Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2014, Article ID 569450, 9 pages
http://dx.doi.org/10.1155/2014/569450

Research Article

Statistical Approximation of q-Bernstein-Schurer-Stancu-Kantorovich Operators

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Received 4 August 2014; Accepted 28 September 2014; Published 12 November 2014

Academic Editor: Changfeng Ma

We introduce two kinds of Kantorovich-type q-Bernstein-Schurer-Stancu operators. We first estimate moments of q-Bernstein-Schurer-Stancu-Kantorovich operators. We also establish the statistical approximation properties of these operators. Furthermore, we study the rates of statistical convergence of these operators by means of modulus of continuity and the functions of Lipschitz class.

1. Introduction

In 1987, Lupaş [1] introduced a q-type of the Bernstein operators and in 1997 another generalization of these operators based on q-integers was introduced by Phillips [2]. Thereafter, an intensive research has been done on the q-parametric operators. Recently the statistical approximation properties have also been investigated for q-analogue polynomials. For instance, in [3] q-Bleimann, Butzer, and Hahn operators; in [4] Kantorovich-type q-Bernstein operators; in [5] q-analogue of MKZ operators; in [6] Kantorovich-type q-Szász-Mirakjan operators; in [7] Kantorovich-type discrete q-Beta operators; in [8] Kantorovich-type q-Bernstein-Stancu operators were introduced and their statistical approximation properties were studied.

The main aim of this paper is to introduce two kinds of Kantorovich-type q-Bernstein-Stancu operators and study the statistical approximation properties of these operators with the help of the Korovkin-type approximation theorem. We also estimate the rate of statistical convergence by means of modulus of continuity and with the help of the elements of the Lipschits classes.

Before proceeding further, let us give some basic definitions and notations. Throughout the present paper, we consider $0 < q < 1$. For any $n = 0, 1, 2, \ldots$, the q-integer $[n]_q$ is defined as (see [2])

$$[n]_q = 1 + q + \cdots + q^{n-1}, \quad (n = 0, 1, 2, \ldots), \ [0]_q = 0 \tag{1}$$

and the q-factorial $[n]_q!$ as

$$[n]_q! = [1]_q [2]_q \cdots [n]_q, \quad (n = 1, 2, \ldots), \ [0]_q! = 1. \tag{2}$$

For the integers n, k, $n \geq k \geq 0$, the q-binomial or the Gaussian coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! \, [n-k]_q!}. \tag{3}$$

For an arbitrary function $f(x)$, the q-differential is given by

$$d_q f(x) = f(qx) - f(x). \tag{4}$$

The q-Jackson integral in the interval $[0, b]$ is defined as (see [9])

$$\int_0^b f(t)\, d_q t = (1-q) \sum_{j=0}^{\infty} f\left(q^j b\right) q^j, \quad 0 < q < 1, \tag{5}$$

provided that sums converge absolutely.

Suppose $0 < a < b$. The q-Jackson integral in a general generic interval $[a, b]$ is defined as

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t, \quad 0 < q < 1. \tag{6}$$

2. Construction of the Operators

For any $n \in N$, p a fixed nonnegative number and α, β a real parameters satisfying the conditions $0 \le \alpha \le \beta$, we introduce the Kantorovich-type q-Bernstein-Schurer-Stancu operators $K_{n,q}^{(\alpha,\beta)}(f;x) : C[0,p+1] \to C[0,1]$ as follows:

$$K_{n,q}^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) \times \int_0^1 f\left(\frac{t}{[n+1+\beta]_q} + \frac{q[k+\alpha]_q}{[n+1+\beta]_q}\right) d_q t, \quad x \in [0,1], \tag{7}$$

where $\overline{p}_{n,k}(q;x) = \left[\begin{smallmatrix} n+p \\ k \end{smallmatrix}\right]_q x^k \prod_{s=0}^{n+p-k-1}(1-q^s x)$.

The moments of these operators $K_{n,q}^{(\alpha,\beta)}(f;x)$ are obtained as follows.

Lemma 1. *For $K_{n,q}^{(\alpha,\beta)}(t^i;x)$, $i = 0,1,2$, one has*

$$K_{n,q}^{(\alpha,\beta)}(1;x) = 1, \tag{8}$$

$$K_{n,q}^{(\alpha,\beta)}(t;x) = \frac{[n+p]_q}{[n+1+\beta]_q} q^{\alpha+1} x + \frac{1}{[n+1+\beta]_q}\left(\frac{1}{[2]_q} + q[\alpha]_q\right), \tag{9}$$

$$\begin{aligned} K_{n,q}^{(\alpha,\beta)}(t^2;x) = {} & \frac{[n+p]_q[n+p-1]_q}{[n+1+\beta]_q^2} q^{2\alpha+3} x^2 \\ & + \frac{[n+p]_q}{[n+1+\beta]_q^2} \times \left(\frac{2}{[2]_q} q^{\alpha+1} + q^{2+\alpha}\left(2[\alpha]_q + q^\alpha\right)\right) x \\ & + \frac{1}{[n+1+\beta]_q^2}\left(\frac{1}{[3]_q} + \frac{2q[\alpha]_q}{[2]_q} + q^2[\alpha]_q^2\right). \end{aligned} \tag{10}$$

Proof. It is obvious that

$$\int_0^1 1\, d_q t = 1, \qquad \int_0^1 t\, d_q t = \frac{1}{[2]_q}, \qquad \int_0^1 t^2 d_q t = \frac{1}{[3]_q}. \tag{11}$$

For $i = 0$, since $\sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) = 1$, (8) holds.
For $i = 1$,

$$\begin{aligned} K_{n,q}^{(\alpha,\beta)}(t;x) &= \sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) \times \int_0^1 \left(\frac{t}{[n+1+\beta]_q} + \frac{q[k+\alpha]_q}{[n+1+\beta]_q}\right) d_q t \\ &= \sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) \int_0^1 \frac{t}{[n+1+\beta]_q} d_q t + \sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) \int_0^1 \frac{q[k+\alpha]_q}{[n+1+\beta]_q} d_q t \\ &= \frac{1}{[2]_q[n+1+\beta]_q} + \sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) \frac{q[k+\alpha]_q}{[n+1+\beta]_q}. \end{aligned} \tag{12}$$

Using the properties of the generalized q-Schurer-Stancu operators ([10, Lemma 2])

$$\sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) \frac{[k+\alpha]_q}{[n+\beta]_q} = \frac{q^\alpha [n+p]_q}{[n+\beta]_q} x + \frac{[\alpha]_q}{[n+\beta]_q}, \tag{13}$$

we have

$$K_{n,q}^{(\alpha,\beta)}(t;x) = \frac{[n+p]_q}{[n+1+\beta]_q} q^{\alpha+1} x + \frac{1}{[n+1+\beta]_q}\left(\frac{1}{[2]_q} + q[\alpha]_q\right). \tag{14}$$

For $i = 2$,

$$\begin{aligned} & \int_0^1 \left(\frac{t}{[n+1+\beta]_q} + \frac{q[k+\alpha]_q}{[n+1+\beta]_q}\right)^2 d_q t \\ &= \frac{1}{[n+1+\beta]_q^2} \times \left(\int_0^1 t^2 d_q t + 2q[k+\alpha]_q \int_0^1 t\, d_q t + q^2[k+\alpha]_q^2 \int_0^1 1\, d_q t\right) \\ &= \frac{1}{[n+1+\beta]_q^2}\left(\frac{1}{[3]_q} + \frac{2q[k+\alpha]_q}{[2]_q} + q^2[k+\alpha]_q^2\right), \end{aligned} \tag{15}$$

we obtain

$$\begin{aligned} & K_{n,q}^{(\alpha,\beta)}(t^2;x) \\ &= \frac{1}{[n+1+\beta]_q^2} \times \sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x)\left(\frac{1}{[3]_q} + \frac{2q[k+\alpha]_q}{[2]_q} + q^2[k+\alpha]_q^2\right) \end{aligned}$$

$$= \frac{1}{[3]_q [n+1+\beta]_q^2} + \frac{[n+\beta]_q}{[n+1+\beta]_q^2} \frac{2q}{[2]_q} \times \sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) \frac{[k+\alpha]_q}{[n+\beta]_q} + \frac{q^2 [n+\beta]_q^2}{[n+1+\beta]_q^2} \sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) \frac{[k+\alpha]_q^2}{[n+\beta]_q^2}. \tag{16}$$

In view of (13) and [10, Lemma 2]

$$\sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) \frac{[k+\alpha]_q^2}{[n+\beta]_q^2} = \frac{[n+p]_q [n+p-1]_q}{[n+\beta]_q^2} q^{2\alpha+1} x^2 + \frac{[n+p]_q q^{\alpha}}{[n+\beta]_q^2} \left(2[\alpha]_q + q^{\alpha}\right) x + \frac{[\alpha]_q^2}{[n+\beta]_q^2}, \tag{17}$$

by simple calculation we can get the stated result (10). □

Lemma 2. *For $\delta_n(q;x) := K_{n,q}^{(\alpha,\beta)}((t-x)^2;x)$, one has*

$$\delta_n(q;x) \le \left(\frac{[n+p]_q}{[n+1+\beta]_q} q^{\alpha+1} - 1 \right)^2 + \frac{2(1+\alpha)[n+p]_q}{[n+1+\beta]_q^2} + \frac{(1+\alpha)^2}{[n+1+\beta]_q^2}. \tag{18}$$

Proof. In view of Lemma 1, we have

$$\begin{aligned} \delta_n(q;x) &= K_{n,q}^{(\alpha,\beta)}(t^2;x) - 2x K_{n,q}^{(\alpha,\beta)}(t;x) + x^2 \\ &= \left(\frac{[n+p]_q [n+p-1]_q q^{2\alpha+3}}{[n+1+\beta]_q^2} - \frac{2q^{\alpha+1}[n+p]_q}{[n+1+\beta]_q} + 1 \right) x^2 \\ &\quad + \left(\frac{[n+p]_q}{[n+1+\beta]_q^2} \left(\frac{2}{[2]_q} q^{\alpha+1} + q^{2+\alpha}\left(2[\alpha]_q + q^{\alpha}\right) \right) - \frac{2}{[n+1+\beta]_q} \left(\frac{1}{[2]_q} + q[\alpha]_q \right) \right) x \\ &\quad + \frac{1}{[n+1+\beta]_q^2} \left(\frac{1}{[3]_q} + \frac{2q[\alpha]_q}{[2]_q} + q^2[\alpha]_q^2 \right) \\ &\le \left(\frac{[n+p]_q}{[n+1+\beta]_q} q^{\alpha+1} - 1 \right)^2 + \frac{2(1+\alpha)[n+p]_q}{[n+1+\beta]_q^2} + \frac{(1+\alpha)^2}{[n+1+\beta]_q^2}. \end{aligned} \tag{19}$$

□

Now, we consider a sequence $q = q_n$ satisfying the following two expressions:

$$\lim_{n\to\infty} q_n = 1, \qquad \lim_{n\to\infty} \frac{1}{[n]_{q_n}} = 0. \tag{20}$$

By Korovkin's theorem, we can state the following theorem.

Theorem 3. *Let $K_{n,q_n}^{(\alpha,\beta)}(f;x)$ be a sequence satisfying (20) for $0 < q_n < 1$. Then for any function $f \in C[0, p+1]$, the following equality*

$$\lim_{n\to\infty} \left\| K_{n,q_n}^{(\alpha,\beta)}(f;\cdot) - f \right\|_{C[0,1]} = 0 \tag{21}$$

is satisfied.

Proof. We know that $K_{n,q}^{(\alpha,\beta)}(f;x)$ is linear positive. By Lemma 1, if we choose the sequence $q = q_n$ satisfying (20) and using the equality

$$[n+p]_{q_n} = [n]_{q_n} + q_n^n [p]_{q_n}, \quad [n+1+\beta]_{q_n} = [n]_{q_n} + q_n^n [\beta+1]_{q_n} \tag{22}$$

we have

$$\lim_{n\to\infty} \left\| K_{n,q_n}^{(\alpha,\beta)}(t^i;x) - x^i \right\|_{C[0,1]} = 0, \quad i = 0,1,2. \tag{23}$$

Because of the linearity and positivity of $K_{n,q_n}^{(\alpha,\beta)}(f;x)$, the proof is complete by the classical Korovkin theorem. □

We now redefine $K_{n,q}^{(\alpha,\beta)}(f;x)$ as

$$\widetilde{K}_{n,q}^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n+p} \overline{p}_{n,k}(q;x) \times \int_0^1 f\left(\frac{t}{[n+1]_q + \beta} + \frac{q\left([k]_q + \alpha\right)}{[n+1]_q + \beta} \right) d_q t. \tag{24}$$

Let us give some lemmas as follows.

Lemma 4. *For $\widetilde{K}_{n,q}^{(\alpha,\beta)}(t^i;x)$, $i = 0,1,2$, one has*

$$\widetilde{K}_{n,q}^{(\alpha,\beta)}(1;x) = 1, \tag{25}$$

$$\widetilde{K}_{n,q}^{(\alpha,\beta)}(t;x) = \frac{[n+p]_q}{[n+1]_q + \beta} qx + \frac{1}{[n+1]_q + \beta} \left(\frac{1}{[2]_q} + q\alpha \right), \tag{26}$$

$$\widetilde{K}_{n,q}^{(\alpha,\beta)}\left(t^2;x\right)=\frac{[n+p]_q\,[n+p-1]_q}{\left([n+1]_q+\beta\right)^2}q^3x^2+\frac{[n+p]_q}{\left([n+1]_q+\beta\right)^2}\left(\frac{2q}{[2]_q}+q^2(2\alpha+1)\right)x+\frac{1}{\left([n+1]_q+\beta\right)^2}\left(\frac{1}{[3]_q}+\frac{2q\alpha}{[2]_q}+q^2\alpha^2\right).\tag{27}$$

Proof. It is obvious that (25) holds.

For $i=1$,

$$\begin{aligned}\widetilde{K}_{n,q}^{(\alpha,\beta)}(t;x)&=\sum_{k=0}^{n+p}\overline{p}_{n,k}(q;x)\times\int_0^1\left(\frac{t}{[n+1]_q+\beta}+\frac{q\left([k]_q+\alpha\right)}{[n+1]_q+\beta}\right)d_qt\\&=\sum_{k=0}^{n+p}\overline{p}_{n,k}(q;x)\int_0^1\frac{t}{[n+1]_q+\beta}d_qt+\sum_{k=0}^{n+p}\overline{p}_{n,k}(q;x)\int_0^1\frac{q\left([k]_q+\alpha\right)}{[n+1+\beta]_q}d_qt\\&=\frac{1}{[2]_q\left([n+1]_q+\beta\right)}+\sum_{k=0}^{n+p}\overline{p}_{n,k}(q;x)\frac{q\left([k]_q+\alpha\right)}{[n+1]_q+\beta}.\end{aligned}\tag{28}$$

Taking into account [11, Lemma 1]

$$\sum_{k=0}^{n+p}\overline{p}_{n,k}(q;x)\frac{[k]_q+\alpha}{[n]_q+\beta}=\frac{[n+p]_q}{[n]_q+\beta}x+\frac{\alpha}{[n]_q+\beta},\tag{29}$$

we have

$$\widetilde{K}_{n,q}^{(\alpha,\beta)}(t;x)=\frac{[n+p]_q}{[n+1]_q+\beta}qx+\frac{1}{[n+1]_q+\beta}\left(\frac{1}{[2]_q}+q\alpha\right).\tag{30}$$

For $i=2$,

$$\begin{aligned}&\int_0^1\left(\frac{t}{[n+1]_q+\beta}+\frac{q\left([k]_q+\alpha\right)}{[n+1]_q+\beta}\right)^2d_qt\\&=\frac{1}{\left([n+1]_q+\beta\right)^2}\times\left(\int_0^1t^2d_qt+2q\left([k]_q+\alpha\right)\times\int_0^1t\,d_qt+q^2\left([k]_q+\alpha\right)^2\int_0^1 1\,d_qt\right)\\&=\frac{1}{\left([n+1]_q+\beta\right)^2}\times\left(\frac{1}{[3]_q}+\frac{2q\left([k]_q+\alpha\right)}{[2]_q}+q^2\left([k]_q+\alpha\right)^2\right),\end{aligned}\tag{31}$$

we obtain

$$\begin{aligned}&\widetilde{K}_{n,q}^{(\alpha,\beta)}\left(t^2;x\right)\\&=\frac{1}{\left([n+1]_q+\beta\right)^2}\times\sum_{k=0}^{n+p}\overline{p}_{n,k}(q;x)\left(\frac{1}{[3]_q}+\frac{2q\left([k]_q+\alpha\right)}{[2]_q}+q^2\left([k]_q+\alpha\right)^2\right)\\&=\frac{1}{[3]_q\left([n+1]_q+\beta\right)^2}+\frac{[n]_q+\beta}{\left([n+1]_q+\beta\right)^2}\frac{2q}{[2]_q}\times\sum_{k=0}^{n+p}\overline{p}_{n,k}(q;x)\frac{[k]_q+\alpha}{[n]_q+\beta}\\&\quad+\frac{q^2\left([n]_q+\beta\right)^2}{\left([n+1]_q+\beta\right)^2}\sum_{k=0}^{n+p}\overline{p}_{n,k}(q;x)\left(\frac{[k]_q+\alpha}{[n]_q+\beta}\right)^2.\end{aligned}\tag{32}$$

From (29) and [11, Lemma 1]

$$\begin{aligned}\sum_{k=0}^{n+p}\overline{p}_{n,k}(q;x)\left(\frac{[k]_q+\alpha}{[n]_q+\beta}\right)^2&=\frac{[n+p]_q\,[n+p-1]_q}{\left([n]_q+\beta\right)^2}qx^2\\&\quad+\frac{[n+p]_q}{\left([n]_q+\beta\right)^2}(2\alpha+1)x+\frac{\alpha^2}{\left([n]_q+\beta\right)^2},\end{aligned}\tag{33}$$

by simple calculation we arrive at the desired result (27). □

Lemma 5. *For* $\widetilde{\delta}_n(q;x):=K_{n,q}^{(\alpha,\beta)}((t-x)^2;x)$, *one has*

$$\widetilde{\delta}_n(q;x)\le\left(\frac{[n+p]_q}{[n+1]_q+\beta}q-1\right)^2+\frac{2(1+\alpha)[n+p]_q}{\left([n+1]_q+\beta\right)^2}+\frac{(1+\alpha)^2}{\left([n+1]_q+\beta\right)^2}.\tag{34}$$

Proof. From Lemma 4, it is immediately seen that

$$\begin{aligned}
&\widetilde{\delta}_n(q;x) \\
&= \widetilde{K}^{(\alpha,\beta)}_{n,q}(t^2;x) - 2x\widetilde{K}^{(\alpha,\beta)}_{n,q}(t;x) + x^2 \\
&= \left(\frac{[n+p]_q[n+p-1]_q q^3}{([n+1]_q+\beta)^2} - \frac{2q[n+p]_q}{[n+1]_q+\beta} + 1\right)x^2 \\
&\quad + \left(\frac{[n+p]_q}{([n+1]_q+\beta)^2}\left(\frac{2}{[2]_q}q + q^2(2\alpha+1)\right)\right. \\
&\quad \left. - \frac{2}{[n+1]_q+\beta}\left(\frac{1}{[2]_q} + q\alpha\right)\right)x \\
&\quad + \frac{1}{([n+1]_q+\beta)^2}\left(\frac{1}{[3]_q} + \frac{2q\alpha}{[2]_q} + q^2\alpha^2\right) \\
&\le \left(\frac{[n+p]_q}{[n+1]_q+\beta}q - 1\right)^2 \\
&\quad + \frac{2(1+\alpha)[n+p]_q}{([n+1]_q+\beta)^2} + \frac{(1+\alpha)^2}{([n+1]_q+\beta)^2}.
\end{aligned} \tag{35}$$

□

We can give the following result, a theorem of Korovkin type.

Theorem 6. *Let $q = q_n$ be a sequence satisfying (20) for $0 < q_n < 1$. Then for any function $f \in C[0, p+1]$, the sequence $\widetilde{K}^{(\alpha,\beta)}_{n,q_n}(f;x)$ converges to $f(x)$ uniformly on $[0,1]$.*

The proof of the above theorem follows along Theorem 3; thus we omit the details.

3. Statistical Approximation of Korovkin Type

Further on, let us recall the concept of statistical convergence which was introduced by Fast [12].

Let us set $K \in N$ and $K_n = \{k \le n : k \in K\}$; the natural density of K is defined by $\delta(K) := \lim_{n\to\infty}(1/n)|K_n|$ if the limit exists (see [13]), where $|K_n|$ denotes the cardinality of the set K_n.

A sequence $x = x_k$ is called statistically convergent to a number L if, for every $\varepsilon > 0$, $\delta\{k \in N : |x_k - L| \ge \varepsilon\} = 0$. This convergence is denoted as $\mathrm{st}-\lim_k x_k = L$. It is known that any convergent sequence is statistically convergent, but its converse is not true. Details can be found in [14].

In approximation theory by linear positive operators, the concept of statistical convergence was used by Gadjiev and Orhan [15]. They proved the following Bohman-Korovkin-type approximation theorem for statistical convergence.

Theorem 7 (see [15]). *If the sequence of linear positive operators $A_n : C[a,b] \to C[a,b]$ satisfies the conditions*

$$st-\lim_n \|A_n(e_i;\cdot) - e_i\|_{C[a,b]} = 0 \tag{36}$$

for $e_i(t) = t^i$, $i = 0,1,2$, then, for any $f \in C[a,b]$,

$$st-\lim_n \|A_n(f;\cdot) - f\|_{C[a,b]} = 0. \tag{37}$$

In this section, we establish the following Korovkin-type statistical approximation theorems.

Theorem 8. *Let $q = q_n$, $0 < q_n < 1$, be a sequence satisfying the following conditions:*

$$\begin{gathered}
st-\lim_n q_n = 1, \\
st-\lim_n q_n^n = a \quad (a < 1), \\
st-\lim_n \frac{1}{[n]_{q_n}} = 0;
\end{gathered} \tag{38}$$

then for $f \in C[0, p+1]$, one has

$$st-\lim_n \left\|K^{(\alpha,\beta)}_{n,q_n}(f;\cdot) - f\right\|_{C[0,1]} = 0. \tag{39}$$

Proof. From Theorem 7, it is enough to prove that $\mathrm{st}-\lim_n \|K^{(\alpha,\beta)}_{n,q_n}(e_i;\cdot) - e_i\|_{C[0,1]} = 0$ for $e_i = t^i$, $i = 0,1,2$.

By (8), we can easily get

$$\mathrm{st}-\lim_n \left\|K^{(\alpha,\beta)}_{n,q_n}(e_0;\cdot) - e_0\right\|_{C[0,1]} = 0. \tag{40}$$

From equality (9) we have

$$\begin{aligned}
&\left\|K^{(\alpha,\beta)}_{n,q_n}(e_1;\cdot) - e_1\right\|_{C[0,1]} \\
&\le \left|\frac{q_n^{\alpha+1}[n+p]_{q_n}}{[n+1+\beta]_{q_n}} - 1\right| \\
&\quad + \frac{1}{[n+1+\beta]_{q_n}}\left(\frac{1}{[2]_{q_n}} + q_n[\alpha]_{q_n}\right) \\
&\le \left|\frac{q_n^{\alpha+1}[n+p]_{q_n}}{[n+1+\beta]_{q_n}} - 1\right| + \frac{1+\alpha}{[n]_{q_n}}.
\end{aligned} \tag{41}$$

Now for a given $\varepsilon > 0$, let us define the following sets:

$$\begin{aligned}
U &= \left\{k : \left\|K^{(\alpha,\beta)}_{n,q_k}(e_1;\cdot) - e_1\right\|_{C[0,1]} \ge \varepsilon\right\}, \\
U_1 &= \left\{k : \frac{q_k^{\alpha+1}[k+p]_{q_k}}{[k+1+\beta]_{q_k}} - 1 \ge \frac{\varepsilon}{2}\right\}, \\
U_2 &= \left\{k : \frac{1+\alpha}{[k]_{q_k}} \ge \frac{\varepsilon}{2}\right\}.
\end{aligned} \tag{42}$$

From (41), one can see that $U \subseteq U_1 \cup U_2$, so we have

$$\delta\left\{k \le n : \left\|K^{(\alpha,\beta)}_{n,q_k}(e_1;\cdot) - e_1\right\|_{C[0,1]} \ge \varepsilon\right\} \le \delta\left\{k \le n : \frac{q_k^{\alpha+1}[k+p]_{q_k}}{[k+1+\beta]_{q_k}} - 1 \ge \frac{\varepsilon}{2}\right\} + \delta\left\{k \le n : \frac{1+\alpha}{[k]_{q_k}} \ge \frac{\varepsilon}{2}\right\}. \quad (43)$$

By (22) and (38) it is clear that

$$\text{st} - \lim_n \left(\frac{q_n^{\alpha+1}[n+p]_{q_n}}{[n+1+\beta]_{q_n}} - 1\right) = 0, \qquad \text{st} - \lim_n \frac{1+\alpha}{[n]_{q_n}} = 0. \quad (44)$$

So we have

$$\text{st} - \lim_n \left\|K^{(\alpha,\beta)}_{n,q_n}(e_1;\cdot) - e_1\right\|_{C[0,1]} = 0. \quad (45)$$

Finally, in view of (10), one can write

$$\begin{aligned}\left\|K^{(\alpha,\beta)}_{n,q_n}(e_2;\cdot) - e_2\right\|_{C[0,1]} &\le \left|\frac{[n+p]_{q_n}[n+p-1]_{q_n}}{[n+1+\beta]^2_{q_n}} q_n^{2\alpha+3} - 1\right| \\ &+ \frac{[n+p]_{q_n}}{[n+1+\beta]^2_{q_n}}\left(\frac{2}{[2]_{q_n}} q_n^{\alpha+1} + q_n^{2+\alpha}\left(2[\alpha]_{q_n} + q_n^{\alpha}\right)\right) \\ &+ \frac{1}{[n+1+\beta]^2_{q_n}}\left(\frac{1}{[3]_{q_n}} + \frac{2q_n[\alpha]_{q_n}}{[2]_{q_n}} + q_n^2[\alpha]^2_{q_n}\right).\end{aligned} \quad (46)$$

Using (22),

$$\begin{aligned}&\frac{2}{[2]_{q_n}} q_n^{\alpha+1} + q_n^{2+\alpha}\left(2[\alpha]_{q_n} + q_n^{\alpha}\right) \le 2 + 2\alpha, \\ &\frac{1}{[3]_{q_n}} + \frac{2q_n[\alpha]_q}{[2]_q} + q^2[\alpha]^2_q \le (1+\alpha)^2, \\ &q_n[n+p-1]_{q_n} = [n+p]_{q_n} - 1,\end{aligned} \quad (47)$$

we can write

$$\begin{aligned}\left\|K^{(\alpha,\beta)}_{n,q_n}(e_2;\cdot) - e_2\right\|_{C[0,1]} &\le \frac{[n+p]_{q_n} q_n^{2\alpha+2}}{[n+1+\beta]^2_{q_n}} + \left|\frac{q_n^{2\alpha+2}[n+p]^2_{q_n}}{[n+1+\beta]^2_{q_n}} - 1\right| \\ &+ \frac{(2+2\alpha)[n+p]_{q_n}}{[n+1+\beta]^2_{q_n}} + \frac{(1+\alpha)^2}{[n+1+\beta]^2_{q_n}} \\ &\le \left|\frac{q_n^{2\alpha+2}[n+p]^2_{q_n}}{[n+1+\beta]^2_{q_n}} - 1\right| \\ &+ \frac{(3+2\alpha)}{[n]_{q_n}}\left(1 + \frac{[p]_{q_n}}{[n]_{q_n}}\right) + \frac{(1+\alpha)^2}{[n]^2_{q_n}} \\ &=: \theta_n + \gamma_n + \eta_n.\end{aligned} \quad (48)$$

Then, from (22) and (38), we have

$$\text{st} - \lim_n \theta_n = \text{st} - \lim_n \gamma_n = \text{st} - \lim_n \eta_n = 0. \quad (49)$$

Here for a given $\varepsilon > 0$, let us define the following sets:

$$\begin{aligned}T &= \left\{k : \left\|K^{(\alpha,\beta)}_{n,q_k}(e_2;\cdot) - e_2\right\|_{C[0,1]} \ge \varepsilon\right\}, \\ T_1 &= \left\{k : \theta_k \ge \frac{\varepsilon}{3}\right\}, \qquad T_2 = \left\{k : \gamma_k \ge \frac{\varepsilon}{3}\right\}, \\ T_3 &= \left\{k : \eta_k \ge \frac{\varepsilon}{3}\right\}.\end{aligned} \quad (50)$$

It is clear that $T \subseteq T_1 \cup T_2 \cup T_3$. So we get

$$\begin{aligned}\delta\left\{k \le n : \left\|K^{(\alpha,\beta)}_{n,q_k}(e_2;\cdot) - e_2\right\|_{C[0,1]} \ge \varepsilon\right\} &\le \delta\left\{k \le n : \theta_k \ge \frac{\varepsilon}{3}\right\} + \delta\left\{k \le n : \gamma_k \ge \frac{\varepsilon}{3}\right\} \\ &+ \delta\left\{k \le n : \eta_k \ge \frac{\varepsilon}{3}\right\}.\end{aligned} \quad (51)$$

By (49), we have

$$\delta\left\{k \le n : \left\|K^{(\alpha,\beta)}_{n,q_k}(e_2;\cdot) - e_2\right\|_{C[0,1]} \ge \varepsilon\right\} = 0, \quad (52)$$

which implies that

$$\text{st} - \lim_n \left\|K^{(\alpha,\beta)}_{n,q_n}(e_2;\cdot) - e_2\right\|_{C[0,1]} = 0. \quad (53)$$

In view of (40), (45), and (53), the proof is complete. □

Theorem 9. *Let $q = q_n$, $0 < q_n < 1$, be a sequence satisfying (38); then for $f \in C[0, p+1]$, one has* $st - \lim_n \|\widetilde{K}^{(\alpha,\beta)}_{n,q_n}(f;\cdot) - f\|_{C[0,1]} = 0$.

Proof. From Theorem 7, it is enough to prove that $\text{st} - \lim_n \|\widetilde{K}^{(\alpha,\beta)}_{n,q_n}(e_i;\cdot) - e_i\|_{C[0,1]} = 0$ for $e_i = t^i$, $i = 0, 1, 2$.

Using (25), we can easily get

$$\text{st} - \lim_n \left\|\widetilde{K}^{(\alpha,\beta)}_{n,q_n}(e_0;\cdot) - e_0\right\|_{C[0,1]} = 0. \quad (54)$$

From equality (26) we have

$$\left\|\widetilde{K}^{(\alpha,\beta)}_{n,q_n}(e_1;\cdot) - e_1\right\|_{C[0,1]} \le \left|\frac{q_n[n+p]_{q_n}}{[n+1]_{q_n} + \beta} - 1\right| + \frac{1+\alpha}{[n]_{q_n}}. \quad (55)$$

Now for a given $\varepsilon > 0$, let us define the following sets:

$$\widetilde{U} = \left\{k : \left\|\widetilde{K}_{n,q_k}^{(\alpha,\beta)}(e_1;\cdot) - e_1\right\|_{C[0,1]} \geq \varepsilon\right\},$$
$$\widetilde{U}_1 = \left\{k : \frac{q_k [k+p]_{q_k}}{[k+1]_{q_k} + \beta} - 1 \geq \frac{\varepsilon}{2}\right\}, \tag{56}$$
$$\widetilde{U}_2 = \left\{k : \frac{1+\alpha}{[k]_{q_k}} \geq \frac{\varepsilon}{2}\right\}.$$

From (55), one can see that $\widetilde{U} \subseteq \widetilde{U}_1 \cup \widetilde{U}_2$, so we have

$$\delta\left\{k \leq n : \left\|\widetilde{K}_{n,q_k}^{(\alpha,\beta)}(e_1;\cdot) - e_1\right\|_{C[0,1]} \geq \varepsilon\right\}$$
$$\leq \delta\left\{k \leq n : \frac{q_k [k+p]_{q_k}}{[k+1]_{q_k} + \beta} - 1 \geq \frac{\varepsilon}{2}\right\} \tag{57}$$
$$+ \delta\left\{k \leq n : \frac{1+\alpha}{[k]_{q_k}} \geq \frac{\varepsilon}{2}\right\}.$$

By (38) it is clear that

$$\mathrm{st} - \lim_n \left(\frac{q_n [n+p]_{q_n}}{[n+1]_{q_n} + \beta} - 1\right) = 0, \qquad \mathrm{st} - \lim_n \frac{1+\alpha}{[n]_{q_n}} = 0. \tag{58}$$

So we have

$$\mathrm{st} - \lim_n \left\|\widetilde{K}_{n,q_n}^{(\alpha,\beta)}(e_1;\cdot) - e_1\right\|_{C[0,1]} = 0. \tag{59}$$

Finally, in view of (27), one can write

$$\left\|\widetilde{K}_{n,q_n}^{(\alpha,\beta)}(e_2;\cdot) - e_2\right\|_{C[0,1]}$$
$$\leq \left|\frac{[n+p]_{q_n}[n+p-1]_{q_n}}{\left([n+1]_{q_n} + \beta\right)^2} q_n^3 - 1\right|$$
$$+ \frac{[n+p]_{q_n}}{\left([n+1]_{q_n} + \beta\right)^2}\left(\frac{2}{[2]_{q_n}} + q_n(2\alpha+1)\right) \tag{60}$$
$$+ \frac{1}{\left([n+1]_{q_n} + \beta\right)^2}\left(\frac{1}{[3]_{q_n}} + \frac{2q_n\alpha}{[2]_{q_n}} + q_n^2\alpha^2\right).$$

Using $q_n[n+p-1]_{q_n} = [n+p]_{q_n} - 1$, then we can write

$$\left|\frac{[n+p]_{q_n}[n+p-1]_{q_n}}{\left([n+1]_{q_n} + \beta\right)^2} q_n^3 - 1\right|$$
$$= \left|\left(\frac{[n+p]_{q_n}}{[n+1]_{q_n} + \beta} q_n\right)^2 - 1\right| + \frac{[n+p]_{q_n} q_n^2}{\left([n+1]_{q_n} + \beta\right)^2}. \tag{61}$$

So,

$$\left\|\widetilde{K}_{n,q_n}^{(\alpha,\beta)}(e_2;\cdot) - e_2\right\|_{C[0,1]}$$
$$\leq \left|\left(\frac{[n+p]_{q_n}}{[n+1]_{q_n} + \beta} q_n\right)^2 - 1\right|$$
$$+ \frac{(3+2\alpha)[n+p]_{q_n}}{\left([n+1]_{q_n} + \beta\right)^2} + \frac{(1+\alpha)^2}{[n]_{q_n}^2} \tag{62}$$
$$=: \widetilde{\theta}_n + \widetilde{\gamma}_n + \widetilde{\eta}_n.$$

Then, from (38), we have

$$\mathrm{st} - \lim_n \widetilde{\theta}_n = \mathrm{st} - \lim_n \widetilde{\gamma}_n = \mathrm{st} - \lim_n \widetilde{\eta}_n = 0. \tag{63}$$

Here for a given $\varepsilon > 0$, let us define the following sets:

$$\widetilde{T} = \left\{k : \left\|\widetilde{K}_{n,q_k}^{(\alpha,\beta)}(e_2;\cdot) - e_2\right\|_{C[0,1]} \geq \varepsilon\right\},$$
$$\widetilde{T}_1 = \left\{k : \widetilde{\theta}_k \geq \frac{\varepsilon}{3}\right\}, \qquad \widetilde{T}_2 = \left\{k : \widetilde{\gamma}_k \geq \frac{\varepsilon}{3}\right\}, \tag{64}$$
$$T_3 = \left\{k : \widetilde{\eta}_k \geq \frac{\varepsilon}{3}\right\}.$$

It is clear that $\widetilde{T} \subseteq \widetilde{T}_1 \cup \widetilde{T}_2 \cup \widetilde{T}_3$. So we get

$$\delta\left\{k \leq n : \left\|\widetilde{K}_{n,q_k}^{(\alpha,\beta)}(e_2;\cdot) - e_2\right\|_{C[0,1]} \geq \varepsilon\right\}$$
$$\leq \delta\left\{k \leq n : \widetilde{\theta}_k \geq \frac{\varepsilon}{3}\right\} + \delta\left\{k \leq n : \widetilde{\gamma}_k \geq \frac{\varepsilon}{3}\right\} \tag{65}$$
$$+ \delta\left\{k \leq n : \widetilde{\eta}_k \geq \frac{\varepsilon}{3}\right\}.$$

By (63), we have

$$\delta\left\{k \leq n : \left\|\widetilde{K}_{n,q_k}^{(\alpha,\beta)}(e_2;\cdot) - e_2\right\|_{C[0,1]} \geq \varepsilon\right\} = 0, \tag{66}$$

which implies that

$$\mathrm{st} - \lim_n \left\|\widetilde{K}_{n,q_n}^{(\alpha,\beta)}(e_2;\cdot) - e_2\right\|_{C[0,1]} = 0. \tag{67}$$

In view of (54), (59), and (67), the proof is complete. □

4. Rates of Statistical Convergence

Let $f \in C[0, p+1]$ for any $t \in [0, p+1]$ and $x \in [0,1]$. Then we have $|f(t) - f(x)| \leq \omega(f, |t-x|)$, so for any $\delta > 0$, we get

$$\omega(f, |t-x|) \leq \begin{cases} \omega(f,\delta), & |t-x| < \delta, \\ \omega\left(f, \dfrac{(t-x)^2}{\delta}\right), & |t-x| \geq \delta. \end{cases} \tag{68}$$

Owing to $\omega(f, \lambda\delta) \leq (1+\lambda)\omega(f,\delta)$ for $\lambda > 0$, it is obvious that we have

$$\left|f(t) - f(x)\right| \leq \left(1 + \delta^{-2}(t-x)^2\right)\omega(f,\delta) \tag{69}$$

for any $t \in [0, p+1]$, $x \in [0,1]$ and $\delta > 0$.

Next we will give the rates of convergence of both $K_{n,q}^{(\alpha,\beta)}(f;x)$ and $\widetilde{K}_{n,q}^{(\alpha,\beta)}(f;x)$ in terms of the modulus of continuity.

Theorem 10. *Let $q = q_n$, $0 < q_n < 1$ be a sequence satisfying (38); then for any function $f \in C[0, p + 1]$, $x \in [0, 1]$, one has*

$$\left|K_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)\right| \leq 2\omega\left(f, \sqrt{\delta_n}\right), \tag{70}$$

where

$$\delta_n = \left(\frac{[n+p]_{q_n}}{[n+1+\beta]_{q_n}} q_n^{\alpha+1} - 1\right)^2 + \frac{2(1+\alpha)[n+p]_{q_n}}{[n+1+\beta]_{q_n}^2} + \frac{(1+\alpha)^2}{[n+1+\beta]_{q_n}^2}. \tag{71}$$

Proof. Using the linearity and positivity of the operator $K_{n,q}^{(\alpha,\beta)}(f;x)$ and inequality (69), for any $f \in C[0, p + 1]$ and $x \in [0, 1]$, we get

$$\begin{aligned}\left|K_{n,q}^{(\alpha,\beta)}(f,x) - f(x)\right| &\leq K_{n,q}^{(\alpha,\beta)}\left(\left|f(t) - f(x)\right|; x\right) \\ &\leq \left(1 + \delta^{-2} K_{n,q}^{(\alpha,\beta)}\left((t-x)^2; x\right)\right)\omega(f,\delta).\end{aligned} \tag{72}$$

In view of Lemma 2, take $q = q_n$, $0 < q_n < 1$ as a sequence satisfying (38) and choose $\delta = \sqrt{\delta_n}$ in (72); the desired result follows immediately. □

Theorem 11. *Let $q = q_n$, $0 < q_n < 1$ be a sequence satisfying (38); then for any function $f \in C[0, p + 1]$, $x \in [0, 1]$, one has*

$$\left|\widetilde{K}_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)\right| \leq 2\omega\left(f, \sqrt{\delta_n^*}\right), \tag{73}$$

where

$$\delta_n^* = \left(\frac{[n+p]_{q_n}}{[n+1]_{q_n} + \beta} q_n - 1\right)^2 + \frac{(2+2\alpha)[n+p]_{q_n}}{\left([n+1]_{q_n} + \beta\right)^2} + \frac{(1+\alpha)^2}{\left([n+1]_{q_n} + \beta\right)^2}. \tag{74}$$

Proof. Using the linearity and positivity of the operator $\widetilde{K}_{n,q}^{(\alpha,\beta)}(f;x)$ and inequality (69), for any $f \in C[0, p + 1]$ and $x \in [0, 1]$, we get

$$\begin{aligned}\left|\widetilde{K}_{n,q}^{(\alpha,\beta)}(f,x) - f(x)\right| &\leq \widetilde{K}_{n,q}^{(\alpha,\beta)}\left(\left|f(t) - f(x)\right|; x\right) \\ &\leq \left(1 + \delta^{-2} \widetilde{K}_{n,q}^{(\alpha,\beta)}\left((t-x)^2; x\right)\right)\omega(f,\delta).\end{aligned} \tag{75}$$

In view of Lemma 5, take $q = q_n$, $0 < q_n < 1$ as a sequence satisfying (38) and choose $\delta = \sqrt{\delta_n^*}$ in (75); the desired result follows immediately. □

Finally, we give the rates of statistical convergence of both $K_{n,q}^{(\alpha,\beta)}(f;x)$ and $\widetilde{K}_{n,q}^{(\alpha,\beta)}(f;x)$ with the help of functions of the Lipschitz class. We recall a function $f \in \mathrm{Lip}_M(\lambda)$ on $[0, p+1]$, if the inequality

$$\left|f(t) - f(x)\right| \leq M\left|t - x\right|^\lambda, \quad t, x \in [0, p+1] \tag{76}$$

holds.

Theorem 12. *Let $f \in \mathrm{Lip}_M(\lambda)$ on $[0, p + 1]$, $0 < \lambda \leq 1$. Let $q = q_n$, $0 < q_n < 1$ be a sequence satisfying the condition given in (38). If we take δ_n as in (71), then one has*

$$\left|K_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)\right| \leq M\delta_n^{\lambda/2}, \quad x \in [0,1]. \tag{77}$$

Proof. Let $f \in \mathrm{Lip}_M(\lambda)$ on $[0, p + 1]$, $0 < \lambda \leq 1$. Since $K_{n,q_n}^{(\alpha,\beta)}(f;x)$ is linear and positive, by using (76), we have

$$\begin{aligned}\left|K_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)\right| &\leq K_{n,q_n}^{(\alpha,\beta)}\left(\left|f(t) - f(x)\right|; x\right) \\ &\leq K_{n,q_n}^{(\alpha,\beta)}\left(\left|t - x\right|^\lambda; x\right).\end{aligned} \tag{78}$$

If we take $p' = 2/\lambda$, $q' = 2/(2-\lambda)$ and apply the Hölder inequality and Lemma 2, then we obtain

$$\begin{aligned}\left|K_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)\right| &\leq M\left(K_{n,q_n}^{(\alpha,\beta)}\left((t-x)^2; x\right)\right)^{\lambda/2} \\ &\leq M\delta_n^{\lambda/2}.\end{aligned} \tag{79}$$

□

Theorem 13. *Let $f \in \mathrm{Lip}_M(\lambda)$ on $[0, p + 1]$, $0 < \lambda \leq 1$. Let $q = q_n$, $0 < q_n < 1$ be a sequence satisfying the condition given in (38). If we take δ_n^* as in (74), then one has*

$$\left|\widetilde{K}_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)\right| \leq M\left(\delta_n^*\right)^{\lambda/2}, \quad x \in [0,1]. \tag{80}$$

Proof. Let $f \in \mathrm{Lip}_M(\lambda)$ on $[0, p + 1]$, $0 < \lambda \leq 1$. Since $\widetilde{K}_{n,q_n}^{(\alpha,\beta)}(f;x)$ is linear and positive, by using (76), we have

$$\begin{aligned}\left|\widetilde{K}_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)\right| &\leq \widetilde{K}_{n,q_n}^{(\alpha,\beta)}\left(\left|f(t) - f(x)\right|; x\right) \\ &\leq \widetilde{K}_{n,q_n}^{(\alpha,\beta)}\left(\left|t - x\right|^\lambda; x\right).\end{aligned} \tag{81}$$

If we take $p' = 2/\lambda$, $q' = 2/(2-\lambda)$ and apply the Hölder inequality and Lemma 5, then we obtain

$$\begin{aligned}\left|\widetilde{K}_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)\right| &\leq M\left(\widetilde{K}_{n,q_n}^{(\alpha,\beta)}\left((t-x)^2; x\right)\right)^{\lambda/2} \\ &\leq M\left(\delta_n^*\right)^{\lambda/2}.\end{aligned} \tag{82}$$

□

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work is supported by the Fundamental Research Funds for the Central Universities (no. N110323010).

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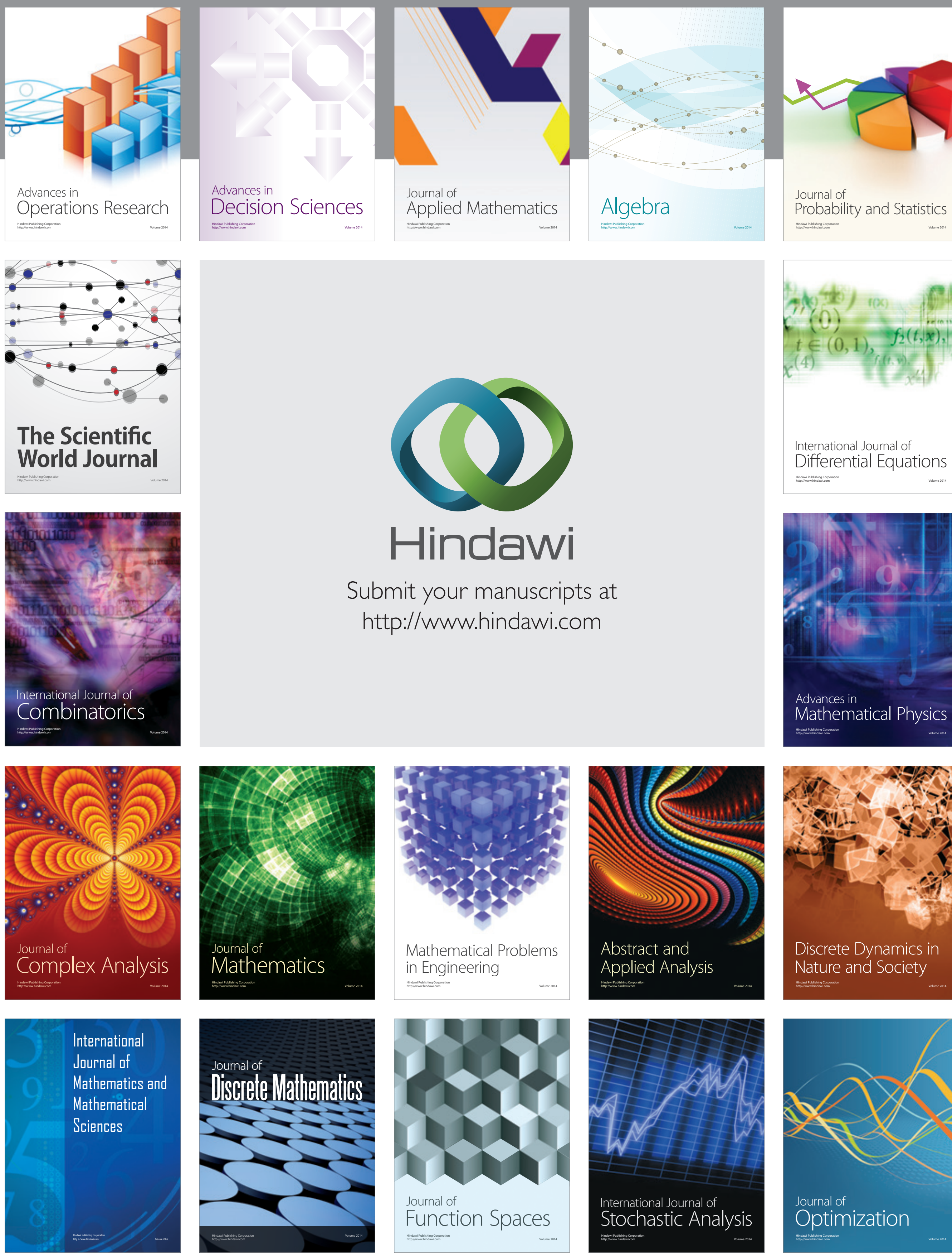
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