{23}$$

where

$$\begin{aligned} &F_1(x,y)(t) \\ &\quad = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), y(s))\, ds - \frac{1}{a+b} \\ &\qquad \times \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x(s), y(s))\, ds - p\right], \\ &F_2(x,y)(t) \\ &\quad = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x(s), y(s))\, ds - \frac{1}{c+d} \\ &\qquad \times \left[\frac{d}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} g(s, x(s), y(s))\, ds - q\right]. \end{aligned} \tag{24}$$

It is easy to know that the existence of solution of the coupled fractional BVP (2) is equivalent to the operator F having a fixed point on E.

Theorem 10. *Suppose that (H1)–(H3) hold. Then the coupled fractional BVP (2) has at least one solution on J.*

Proof. We will use Schaefer's fixed point theorem to prove that F has a fixed point. The proof is divided into several steps.

Firstly, F is continuous. Let $\{(x_n, y_n)\}$ be a sequence such that $(x_n, y_n) \to (x, y)$ in E. For each $t \in J$, we have

$$\begin{aligned} &\|F_1(x_n,y_n)(t) - F_1(x,y)(t)\| \\ &\quad \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\qquad \times \|f(s, x_n(s), y_n(s)) - f(s, x(s), y(s))\|\, ds \\ &\qquad + \frac{|b|}{|a+b|\,\Gamma(\alpha)} \\ &\qquad \times \int_0^T (T-s)^{\alpha-1} \\ &\qquad \times \|f(s, x_n(s), y_n(s)) - f(s, x(s), y(s))\|\, ds \\ &\quad \le \frac{1}{\Gamma(\alpha)} \left[\int_0^t (t-s)^{\alpha-1} ds + \frac{|b|}{|a+b|} \int_0^T (T-s)^{\alpha-1} ds\right] \\ &\qquad \times \|f(\cdot, x_n, y_n) - f(\cdot, x, y)\| \\ &\quad \le \frac{T^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|}\right) \|f(\cdot, x_n, y_n) - f(\cdot, x, y)\|. \end{aligned} \tag{25}$$

Similarly, we obtain

$$\begin{aligned} &\|F_2(x_n,y_n)(t) - F_2(x,y)(t)\| \\ &\quad \le \frac{T^\beta}{\Gamma(\beta+1)} \left(1 + \frac{|d|}{|c+d|}\right) \|g(\cdot, x_n, y_n) - g(\cdot, x, y)\|. \end{aligned} \tag{26}$$

Since f, g are continuous ((H1)), we have by (25) and (26) that

$$\|F_1(x_n, y_n) - F_1(x, y)\| \longrightarrow 0,$$
$$\|F_2(x_n, y_n) - F_2(x, y)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{27}$$

Thus, we get

$$\|F(x_n, y_n) - F(x, y)\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{28}$$

Secondly, we will prove that F maps bounded sets into bounded sets in E.

By (H2), for any $m > 0$, we have for each $t \in J$ and $(x, y) \in B_m = \{(x, y) \in E : \|(x, y)\| \le m\}$ that

$$\begin{aligned}
&\|F_1(x, y)(t)\| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x(s), y(s))\| ds + \frac{|b|}{|a+b|\Gamma(\alpha)} \\
&\quad \times \int_0^T (T-s)^{\alpha-1} \|f(s, x(s), y(s))\| ds + \frac{|p|}{|a+b|} \\
&\le \frac{M_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(1 + \|x(s)\|^{\lambda_1} + \|y(s)\|^{\lambda_2}\right) ds \\
&\quad + \frac{|b| M_1}{|a+b|\Gamma(\alpha)} \\
&\quad \times \int_0^T (T-s)^{\alpha-1} \left(1 + \|x(s)\|^{\lambda_1} + \|y(s)\|^{\lambda_2}\right) ds \\
&\quad + \frac{|p|}{|a+b|} \\
&= \frac{M_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{|b| M_1}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds \\
&\quad + \frac{|p|}{|a+b|} + \frac{M_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x(s)\|^{\lambda_1} ds \\
&\quad + \frac{M_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y(s)\|^{\lambda_2} ds \\
&\quad + \frac{|b| M_1}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|x(s)\|^{\lambda_1} ds \\
&\quad + \frac{|b| M_1}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|y(s)\|^{\lambda_2} ds \\
&\le \frac{M_1 T^\alpha}{\Gamma(\alpha+1)} + \frac{|b| M_1 T^\alpha}{|a+b|\Gamma(\alpha+1)} + \frac{|p|}{|a+b|} + \frac{M_1 T^\alpha m^{\lambda_1}}{\Gamma(\alpha+1)} \\
&\quad + \frac{M_1 T^\alpha m^{\lambda_2}}{\Gamma(\alpha+1)} + \frac{|b| M_1 T^\alpha m^{\lambda_1}}{|a+b|\Gamma(\alpha+1)} \\
&\quad + \frac{|b| M_1 T^\alpha m^{\lambda_2}}{|a+b|\Gamma(\alpha+1)} := N_1,
\end{aligned} \tag{29}$$

which implies that $\|F_1(x, y)\| \le N_1$. Similarly, we can obtain that $\|F_2(x, y)\| \le N_2$, where

$$\begin{aligned}
N_2 &= \frac{M_2 T^\beta}{\Gamma(\beta+1)} + \frac{|d| M_2 T^\beta}{|c+d|\Gamma(\beta+1)} + \frac{|q|}{|c+d|} + \frac{M_2 T^\beta m^{\mu_1}}{\Gamma(\beta+1)} \\
&\quad + \frac{M_2 T^\beta m^{\mu_2}}{\Gamma(\beta+1)} + \frac{|d| M_2 T^\beta m^{\mu_1}}{|c+d|\Gamma(\beta+1)} + \frac{|d| M_2 T^\beta m^{\mu_2}}{|c+d|\Gamma(\beta+1)}.
\end{aligned} \tag{30}$$

Thus, we get

$$\|F(x, y)\| \le \max\{N_1, N_2\}. \tag{31}$$

Thirdly, F maps bounded sets into equicontinuous sets of E. Let $0 \le t_1 < t_2 \le T$, $(x, y) \in B_m$. By (H2), we have

$$\begin{aligned}
&\|F_1(x, y)(t_2) - F_1(x, y)(t_1)\| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right] \\
&\qquad \times \|f(s, x(s), y(s))\| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \|f(s, x(s), y(s))\| ds \\
&\le \frac{M_1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right] \\
&\qquad \times \left(1 + \|x(s)\|^{\lambda_1} + \|y(s)\|^{\lambda_2}\right) ds \\
&\quad + \frac{M_1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \left(1 + \|x(s)\|^{\lambda_1} + \|y(s)\|^{\lambda_2}\right) ds \\
&\le \frac{M_1}{\Gamma(\alpha)} \left(1 + m^{\lambda_1} + m^{\lambda_2}\right) \int_0^{t_1} \left[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right] ds \\
&\quad + \frac{M_1}{\Gamma(\alpha)} \left(1 + m^{\lambda_1} + m^{\lambda_2}\right) \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds \\
&\le \frac{M_1}{\Gamma(\alpha+1)} \left(1 + m^{\lambda_1} + m^{\lambda_2}\right) \\
&\quad \times \left[\left|t_1^\alpha - t_2^\alpha\right| + 2(t_2 - t_2)^\alpha\right] \longrightarrow 0, \quad \text{as } t_2 \longrightarrow t_1.
\end{aligned} \tag{32}$$

Similarly, we obtain

$$\begin{aligned}
&\|F_2(x, y)(t_2) - F_2(x, y)(t_1)\| \\
&\le \frac{M_2}{\Gamma(\beta+1)} \left(1 + m^{\mu_1} + m^{\mu_2}\right) \\
&\quad \times \left[\left|t_1^\beta - t_2^\beta\right| + 2(t_2 - t_2)^\beta\right] \longrightarrow 0, \quad \text{as } t_2 \longrightarrow t_1.
\end{aligned} \tag{33}$$

Hence, F is equicontinuous.

Let $\{(x_n, y_n)\}$, $n = 1, 2, \ldots$, be a sequence on B_m, and

$$\begin{aligned}
F_1(x_n, y_n)(t) &= G_1(x_n, y_n)(t) + G_2(x_n, y_n)(T), \quad t \in J, \\
F_2(x_n, y_n)(t) &= H_1(x_n, y_n)(t) + H_2(x_n, y_n)(T), \quad t \in J,
\end{aligned} \tag{34}$$

where

$$G_1(x_n, y_n)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_n(s), y_n(s))\, ds, \quad t \in J, \tag{35}$$

$$\begin{aligned} G_2(x_n, y_n)(T) &= -\frac{1}{a+b} \\ &\times \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x_n(s), y_n(s))\, ds - p \right], \\ H_1(x_n, y_n)(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x_n(s), y_n(s))\, ds, \quad t \in J, \\ H_2(x_n, y_n)(T) &= -\frac{1}{c+d} \\ &\times \left[\frac{d}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} g(s, x_n(s), y_n(s))\, ds - q \right]. \end{aligned} \tag{36}$$

According to the condition (H3) and Mazur Lemma [18], we know that $\overline{\text{conv}}K_1$ is compact. For any $t_* \in J$,

$$\begin{aligned} G_1(x_n, y_n)(t_*) &= \frac{1}{\Gamma(\alpha)} \int_0^{t_*} (t_* - s)^{\alpha-1} f(s, x_n(s), y_n(s))\, ds \\ &= \frac{t_*}{\Gamma(\alpha)} \xi_n, \end{aligned} \tag{37}$$

where

$$\xi_n = \lim_{k\to\infty} \sum_{i=1}^{k} \frac{1}{k} \left(t_* - \frac{it_*}{k} \right)^{\alpha-1} f\left(\frac{it_*}{k}, x_n\left(\frac{it_*}{k}\right), y_n\left(\frac{it_*}{k}\right) \right). \tag{38}$$

Since $\overline{\text{conv}}K_1$ is convex and compact, we have that $\xi_n \in \overline{\text{conv}}K_1$. Thus, for any $t_* \in J$, the set $\{G_1(x_n, y_n)(t_*)\}$ is relatively compact. From Ascoli-Arzela theorem [17], every $\{G_1(x_n, y_n)(t)\}$ contains a uniformly convergent subsequence $\{G_1(x_{n_k}, y_{n_k})(t)\}$, $k = 1, 2, \ldots$, on J. Hence, the set $\{G_1(x, y) : (x, y) \in B_m\}$ is relatively compact. Similarly, one can obtain that $\{G_2(x_n, y_n)(T)\}$ contains a uniformly convergent subsequence $\{G_2(x_{n_k}, y_{n_k})(T)\}$, $k = 1, 2, \ldots$. Thus, the set $\{F_1(x, y) : (x, y) \in B_m\}$ is relatively compact. Similar to the above process, we can get that the set $\{F_2(x, y) : (x, y) \in B_m\}$ is relatively compact. Thus, the set $\{F(x, y), (x, y) \in B_m\}$ is relatively compact.

From the above three steps, we can conclude that F is continuous and completely compact.

Finally, we will show that the set

$$E(F) = \{(x, y) \in E : (x, y) = \lambda F(x, y),\ \text{for some } \lambda \in (0, 1)\} \tag{39}$$

is bounded.

Let $(x, y) \in E(F)$; then $(x, y) = \lambda F(x, y)$ for some $\lambda \in (0, 1)$. Hence, for any $t \in J$, we obtain

$$\begin{aligned} x(t) &= \lambda \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), y(s))\, ds - \frac{1}{a+b} \right. \\ &\quad \left. \times \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x(s), y(s))\, ds - p \right] \right), \\ y(t) &= \lambda \left(\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x(s), y(s))\, ds - \frac{1}{c+d} \right. \\ &\quad \left. \times \left[\frac{d}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} g(s, x(s), y(s))\, ds - q \right] \right). \end{aligned} \tag{40}$$

For each $t \in J$, we obtain

$$\begin{aligned} \|x(t)\| &\le \frac{M_1 T^\alpha}{\Gamma(\alpha+1)} + \frac{|b| M_1 T^\alpha}{|a+b| \Gamma(\alpha+1)} + \frac{|p|}{|a+b|} \\ &\quad + \frac{M_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|x(s)\|^{\lambda_1} ds \\ &\quad + \frac{M_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y(s)\|^{\lambda_2} ds \\ &\quad + \frac{M_1 |b|}{|a+b| \Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|x(s)\|^{\lambda_1} ds \\ &\quad + \frac{M_1 |b|}{|a+b| \Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|y(s)\|^{\lambda_2} ds, \\ \|y(t)\| &\le \frac{M_2 T^\beta}{\Gamma(\beta+1)} + \frac{|d| M_2 T^\beta}{|c+d| \Gamma(\beta+1)} + \frac{|q|}{|c+d|} \\ &\quad + \frac{M_2}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \|x(s)\|^{\mu_1} ds \\ &\quad + \frac{M_2}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \|y(s)\|^{\mu_2} ds \\ &\quad + \frac{M_2 |d|}{|c+d| \Gamma(\beta)} \int_0^T (T-s)^{\beta-1} \|x(s)\|^{\mu_1} ds \\ &\quad + \frac{M_2 |d|}{|c+d| \Gamma(\beta)} \int_0^T (T-s)^{\beta-1} \|y(s)\|^{\mu_2} ds. \end{aligned} \tag{41}$$

By Lemma 7, there exists a $M^* > 0$ such that

$$\|x(t)\| \le M^*, \quad \|y(t)\| \le M^*, \quad t \in J. \tag{42}$$

Hence for any $t \in J$, we obtain

$$\|(x, y)\| = \max\{\|x\|, \|y\|\} \leq M^*, \tag{43}$$

which implies that the set $E(F)$ is bounded. From Theorem 8 (Schaefer's fixed point theorem), we have that F has a fixed point which is a solution of the fractional BVP (2). □

Next, we give the second result of this paper, which applies Theorem 9. We firstly introduce the following assumption.

(H4) There exist functions $h_i, k_i \in L^\infty(J, X)$ and nondecreasing functions $\varphi_i, \psi_i : X \to X$ $(i = 1, 2)$ such that

$$\begin{aligned} \|f(t, x, y)\| &\leq h_1(t)\varphi_1(\|x\|) + h_2(t)\varphi_2(\|y\|), \\ &\qquad \text{for } t \in J, \quad (x, y) \in E, \\ \|g(t, x, y)\| &\leq k_1(t)\psi_1(\|x\|) + k_2(t)\psi_2(\|y\|), \\ &\qquad \text{for } t \in J, \quad (x, y) \in E. \end{aligned} \tag{44}$$

For convenience, let

$$A_1 = \frac{T^\alpha}{\Gamma(\alpha)} \max\left\{1, \frac{|b|}{|a+b|}\right\} \left(\|h_1\|_{L^\infty} + \|h_2\|_{L^\infty}\right), \tag{45}$$

$$A_2 = \frac{T^\beta}{\Gamma(\beta)} \max\left\{1, \frac{|d|}{|c+d|}\right\} \left(\|k_1\|_{L^\infty} + \|k_2\|_{L^\infty}\right), \tag{46}$$

$$\Phi(t) = \max\{\max\{\varphi_1(t), \varphi_2(t)\}, \ \max\{\psi_1(t), \psi_2(t)\}\}. \tag{47}$$

Theorem 11. *Let (H1), (H3), and (H4) hold. Assume that there exists $r > 0$, with*

$$\frac{r}{A\Phi(r) + B} > 1, \tag{48}$$

where

$$A = \max\{A_1, A_2\}, \quad B = \max\left\{\frac{|p|}{|a+b|}, \frac{|q|}{|c+d|}\right\}. \tag{49}$$

Then problem (2) has at least one solution.

Proof. Firstly, we prove that F maps sets into bounded sets in E. Let D be a bounded subset of E. For each $t \in J$, and $(x, y) \in D$ with $\|(x, y)\| \leq r$ $(r > 0)$, we have

$$\begin{aligned} &\|F_1(x, y)(t)\| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x(s), y(s))\| ds + \frac{|b|}{|a+b|\Gamma(\alpha)} \\ &\quad \times \int_0^T (T-s)^{\alpha-1} \|f(s, x(s), y(s))\| ds + \frac{|p|}{|a+b|} \\ &\leq \frac{1}{\Gamma(\alpha)} \left[\|h_1\|_{L^\infty} \varphi_1(\|x\|) + \|h_2\|_{L^\infty} \varphi_2(\|y\|)\right] \\ &\quad \times \int_0^t (t-s)^{\alpha-1} ds + \frac{|b|}{|a+b|\Gamma(\alpha)} \\ &\quad \times \left[\|h_1\|_{L^\infty} \varphi_1(\|x\|) + \|h_2\|_{L^\infty} \varphi_2(\|y\|)\right] \\ &\quad \times \int_0^T (T-s)^{\alpha-1} ds + \frac{|p|}{|a+b|} \\ &\leq \frac{T^\alpha}{\Gamma(\alpha)} \max\left\{1, \frac{|b|}{|a+b|}\right\} \left(\|h_1\|_{L^\infty} + \|h_2\|_{L^\infty}\right) \\ &\quad \times \max\{\varphi_1(\|(x, y)\|), \varphi_2(\|(x, y)\|)\} + \frac{|p|}{|a+b|}, \end{aligned} \tag{50}$$

which implies that

$$\|F_1(x, y)\| \leq A_1 \max\{\varphi_1(\|(x, y)\|), \varphi_2(\|(x, y)\|)\} + \frac{|p|}{|a+b|}, \tag{51}$$

where A_1 is as in (45). Similarly, we have

$$\begin{aligned} \|F_2(x, y)\| \leq{}& A_2 \max\{\psi_1(\|(x, y)\|), \psi_2(\|(x, y)\|)\} \\ &+ \frac{|q|}{|c+d|}, \end{aligned} \tag{52}$$

where A_2 is as in (46). Combining (47), (51), and (52), we obtain

$$\|F(x, y)\| \leq A\Phi(\|(x, y)\|) + B \leq A\Phi(r) + B, \tag{53}$$

where A and B are as in (49). This implies that $F(D)$ is bounded in E.

Secondly, we claim that F is continuous and completely continuous. The proof of this claim is the same as the corresponding part in the proof of Theorem 10 by the conditions (H1), (H3), and (H4).

Finally, let $(x, y) = \nu F(x, y)$ for some $\nu \in (0, 1)$. Then for any $t \in J$, we have by (53) that

$$\frac{\|(x, y)\|}{A\Phi(\|(x, y)\|) + B} \leq 1. \tag{54}$$

By (48), we know that there exists $r > 0$ such that $\|(x, y)\| \neq r$. Let

$$U = \{(x, y) \in E : \|(x, y)\| < r\}. \tag{55}$$

From the choice of U, there is no $(x, y) \in \partial U$ such that $(x, y) = \nu F(x, y)$ for some $\nu \in (0, 1)$. Therefore, Theorem 9 guarantees that F has a fixed point $(x, y) \in \overline{U}$ which is a solution of (2). This completes the proof. □

4. An Example

In this section, we give an example to illustrate the main results.

Example 1. Consider the following fractional boundary value problem:

$$\begin{aligned} {}_0^C D_t^{\alpha} x(t) &= \frac{1+\sqrt{t}\,|x(t)|^{\lambda_1}+t^2\,|y(t)|^{\lambda_2}}{(1+e^t)(1+|x(t)|)(1+|y(t)|)}, \\ &\qquad 0<\alpha<1, \quad t\in J:=[0,1], \\ {}_0^C D_t^{\beta} y(t) &= \frac{t+\sin t\,|x(t)|^{\mu_1}+\cos t\,|y(t)|^{\mu_2}}{e^t(2+|x(t)|)(1+|y(t)|)}, \\ &\qquad 0<\beta<1, \quad t\in J:=[0,1], \\ &x(0)+2x(1)=0, \\ &y(0)-y(1)=0. \end{aligned} \tag{56}$$

Set

$$\begin{aligned} f(t,x,y) &= \frac{1+\sqrt{t}\,|x|^{\lambda_1}+t^2\,|y|^{\lambda_2}}{(1+e^t)(1+|x|)(1+|y|)}, \\ &\qquad (t,x,y)\in J\times[1,2]\times[1,2], \\ g(t,x,y) &= \frac{t+\sin t\,|x|^{\mu_1}+\cos t\,|y|^{\mu_2}}{e^t(2+|x|)(1+|y|)}, \\ &\qquad (t,x,y)\in J\times[1,2]\times[1,2]. \end{aligned} \tag{57}$$

For each $t\in J$ and $(x,y)\in[1,2]\times[1,2]$, we have

$$\begin{aligned} |f(t,x,y)| &\le \frac{1+\sqrt{t}\,|x|^{\lambda_1}+t^2\,|y|^{\lambda_2}}{(1+e^t)(1+|x|)(1+|y|)} \\ &\le \frac{1}{8}\left(1+|x|^{\lambda_1}+|y|^{\lambda_2}\right), \\ |g(t,x,y)| &\le \frac{1+|\sin t|\,|x|^{\mu_1}+|\cos t|\,|y|^{\mu_2}}{e^t(2+|x|)(1+|y|)} \\ &\le \frac{1}{6}\left(1+|x|^{\mu_1}+|y|^{\mu_2}\right). \end{aligned} \tag{58}$$

On the other hand, we easily see that

$$\begin{aligned} &\int_0^t (t-s)^{\alpha-1}\frac{1+\sqrt{s}\,|x(s)|^{\lambda_1}+s^2\,|y(s)|^{\lambda_2}}{(1+e^s)(1+|x(s)|)(1+|y(s)|)}ds \\ &\le \int_0^t (t-s)^{\alpha-1}\left(\frac{1}{8}+\frac{1}{4}\sqrt{s}+\frac{1}{4}s^2\right)ds \\ &= \frac{1}{8\alpha}t^{\alpha}+\frac{\Gamma(\alpha)\Gamma(3/2)}{4\Gamma(\alpha+3/2)}t^{\alpha+1/2}+\frac{\Gamma(\alpha)\Gamma(3)}{4\Gamma(\alpha+3)}t^{\alpha+2} \\ &\le \frac{1}{8\alpha}+\frac{\Gamma(\alpha)\Gamma(3/2)}{4\Gamma(\alpha+3/2)}+\frac{\Gamma(\alpha)}{2\Gamma(\alpha+3)}, \\ &\int_0^t (t-s)^{\beta-1}\frac{s+|\sin s|\,|x(s)|^{\mu_1}+|\cos s|\,|y(s)|^{\mu_2}}{e^s(2+|x(s)|)(1+|y(s)|)} \\ &\le \int_0^t (t-s)^{\beta-1}\left(\frac{s}{6}+\frac{1}{2}+\frac{1}{3}\right)ds \\ &= \frac{\Gamma(\beta)\Gamma(2)}{6\Gamma(\beta+2)}t^{\beta+1}+\frac{5}{6\beta}t^{\beta} \\ &\le \frac{\Gamma(\beta)}{6\Gamma(\beta+2)}+\frac{5}{6\beta}. \end{aligned} \tag{59}$$

Thus the sets

$$\begin{aligned} K_1 &= \left\{(t-s)^{\alpha-1}\frac{1+\sqrt{s}\,|x(s)|^{\lambda_1}+s^2\,|y(s)|^{\lambda_2}}{(1+e^s)(1+|x(s)|)(1+|y(s)|)} : \right. \\ &\qquad \left. x,y\in C(J,[1,2]),\ s\in[0,t]\right\}, \\ K_2 &= \left\{(t-s)^{\beta-1}\frac{s+|\sin s|\,|x(s)|^{\mu_1}+|\cos s|\,|y(s)|^{\mu_2}}{e^s(2+|x(s)|)(1+|y(s)|)} : \right. \\ &\qquad \left. x,y\in C(J,[1,2]),\ s\in[0,t]\right\} \end{aligned} \tag{60}$$

are bounded and closed which implies that K_1 and K_2 are compact. Hence, all the assumptions in Theorem 10 are satisfied. By Theorem 10, the fractional boundary value problem (56) has at least one solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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