

Research Article

Convergence of Viscosity Iteration Process for a Finite Family of Generalized Asymptotically Quasi-Nonexpansive Mappings

Zhiming Cheng

College of Mathematics and Computer Science, Yangtze Normal University, Chongqing 408100, China

Correspondence should be addressed to Zhiming Cheng; fsxczm@yznu.cn

Received 22 August 2013; Revised 10 December 2013; Accepted 18 December 2013; Published 2 January 2014

Academic Editor: Fernando Simões

Copyright © 2014 Zhiming Cheng. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We introduce a general iteration method for a finite family of generalized asymptotically quasi-nonexpansive mappings. The results presented in the paper extend and improve some recent results in the works by Shahzad and Udomene (2006); L. Qihou (2001); Khan et al. (2008).

1. Introduction and Preliminaries

Let *C* be a nonempty subset of a real Banach space *E* and *T* a self-mapping of *C*. The set of fixed points of *T* is denoted by F(T) and we assume that $F(T) \neq \emptyset$. The mapping *T* is said to be

- (i) contractive mapping if there exists a constant α in
 (0,1) such that || f(x) f(y) || ≤ α ||x y||, for all x, y ∈ C;
- (ii) asymptotically nonexpansive mapping if there exists a sequence $\{u_n\}$ in $[0, \infty)$ with $\lim_{n \to \infty} u_n = 0$ such that $\|T^n x T^n y\| \le (1 + u_n) \|x y\|$, for all $x, y \in C$ and $n = 1, 2, 3, \ldots$;
- (iii) asymptotically quasi-nonexpansive if there exists a sequence $\{u_n\}$ in $[0, \infty)$ with $\lim_{n \to \infty} u_n = 0$ such that $||T^n x p|| \le (1+u_n)||x p||$, for all $x \in C$, $p \in F(T)$ and $n = 1, 2, 3, \ldots$;
- (iv) generalized asymptotically quasi-nonexpansive [1] if there exist two sequences $\{u_n\}$, $\{h_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ and $\lim_{n\to\infty} h_n = 0$ such that

$$||T^{n}x - p|| \le (1 + u_{n}) ||x - p|| + h_{n}, \quad \forall x \in C, \ p \in F(T),$$
(1)

where n = 1, 2, 3, ...;

(v) uniformly *L*-Lipschitzian if there exists a constant L > 0 such that $||T^n x - T^n y|| \le L ||x - y||$, for all $x, y \in C$ and n = 1, 2, 3, ...;

- (vi) $(L-\gamma)$ uniform *L*-Lipschitz if there are constants L > 0and $\gamma > 0$ such that $||T^n x - T^n y|| \le L ||x - y||^{\gamma}$, for all $x, y \in C$ and n = 1, 2, 3, ...;
- (vii) semicompact if for a sequence $\{x_n\}$ in *C* with $\lim_{n \to \infty} ||x_n Tx_n|| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to p \in C$.

In (1), if $h_n = 0$ for all $n \ge 1$, then *T* becomes an asymptotically quasi-nonexpansive mapping; if $u_n = 0$ and $h_n = 0$ for all $n \ge 1$, then *T* becomes a quasi-nonexpansive mapping. It is known that an asymptotically nonexpansive mapping is an asymptotically quasi-nonexpansive and a uniformly *L*-Lipschitzian mapping is (L - 1) uniform *L*-Lipschitz.

The mapping $T : C \rightarrow E$ is said to be demiclosed at 0 if for each sequence $\{x_n\} \in C$ converging weakly to x_0 and $\{Tx_n\}$ converging strongly to 0, we have $Tx_0 = 0$.

A Banach space *E* is said to satisfy Opial's property if for each $x \in E$ and each sequence $\{x_n\}$ weakly convergent to *x*, the following condition holds for all $x \neq y$:

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|.$$
(2)

Let *C* be a nonempty closed convex subset of a real Banach space *E* and $\{T_i : i = 1, 2, ..., k\}$ a finite family of asymptotically nonexpansive mappings of *C* into itself. Suppose that $\alpha_{in} \in [0, 1], n = 1, 2, 3, \dots$, and $i = 1, 2, \dots k$. Then we consider the following mapping of *C* into itself:

$$U_{1n} = (1 - \alpha_{1n}) I + \alpha_{1n} T_1^n U_{0n},$$

$$U_{2n} = (1 - \alpha_{2n}) I + \alpha_{2n} T_2^n U_{1n},$$

$$\vdots$$

$$U_{(k-1)n} = (1 - \alpha_{(k-1)n}) I + \alpha_{(k-1)n} T_{k-1}^n U_{(k-2)n},$$

$$W_n = U_{kn} = (1 - \alpha_{kn}) I + \alpha_{kn} T_k^n U_{(k-1)n},$$
(3)

where $U_{0n} = I$ (identity mapping). Such a mapping W_n is called the modified *W*-mapping generated by T_1, T_2, \ldots, T_k and $\alpha_{1n}, \alpha_{2n}, \ldots, \alpha_{kn}$ (see [2, 3]).

In the sequel, we assume that $F = \bigcap_{i=1}^{k} F(T_i)$.

In 2008, Khan et al. [4] introduced the following iteration process for a family of asymptotically quasi-nonexpansive mappings, for an arbitrary $x_1 \in C$:

$$y_{1n} = (1 - \alpha_{1n}) x_n + \alpha_{1n} T_1^n y_{0n},$$

$$y_{2n} = (1 - \alpha_{2n}) x_n + \alpha_{2n} T_2^n y_{1n},$$

$$\vdots$$

$$y_{(k-1)n} = (1 - \alpha_{(k-1)n}) x_n + \alpha_{(k-1)n} T_{k-1}^n y_{(k-2)n},$$

$$x_{n+1} = (1 - \alpha_{kn}) x_n + \alpha_{kn} T_k^n y_{(k-1)n},$$

(4)

where
$$y_{0n} = x_n$$
, $\alpha_{in} \in [0, 1]$, $i = 1, 2, ..., k$, $n = 1, 2, ...$ and
proved that the iterative sequence $\{x_n\}$ defined by (4) con-
verges strongly to a common fixed point of the family of map-
pings if and only if lim $\inf_{n \to \infty} d(x_n, F) = 0$, where $d(x, F) =$
 $\inf_{p \in F} ||x - p||$. With the help of (3), we write (4) as

$$x_{n+1} = W_n x_n. \tag{5}$$

Recently, Chang et al. [5] introduced the following iteration process of asymptotically nonexpansive mappings in Banach space:

$$\begin{aligned} x_{n+1} &= \lambda_n f\left(x_n\right) + \left(1 - \lambda_n\right) T^n y_n, \\ y_n &= \beta_n x_n + \left(1 - \beta_n\right) T^n x_n, \end{aligned} \tag{6}$$

where $\{\lambda_n\}, \{\beta_n\} \in [0, 1]$ and f is a fixed contractive mapping, and necessary and sufficient conditions are given for the iterative sequence $\{x_n\}$ to converge to the fixed points of T.

For a family of mappings, it is quite significant to devise a general iteration scheme which extends the iteration processes (4) and (6), simultaneously. Thereby, to achieve this goal, we introduce a new iteration process for a family of mappings as follows.

Let *C* be a nonempty closed convex subset of a real Banach space *E*, $\{T_i : C \rightarrow C, i = 1, 2, ..., k\}$ a family of generalized asymptotically quasi-nonexpansive mappings, and $f : C \rightarrow C$ a fixed contractive mapping with contractive coefficient $\alpha \in$ (0, 1). For a given $x_1 \in C$, the iteration scheme is defined as follows:

$$x_{n+1} = \lambda_n f(x_n) + (1 - \lambda_n) W_n x_n, \tag{7}$$

where $\{\lambda_n\} \in [0, 1]$ and W_n is the modified *W*-mapping generated by T_1, T_2, \ldots, T_k , and $\alpha_{1n}, \alpha_{2n}, \ldots, \alpha_{kn}$ for all positive integers *n*.

The purpose of this paper is to study the convergence problem of the iterative sequences $\{x_n\}$ defined by (7). The obtained results extend the corresponding results in [4–8], and Lemma 11 partly improves the method of proof of Lemma 3.1 in [4].

In what follows, we need the following useful known lemmas.

Lemma 1 (see [9]). Let $\{a_n\}$, $\{\delta_n\}$, and $\{\gamma_n\}$ be nonnegative real sequences satisfying the following condition:

$$a_{n+1} \le \left(1 + \delta_n\right) a_n + \gamma_n,\tag{8}$$

where $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$; then $\lim_{n \to \infty} a_n$ exists.

Moreover, if in addition, $\lim \inf_{n \to \infty} a_n = 0$, then $\lim_{n \to \infty} a_n = 0$.

Lemma 2 (see [4]). Let *E* be a uniformly convex Banach space, $0 < b \le t_n \le c < 1$ for all $n \ge 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences in *E*. Assume that $\limsup_{n\to\infty} ||x_n|| \le a$, $\limsup_{n\to\infty} ||y_n|| \le a$, and $\lim_{n\to\infty} ||t_nx_n + (1-t_n)y_n|| = a$ for some $a \ge 0$. Then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

2. Main Results

Lemma 3. Let *C* be a nonempty closed convex subset of a real Banach space *E* and *T* an asymptotically quasi-nonexpansive self-mapping of *C* with $\{u_n\} \in [0, \infty)$ for all $n \ge 1$. Suppose $F(T) \neq \phi$. Then F(T) is a closed subset in *C*.

Proof. Let $\{z_n\}$ be an arbitrary sequence of F(T) and $z_n \to z_0$ as $n \to \infty$. Since *C* is closed, we have $z_0 \in C$. For any $\epsilon > 0$, there exists a natural number *N* such that

$$\|z_n - z_0\| < \frac{\epsilon}{2 + u_1}, \quad \forall n \ge N.$$
(9)

Thus, we get

$$\|Tz_0 - z_0\| \le \|Tz_0 - z_N\| + \|z_N - z_0\|$$

$$\le (1 + u_1) \|z_N - z_0\| + \|z_N - z_0\| \qquad (10)$$

$$= (2 + u_1) \|z_N - z_0\| < \epsilon.$$

Since ϵ is arbitrary, it follows that $||Tz_0 - z_0|| = 0$; that is, $Tz_0 = z_0$. Hence $z_0 \in F(T)$ and F(T) is closed. This completes the proof.

Lemma 4. Let *C* be a nonempty closed convex subset of a real Banach space *E*. Let { $T_i : i = 1, 2, ..., k$ } be *k* generalized asymptotically quasi-nonexpansive self-mappings of *C* with { u_{in} }, { h_{in} } $\subset [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} h_{in} < \infty$ for all $i \in \{1, 2, 3, ..., k\}$. Suppose $F \neq \emptyset$ and { α_{in} } $_{n\geq 1} \subset [0, 1]$ for all $i \in \{1, 2, 3, ..., k\}$. Let W_n be the modified *W*-mapping generated by $T_1, T_2, ..., T_k$ and $\alpha_{1n}, \alpha_{2n}, ..., \alpha_{kn}$. Let the sequence { x_n } be defined by (7) and assuming $\sum_{n=1}^{\infty} \lambda_n < \infty$, then (1) there exist two sequences $\{\nu_n\}$ and $\{\xi_n\}$ in $[0, \infty)$ with $\sum_{n=1}^{\infty} \nu_n < \infty, \sum_{n=1}^{\infty} \xi_n < \infty$ such that

$$\|x_{n+1} - p\| \le (1 + \nu_n)^k \|x_n - p\| + \xi_n, \quad \forall p \in F, \ n \ge 1;$$
(11)

(2) there exists a constant $M_1 > 0$, such that

$$\|x_{n+m} - p\| \le M_1 \|x_n - p\| + M_1 \sum_{i=n}^{\infty} \xi_i,$$

$$\forall p \in F, \ n, m = 1, 2, 3, \dots,$$
(12)

where $\{\xi_n\} \in [0, \infty)$ and $\sum_{n=1}^{\infty} \xi_n < \infty$.

Proof. (1) Let $v_n = \max_{1 \le i \le k} u_{in}$, for all *n*. Since $\sum_{n=1}^{\infty} u_{in} < \infty$ for each *i*, we can get $\sum_{n=1}^{\infty} v_n < \infty$. For all $p \in F$, it follows from (3) that

$$\begin{aligned} \|U_{1n}x_n - p\| &\leq (1 - \alpha_{1n}) \|x_n - p\| + \alpha_{1n} \|T_1^n x_n - p\| \\ &\leq (1 - \alpha_{1n}) \|x_n - p\| \\ &+ \alpha_{1n} \left[(1 + u_{1n}) \|x_n - p\| + h_{1n} \right] \\ &\leq (1 + u_{1n}) \|x_n - p\| + h_{1n} \\ &\leq (1 + \nu_n) \|x_n - p\| + h_{1n}. \end{aligned}$$
(13)

Assume that $||U_{jn}x_n - p|| \le (1 + \nu_n)^j ||x_n - p|| + (1 + \nu_n)^{j-1} \sum_{i=1}^j h_{in}$ for some $1 \le j \le k - 1$. Then

$$\begin{split} \left\| U_{(j+1)n} x_n - p \right\| \\ &\leq \left(1 - \alpha_{(j+1)n} \right) \left\| x_n - p \right\| \\ &+ \alpha_{(j+1)n} \left\| T_{j+1}^n U_{jn} x_n - p \right\| \\ &\leq \left(1 - \alpha_{(j+1)n} \right) \left\| x_n - p \right\| \\ &+ \alpha_{(j+1)n} \left(\left(1 + u_{(j+1)n} \right) \left\| U_{jn} x_n - p \right\| + h_{(j+1)n} \right) \right) \\ &\leq \left(1 - \alpha_{(j+1)n} \right) \left\| x_n - p \right\| + \alpha_{(j+1)n} h_{(j+1)n} \\ &+ \alpha_{(j+1)n} \left(1 + u_{(j+1)n} \right) \\ &\times \left(\left(1 + \nu_n \right)^j \left\| x_n - p \right\| + \left(1 + \nu_n \right)^{j-1} \sum_{i=1}^j h_{in} \right) \\ &\leq \left(\left(1 - \alpha_{(j+1)n} \right) + \alpha_{(j+1)n} (1 + \nu_n)^{j+1} \right) \left\| x_n - p \right\| \\ &+ \left(1 + \nu_n \right)^j \sum_{i=1}^j h_{in} + h_{(j+1)n} \\ &\leq \left(\left(1 - \alpha_{(j+1)n} \right) \left(1 + \nu_n \right)^{j+1} + \alpha_{(j+1)n} (1 + \nu_n)^{j+1} \right) \end{split}$$

$$\times \|x_n - p\| + (1 + \nu_n)^j \sum_{i=1}^{j+1} h_{in}$$

$$\leq (1 + \nu_n)^{j+1} \|x_n - p\| + (1 + \nu_n)^j \sum_{i=1}^{j+1} h_{in}.$$
(14)

Thus, by induction, we have

$$\left\| U_{jn} x_n - p \right\| \le \left(1 + \nu_n \right)^j \left\| x_n - p \right\| + \left(1 + \nu_n \right)^{j-1} \sum_{i=1}^j h_{in},$$
(15)

for all j = 1, 2, ..., k. Hence,

$$\|W_n x_n - p\| = \|U_{kn} x_n - p\| \le (1 + \nu_n)^k \|x_n - p\| + (1 + \nu_n)^{k-1} \sum_{i=1}^k h_{in}.$$
(16)

By (7) and (16), we obtain

$$\begin{aligned} \|x_{n+1} - p\| &\leq \lambda_n \|f(x_n) - p\| + (1 - \lambda_n) \|W_n x_n - p\| \\ &\leq \lambda_n \|f(x_n) - f(p)\| + \lambda_n \|f(p) - p\| \\ &+ (1 - \lambda_n) \|W_n x_n - p\| \\ &\leq \lambda_n \alpha \|x_n - p\| + \lambda_n \|f(p) - p\| + (1 - \lambda_n) \\ &\times \left[(1 + \nu_n)^k \|x_n - p\| + (1 + \nu_n)^{k-1} \sum_{i=1}^k h_{in} \right] \\ &\leq (1 + \nu_n)^k \|x_n - p\| \\ &+ (1 - \lambda_n) (1 + \nu_n)^{k-1} \sum_{i=1}^k h_{in} \\ &+ \lambda_n \|f(p) - p\|. \end{aligned}$$
(17)

Since $\sum_{n=1}^{\infty} \nu_n < \infty$, $\{\nu_n\}_{n=1}^{\infty}$ is bounded. Setting $M = \max\{\sup_n (1 + \nu_n)^{k-1}, \|f(p) - p\|\}$, we get that

$$\|x_{n+1} - p\| \le (1 + \nu_n)^k \|x_n - p\| + \xi_n, \quad \forall p \in F, \ n \ge 1,$$
(18)

where $\xi_n = M(\sum_{i=1}^k h_{in} + \lambda_n)$ and $\sum_{n=1}^{\infty} \xi_n < \infty$. This completes the proof of (1).

(2) If $t \ge 0$, then $1+t \le e^t$ and consequently, $(1+t)^k \le e^{kt}$, $k = 1, 2, \dots$ Thus, from part (1), we get

$$\|x_{n+m} - p\| \le (1 + \nu_{n+m-1})^k \|x_{n+m-1} - p\| + \xi_{n+m-1}$$

$$\le \exp \{k\nu_{n+m-1}\} \|x_{n+m-1} - p\| + \xi_{n+m-1}$$

$$\le \exp \{k\nu_{n+m-1}\}$$

$$\times (\exp \{k\nu_{n+m-2}\} \|x_{n+m-2} - p\| + \xi_{n+m-2})$$

$$+ \xi_{n+m-1}$$

$$\leq \exp \{k (v_{n+m-1} + v_{n+m-2})\} \|x_{n+m-2} - p\| + \exp \{kv_{n+m-1}\} (\xi_{n+m-2} + \xi_{n+m-1})\}$$

$$\leq \exp \{k\sum_{i=n}^{n+m-1} v_i\} \|x_n - p\| + \sum_{i=n}^{n+m-1} v_i\| +$$

$$+ \exp\left\{k\sum_{i=n+1}^{n+m-1} \nu_{i}\right\} \sum_{i=n}^{n+m-1} \xi_{i}$$

$$\leq M_{1} \|x_{n} - p\| + M_{1} \sum_{i=n}^{\infty} \xi_{i},$$
(19)

for any positive integers *m*, *n*, where $M_1 = \exp\{k \sum_{i=1}^{\infty} v_i\}, \sum_{i=1}^{\infty} \xi_i < \infty$. This completes the proof of (2).

Remark 5. Lemma 4 generalizes Lemma 2.1 in [4].

Theorem 6. Let *C* be a nonempty closed convex subset of a real Banach space *E*. Let $\{T_i : i = 1, 2, ..., k\}$ be *k* generalized asymptotically quasi-nonexpansive self-mappings of *C* with $\{u_{in}\}, \{h_{in}\} \in [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} h_{in} < \infty$ for all $i \in \{1, 2, 3, ..., k\}$. Let $\{\alpha_{in}\}_{n\geq 1} \subset [0, 1]$ for all $i \in \{1, 2, 3, ..., k\}$ and let W_n be a modified *W*-mapping generated by $T_1, T_2, ..., T_k$ and $\alpha_{1n}, \alpha_{2n}, ..., \alpha_{kn}$. Suppose that $F \neq \emptyset$ is closed and $\sum_{n=1}^{\infty} \lambda_n < \infty$. Starting from arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (7); then the sequence $\{x_n\}$ converges strongly to $p \in F$ if and only if lim $\inf_{n\to\infty} d(x_n, F) = 0$.

Proof. We will only prove the sufficiency; the necessity is obvious. From Lemma 4(1), we have

$$\|x_{n+1} - p\| \le (1 + \nu_n)^k \|x_n - p\| + \xi_n,$$
 (20)

for all $p \in F$ and all *n*. Therefore,

$$d(x_{n+1}, F) \le (1 + \nu_n)^k d(x_n, F) + \xi_n$$

= $\left(1 + \sum_{r=1}^k \frac{k(k-1)\cdots(k-r+1)}{r!}\nu_n^r\right)$ (21)
 $\times d(x_n, F) + \xi_n.$

As $\sum_{n=1}^{\infty} v_n < \infty$, so $\sum_{r=1}^{k} (k(k-1)\cdots(k-r+1)/r!)v_n^r < \infty$. By Lemma 1 and $\lim \inf_{n\to\infty} d(x_n, F) = 0$, we get that $\lim_{n\to\infty} d(x_n, F) = 0$. Next, we prove that $\{x_n\}$ is a Cauchy sequence. From Lemma 4(2), we have

$$\|x_{n+m} - p\| \le M_1 \|x_n - p\| + M_1 \sum_{i=n}^{\infty} \xi_i$$

$$\forall p \in F, \ n, m \ge 1.$$
(22)

Hence, for all integers $m \ge 1$ and all $p \in F$,

$$\|x_{n+m} - x_n\| \le \|x_{n+m} - p\| + \|x_n - p\|$$

$$\le (M_1 + 1) \|x_n - p\| + M_1 \sum_{j=n}^{\infty} \xi_j.$$
 (23)

Taking infimum over $p \in F$ in (23) gives

$$\|x_{n+m} - x_n\| \le (M_1 + 1) d(x_n, F) + M_1 \sum_{j=n}^{\infty} \xi_j.$$
 (24)

Now, since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{j=1}^{\infty} \xi_j < \infty$, given $\epsilon > 0$, there exists an integer $N_1 > 0$ such that for all $n \ge N_1$, $d(x_n, F) < \epsilon/(2(M_1 + 2))$ and $\sum_{j=n}^{\infty} \xi_n < \epsilon/(2(M_1 + 1))$. So for all integers $n \ge N_1$, $m \ge 1$, we obtain from (24) that

$$\|x_{n+m} - x_n\| < \epsilon, \quad \forall n \ge N_1, \ m \ge 1.$$
(25)

Hence, $\{x_n\}$ is a Cauchy sequence in *E*. Since *E* is complete, there exists $q \in E$ such that $\lim_{n \to \infty} x_n = q$. We now show that $q \in F$. Since $d(x_n, F) \to 0$ and $x_n \to q$ as $n \to \infty$, for each $\overline{\epsilon} > 0$, there exists an integer $N_2 > 0$ such that, $d(x_n, F) =$ $\inf_{p \in F} ||x_n - p|| < \overline{\epsilon}/3$ and $||x_n - q|| < \overline{\epsilon}/2$ for all $n \ge N_2$. In particular, we have $d(x_{N_2}, F) = \inf_{p \in F} ||x_{N_2} - p|| < \overline{\epsilon}/3$; that is, there exists a $\overline{p} \in F$ such that $||x_{N_2} - \overline{p}|| < \overline{\epsilon}/2$; hence

$$\left\|q - \overline{p}\right\| \le \left\|x_{N_2} - q\right\| + \left\|x_{N_2} - \overline{p}\right\| < \overline{\epsilon}.$$
 (26)

Since *F* is a closed subset of *E*, we obtain $q \in F$. This completes the proof.

Remark 7. Theorem 6 generalizes and extends Theorem 2.2 of Khan et al. [4], Theorem 3.1 of Ghosh and Debnath [8], Theorem 3.2 of Shahzad and Udomene [6], and Theorem 1 of Qihou [7] together with its Corollaries 1 and 2.

Asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings are all generalized asymptotically quasi-nonexpansive, by Theorem 6 and Lemma 3, so we have

Corollary 8. Let *C* be a nonempty closed convex subset of a real Banach space *E*. Let $\{T_i : i = 1, 2, ..., k\}$ be *k* asymptotically quasi-nonexpansive self-mappings of *C* with $\{u_{in}\} \in [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in \{1, 2, 3, ..., k\}$. Let $\{\alpha_{in}\}_{n\geq 1} \subset [0, 1]$ for all $i \in \{1, 2, 3, ..., k\}$ and let W_n be a modified *W*-mapping generated by $T_1, T_2, ..., T_k$ and $\alpha_{1n}, \alpha_{2n}, ..., \alpha_{kn}$. Suppose $F \neq \emptyset$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$. Starting from arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (7). Then the sequence $\{x_n\}$ converges strongly to $p \in F$ if and only if $\lim \inf_{n \to \infty} d(x_n, F) = 0$.

Corollary 9. Let C be a nonempty closed convex subset of a real Banach space E. Let $\{T_i : i = 1, 2, ..., k\}$ be k asymptotically nonexpansive self-mappings of C with $\{u_{in}\} \in [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in \{1, 2, 3, ..., k\}$. Let $\{\alpha_{in}\}_{n\geq 1} \in [0, 1]$ for all $i \in \{1, 2, 3, ..., k\}$ and let W_n be a modified W-mapping generated by $T_1, T_2, ..., T_k$ and $\alpha_{1n}, \alpha_{2n}, ..., \alpha_{kn}$.

Suppose $F \neq \emptyset$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$. Starting from arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (7). Then the sequence $\{x_n\}$ converges strongly to $p \in F$ if and only if $\liminf_{n \to \infty} d(x_n, F) = 0$.

Corollary 10. Let *C* be a nonempty closed convex subset of a real Banach space *E*. Let $\{T_i : i = 1, 2, ..., k\}$ be *k* generalized asymptotically quasi-nonexpansive self-mappings of *C* with $\{u_{in}\}, \{h_{in}\} \in [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} h_{in} < \infty$ for all $i \in \{1, 2, 3, ..., k\}$. Let $\{\alpha_{in}\}_{n\geq 1} \subset [0, 1]$ for all $i \in \{1, 2, 3, ..., k\}$ and let W_n be a modified *W*-mapping generated by $T_1, T_2, ..., T_k$ and $\alpha_{1n}, \alpha_{2n}, ..., \alpha_{kn}$. Suppose that $F \neq \emptyset$ is closed and $\sum_{n=1}^{\infty} \lambda_n < \infty$. Starting from arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (7). Then the sequence $\{x_n\}$ of $\{x_n\}$ which converges to *p*.

3. Results in Uniformly Convex Banach Spaces

Lemma 11. Let *C* be a nonempty closed convex subset of a uniformly convex Banach space *E*. Let $\{T_i : i = 1, 2, ..., k\}$ be $k (L - \gamma)$ uniform Lipschitz and generalized asymptotically quasi-nonexpansive self-mappings of *C* with $\{u_{in}\}, \{h_{in}\} \in [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} h_{in} < \infty$ for all $i \in \{1, 2, 3, ..., k\}$. Let $\alpha_{in} \in [\delta, 1 - \delta]$ for some $\delta \in (0, 1/2)$ and let W_n be a modified *W*-mapping generated by $T_1, T_2, ..., T_k$ and $\alpha_{1n}, \alpha_{2n}, ..., \alpha_{kn}$. Suppose $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$. Starting from arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (7). Then $\lim_{n \to \infty} ||x_n - T_j x_n|| = 0$ for each $j \in \{1, 2, 3, ..., k\}$.

Proof. Let $p \in F$ and $v_n = \max_{1 \le i \le k} u_{in}$, for all *n*. By Lemma 1 and Lemma 4(1), it follows that $\lim_{n \to \infty} ||x_n - p||$ exists for all $p \in F$. Assume that

$$\lim_{n \to \infty} \|x_n - p\| = c.$$
⁽²⁷⁾

From (2) and (27) we obtain that

$$\limsup_{n \to \infty} \left\| U_{jn} x_n - p \right\| \le c, \quad \forall 1 \le j \le k.$$
(28)

From (7), we have

$$\|x_{n+1} - p\| = \|\lambda_n f(x_n) + (1 - \lambda_n) W_n x_n - p\|$$

$$\leq \lambda_n \alpha \|x_n - p\| + \lambda_n \|f(p) - p\|$$

$$+ (1 - \lambda_n) \|U_{kn} x_n - p\|;$$
(29)

therefore,

$$\liminf_{n \to \infty} \|U_{kn} x_n - p\| \ge c. \tag{30}$$

From (28) and (30) we can obtain that

$$\lim_{n \to \infty} \left\| U_{kn} x_n - p \right\| = c. \tag{31}$$

Suppose that $\lim_{n\to\infty} ||U_{(j+1)n}x_n - p|| = c$ for some $1 \le j \le k - 1$. Since

$$\begin{aligned} \|U_{(j+1)n}x_n - p\| &\leq (1 - \alpha_{(j+1)n}) \|x_n - p\| \\ &+ \alpha_{(j+1)n} \|T_{j+1}^n U_{jn}x_n - p\| \\ &\leq (1 - \alpha_{(j+1)n}) \|x_n - p\| + \alpha_{(j+1)n} \\ &\times \left[(1 + u_{(j+1)n}) \|U_{jn}x_n - p\| + h_{(j+1)n} \right], \end{aligned}$$
(32)

so we obtain that

$$\liminf_{n \to \infty} \left\| U_{jn} x_n - p \right\| \ge c.$$
(33)

From (28) and (33), we have that

....

$$\lim_{n \to \infty} \left\| U_{jn} x_n - p \right\| = c.$$
(34)

Thus, by induction, we have

$$\lim_{n \to \infty} \left\| U_{jn} x_n - p \right\| = c, \tag{35}$$

for each j = 1, 2, 3, ..., k. That is,

$$\lim_{n \to \infty} \left\| \left(1 - \alpha_{jn} \right) \left(x_n - p \right) + \alpha_{jn} \left(T_j^n U_{(j-1)n} x_n - p \right) \right\| = c,$$
(36)

for each j = 1, 2, 3, ..., k. From (28), we obtain

$$\lim_{n \to \infty} \sup_{j \to \infty} \left\| T_j^n U_{(j-1)n} x_n - p \right\| \le c, \tag{37}$$

for each $j = 1, 2, 3, \ldots, k$. By Lemma 2, we get

$$\lim_{n \to \infty} \left\| T_{j}^{n} U_{(j-1)n} x_{n} - x_{n} \right\| = 0, \quad \forall 1 \le j \le k.$$
(38)

If j = 1, from (38), we have

$$\lim_{n \to \infty} \|T_1^n x_n - x_n\| = 0.$$
(39)

If j = 2, 3, ..., k, then we have

$$\begin{aligned} \left\| T_{j}^{n} x_{n} - x_{n} \right\| &\leq \left\| T_{j}^{n} x_{n} - T_{j}^{n} U_{(j-1)n} x_{n} \right\| + \left\| T_{j}^{n} U_{(j-1)n} x_{n} - x_{n} \right\| \\ &\leq L \left\| x_{n} - U_{(j-1)n} x_{n} \right\|^{\gamma} \\ &+ \left\| T_{j}^{n} U_{(j-1)n} x_{n} - x_{n} \right\| \\ &= L \left(\alpha_{(j-1)n} \left\| x_{n} - T_{j-1}^{n} U_{(j-2)n} x_{n} \right\| \right)^{\gamma} \\ &+ \left\| T_{j}^{n} U_{(j-1)n} x_{n} - x_{n} \right\|. \end{aligned}$$

$$(40)$$

Hence,

$$\lim_{n \to \infty} \left\| T_j^n x_n - x_n \right\| = 0, \quad \forall 1 \le j \le k.$$
(41)

Note that $||x_{n+1}|$

$$\begin{aligned} x_{n+1} - x_n \| &= \|\lambda_n f(x_n) + (1 - \lambda_n) W_n x_n - x_n \| \\ &\leq \lambda_n \left(\alpha \|x_n - p\| + \|f(p) - p\| + \|x_n - p\| \right) \end{aligned}$$

$$+ (1 - \lambda_{n}) \|W_{n}x_{n} - x_{n}\|$$

$$= \lambda_{n} (\alpha \|x_{n} - p\| + \|f(p) - p\| + \|x_{n} - p\|)$$

$$+ (1 - \lambda_{n}) \alpha_{kn} \|T_{k}^{n}U_{(k-1)n}x_{n} - x_{n}\|;$$
(42)

therefore, we have

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.$$
(43)

Now, we observe that

$$\begin{aligned} \left\| x_{n} - T_{j} x_{n} \right\| &\leq \left\| x_{n} - x_{n+1} \right\| + \left\| x_{n+1} - T_{j}^{n+1} x_{n+1} \right\| \\ &+ \left\| T_{j}^{n+1} x_{n+1} - T_{j}^{n+1} x_{n} \right\| + \left\| T_{j}^{n+1} x_{n} - T_{j} x_{n} \right\| \\ &\leq \left\| x_{n} - x_{n+1} \right\| + \left\| x_{n+1} - T_{j}^{n+1} x_{n+1} \right\| \\ &+ L \left\| x_{n+1} - x_{n} \right\|^{\gamma} + L \left\| T_{j}^{n} x_{n} - x_{n} \right\|^{\gamma}. \end{aligned}$$

$$(44)$$

By (41) and (43), we have

$$\lim_{n \to \infty} \left\| x_n - T_j x_n \right\| = 0, \tag{45}$$

for j = 1, 2, 3, ..., k. This completes the proof.

Theorem 12. Let *C* be a nonempty closed convex subset of a uniformly convex Banach space *E*. Let $\{T_i : i = 1, 2, ..., k\}$ be $k (L - \gamma)$ uniform Lipschitz and generalized asymptotically quasi-nonexpansive self-mappings of *C* with $\{u_{in}\}, \{h_{in}\} \in$ $[0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} h_{in} < \infty$ for all $i \in$ $\{1, 2, 3, ..., k\}$. Let $\alpha_{in} \in [\delta, 1 - \delta]$ for some $\delta \in (0, 1/2)$ and let W_n be a modified *W*-mapping generated by $T_1, T_2, ..., T_k$ and $\alpha_{1n}, \alpha_{2n}, ..., \alpha_{kn}$. Suppose $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$, $\sum_{n=1}^{\infty} \lambda_n < \infty$ and there exists one member in $\{T_i^m : i = 1, 2, ..., k\}$ which is semicompact for some positive integer *m*. Starting from arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (7). Then $\{x_n\}$ converges strongly to some common fixed point of the family $\{T_i : i = 1, 2, ..., k\}$.

Proof. By Lemma 11, we have

$$\lim_{n \to \infty} \left\| x_n - T_j x_n \right\| = 0, \tag{46}$$

for each j = 1, 2, 3, ...k. Without loss of generality, we may assume that T_1^m is semicompact for some $m \ge 1$; then we have

$$\begin{aligned} \|T_1^m x_n - x_n\| &\leq \|T_1^m x_n - T_1^{m-1} x_n\| + \|T_1^{m-1} x_n - T_1^{m-2} x_n\| \\ &+ \dots + \|T_1 x_n - x_n\| \\ &\leq \|T_1 x_n - x_n\| + (m-1) L \|T_1 x_n - x_n\|^{\gamma} \longrightarrow 0. \end{aligned}$$
(47)

Since T_1^m is semicompact, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to q \in C$. Hence, we have

$$\|q - T_i q\| = \lim_{j \to \infty} \|x_{n_j} - T_j x_{n_j}\| = 0,$$
 (48)

for each i = 1, 2, 3, ..., k. This implies that $q \in F$. By Corollary 10, $\{x_n\}$ converges strongly to some common fixed point of the family $\{T_i : i = 1, 2, ..., k\}$.

Theorem 13. Let *C* be a nonempty closed convex subset of a uniformly convex Banach space *E*. Let $\{T_i : i = 1, 2, ..., k\}$ be k $(L - \gamma)$ uniform Lipschitz and generalized asymptotically quasi-nonexpansive self-mappings of *C* with $\{u_{in}\}, \{h_{in}\} \in [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} h_{in} < \infty$ for all $i \in \{1, 2, 3, ..., k\}$. Let $\alpha_{in} \in [\delta, 1 - \delta]$ for some $\delta \in (0, 1/2)$ and let W_n be a modified *W*-mapping generated by $T_1, T_2, ..., T_k$ and $\alpha_{1n}, \alpha_{2n}, ..., \alpha_{kn}$. Suppose $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$, $\sum_{n=1}^{\infty} \lambda_n < \infty$ and each $I - T_i$, i = 1, 2, ..., k, is demiclosed at 0. If *E* satisfies Opial's condition, then the sequence $\{x_n\}$ defined by (7) converges weakly to a common fixed point of the family $\{T_i : i = 1, 2, ..., k\}$.

Proof. From the proof of Lemma 11, we know that $\{x_n\}$ is a bounded sequence in *C*. Since *E* is uniformly convex, it must be reflexive. Therefore, there exists a subsequence $\{x_{n_j}\}$ in $\{x_n\}$ converging weakly to $u \in C$. By Lemma 11, $\lim_{j\to\infty} ||x_{n_j} - T_i x_{n_j}|| = 0$ and $I - T_i$ is demiclosed at 0 for i = 1, 2, ..., k, so we obtain $T_i u = u$. That is, $u \in F$. Suppose that there exists another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to $v \in C$. As above, we can prove $v \in F$. By (27) we know that $\lim_{n\to\infty} ||x_n - u||$ and $\lim_{n\to\infty} ||x_n - v||$ exist. Assume $v \neq u$. Then by the Opial's condition, we have

$$\lim_{n \to \infty} \|x_n - u\| = \lim_{n_j \to \infty} \|x_{n_j} - u\| < \lim_{n_j \to \infty} \|x_{n_j} - v\|$$
$$= \lim_{n \to \infty} \|x_n - v\| = \lim_{n_k \to \infty} \|x_{n_k} - v\| \qquad (49)$$
$$< \lim_{n_k \to \infty} \|x_{n_k} - u\| = \lim_{n \to \infty} \|x_n - u\|,$$

which is a contradiction. Hence u = v. This implies that $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i : i = 1, 2, ..., k\}$.

Remark 14. Lemma 11, Theorem 12, and Theorem 13 extend Lemma 3.1, Theorem 3.3, and Theorem 3.2 of Khan et al. [4], respectively.

Conflict of Interests

The author declares that there is no conflict of interests.

References

- N. Shahzad and H. Zegeye, "Strong convergence of an implicit iteration process for a finite family of generalized asymptotically quasi-nonexpansive maps," *Applied Mathematics and Computation*, vol. 189, no. 2, pp. 1058–1065, 2007.
- [2] K. Nakajo, K. Shimoji, and W. Takahashi, "On strong convergence by the hybrid method for families of mappings in Hilbert spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 1-2, pp. 112–119, 2009.

- [3] W. Takahashi and K. Shimoji, "Convergence theorems for nonexpansive mappings and feasibility problems," *Mathematical* and Computer Modelling, vol. 32, no. 11-13, pp. 1463–1471, 2000.
- [4] A. R. Khan, A.-A. Domlo, and H. Fukhar-ud-din, "Common fixed points Noor iteration for a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 1, pp. 1–11, 2008.
- [5] S. S. Chang, H. W. J. Lee, C. K. Chan, and J. K. Kim, "Approximating solutions of variational inequalities for asymptotically nonexpansive mappings," *Applied Mathematics and Computation*, vol. 212, no. 1, pp. 51–59, 2009.
- [6] N. Shahzad and A. Udomene, "Approximating common fixed points of two asymptotically quasi-nonexpansive mappings in Banach spaces," *Fixed Point Theory and Applications*, vol. 2006, Article ID 18909, 10 pages, 2006.
- [7] L. Qihou, "Iterative sequences for asymptotically quasi-nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 259, no. 1, pp. 1–7, 2001.
- [8] M. K. Ghosh and L. Debnath, "Convergence of Ishikawa iterates of quasi-nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 207, no. 1, pp. 96–103, 1997.
- [9] H. K. Xu, "Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process," *Journal of Mathematical Analysis and Applications*, vol. 178, no. 2, pp. 301–308, 1993.



The Scientific World Journal





Decision Sciences







Journal of Probability and Statistics



Hindawi Submit your manuscripts at





International Journal of Differential Equations





International Journal of Combinatorics





Mathematical Problems in Engineering



Abstract and Applied Analysis



Discrete Dynamics in Nature and Society







Journal of Function Spaces



International Journal of Stochastic Analysis



Journal of Optimization