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Research Article

Local C^r Stability for Iterative Roots of Orientation-Preserving Self-Mappings on the Interval

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Stability of iterative roots is important in their numerical computation. It is known that under some conditions iterative roots of orientation-preserving self-mappings are both globally C^0 stable and locally C^1 stable but globally C^1 unstable. Although the global C^1 instability implies the general global C^r $(r \geq 2)$ instability, the local C^1 stability does not guarantee the local C^r $(r \geq 2)$ stability. In this paper we generally prove the local C^r $(r \geq 2)$ stability for iterative roots. For this purpose we need a uniform estimate for the approximation to the conjugation in C^r linearization, which is given by improving the method used for the C^1 case.

1. Introduction

Let $I := [0, 1]$ and let $C^r(I, I)$, $r \geq 0$, be the set of all C^r self-mappings defined on I. An iterative root of order k of a given self-mapping $F : I \to I$ is a self-mapping $f : I \to I$ such that

$$f^k(x) = F(x), \quad \forall x \in I, \tag{1}$$

where f^k denotes the kth iterate of f, defined by $f^k(x) := f(f^{k-1}(x))$ and $f^0(x) = x$ for all $x \in I$ inductively. The study of iterative roots was started long ago, at least about two hundred years ago when Babbage published his paper [1]. In recent decades, regarded as a weak version of the embedding flow problem for dynamical systems [2, 3], the problem of iterative roots attracted great attention in the field of dynamical systems [3, 4] and functional equations [5–8]. Based on the work for monotonic mappings [6, 7], advances have been made to nonmonotonic cases [8–11], self-mappings on circles [12, 13], set-valued functions [14, 15], and high-dimensional mappings [16, 17].

Because of the potential in extensive applications (e.g., to information processing [18, 19] and graph theory [20]), numerical computation [21, 22] of iterative roots became an important task, which demands approximation to iterative roots and considers stability of iterative roots. In 2007 Xu and Zhang [23] proved C^0 stability for iterative roots on a closed interval with exact one fixed point at an endpoint. This result is substantially a local C^0 stability because the stability is totally decided by the behaviors of the iterative root in a sufficiently small neighborhood of the fixed point. In [24] results of global C^0 stability are given, where the global sense means the stability on a closed interval bounded by two fixed points. Recently, it was proved in [25] that iterative roots of every orientation-preserving self-mapping on the interval are locally C^1 stable but globally C^1 unstable.

In this paper we generally consider C^r $(r \geq 1)$ stability of iterative roots. It is clear that the global C^1 instability given in [25] implies the general global C^r $(r \geq 2)$ instability because C^1 approximation is the most fundamental requirement for C^r approximation. However, the above result of local C^1 stability does not guarantee the local C^r $(r \geq 2)$ stability. In this paper we concentrate on the local C^r $(r \geq 2)$ stability for iterative roots of orientation-preserving self-mapping on I. Clearly, the given mapping is a strictly increasing function. The local C^r $(r \geq 2)$ stability is proved by approximation to the conjugation in C^r linearization. In order to give an estimate to the approximation uniformly with respect to the order of iteration, we improve the method used in [25] to obtain lower growth rate for given functions under iteration.

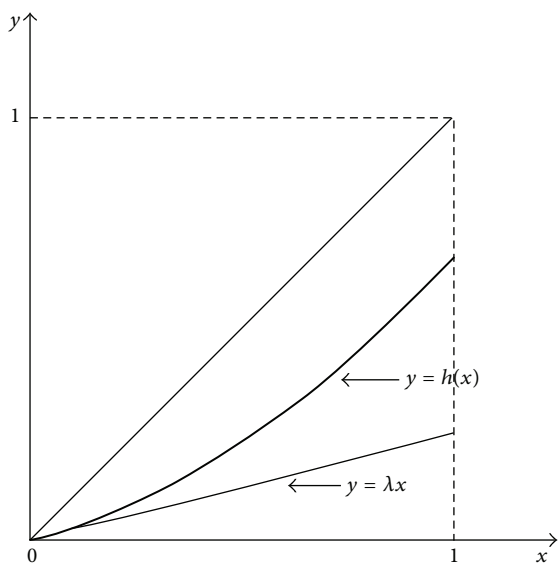

FIGURE 1: $h \in \mathscr{H}^r_-(\lambda)$.

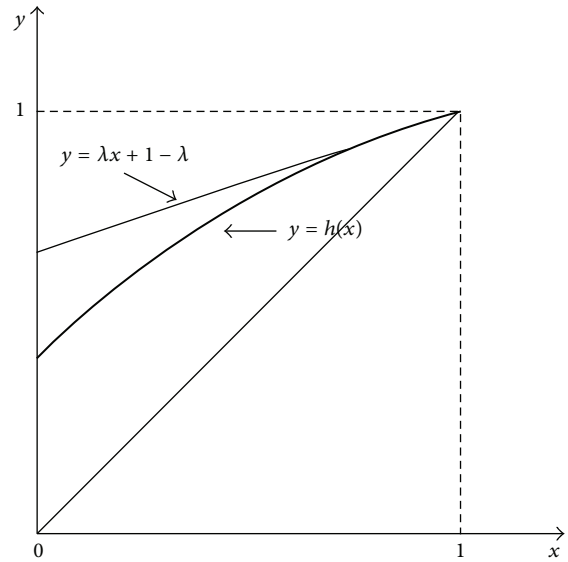

FIGURE 2: $h \in \mathscr{H}^r_+(\lambda)$.

2. Preliminaries

In order to state our results clearly, we first pay attention to the existence of C^r $(r \geq 2)$ iterative roots of increasing C^r self-mappings on a compact interval including exactly one fixed point which is hyperbolic. In some sense, this is a local case. For each $\lambda \in (0,1)$ and integer $r \geq 0$, let

$$\begin{aligned}\mathscr{H}^r_-(\lambda) &:= \Big\{h \in C^r(I,I) : h(0) = 0,\ h'(0) = \lambda,\ h'(x) > 0,\\ &\qquad h(x) < x, \forall x \in (0,1]\Big\},\\ \mathscr{H}^r_+(\lambda) &:= \Big\{h \in C^r(I,I) : h(1) = 1,\ h'(1) = \lambda,\ h'(x) > 0,\\ &\qquad h(x) > x, \forall x \in [0,1)\Big\}\end{aligned} \tag{2}$$

(cf. Figures 1 and 2) together with the norm

$$\|h\|_r := \sup_{x\in I}|h(x)| + \cdots + \sup_{x\in I}\left|h^{(r)}(x)\right|. \tag{3}$$

In what follows we only discuss the first class because $\mathscr{H}^r_+(\lambda)$ can be converted to $\mathscr{H}^r_-(\lambda)$ by considering $G(x) = 1-F(1-x)$.

Given integers $k, r \geq 2$, a function F belonging to the class $\bigcup_{\lambda\in(0,1)}\mathscr{H}^r_-(\lambda)$ has a unique kth order C^r iterative root f defined on I, which is strictly increasing and is given by the formula

$$f(x) := \varphi^{-1}\left(\lambda^{1/k}\varphi(x)\right), \tag{4}$$

where $\varphi : I \to \mathbb{R}$ is the *principal solution* of Schröder's equation

$$\varphi(F(x)) = \lambda\varphi(x). \tag{5}$$

The principle solution is given by $\varphi(x) = \lim_{n\to+\infty}\lambda^{-n}F^n(x)$, satisfying C^r differentiable in I with $\varphi'(0) = 1$ and $\varphi(0) = 0$ and strictly increasing by Theorem 6.1 in [6]. Moreover, the proofs of Theorem 3.5.1 in [7] and Theorem 4.5 in [6] show that

$$\varphi^{(r)}(x) = \lim_{n\to+\infty}\lambda^{-n}\left(F^n\right)^{(r)}(x). \tag{6}$$

Note that the aim of this paper is to consider the local C^r $(r \geq 2)$ stability for iterative roots. Then we recall the formula for higher order derivatives of composition ([26, page 3]). Namely, for integer $m \geq 1$,

$$(G\circ H)^{(m)}(x) = \sum_{\omega\in\Omega(m)} c_\omega G^{(p)}(H(x))\prod_{q=1}^{p} H^{(j_q)}(x), \tag{7}$$

where $\Omega(m) := \{(p; j_1,\ldots,j_p) : 1 \leq p \leq m,\ j_1+\cdots+j_p = m$ and $j_q \geq 1$ for all $q = 1,\ldots,p\}$. Here c_ω is a positive universal constant, which is independent of G and H. Then we have the following lemma.

Lemma 1. *Let $F \in \mathscr{H}^r_-(\lambda)$ with some $r \geq 2$ and $\lambda \in (0,1)$ and let (F_i) be a sequence of functions in $\mathscr{H}^r_-(\lambda)$ satisfying condition*

$$\lim_{i\to+\infty}\|F_i - F\|_r = 0. \tag{8}$$

Then, for a given number $\mu \in (\lambda, 1)$, there exist an $L > 1$, an $\varepsilon > 0$, and an $N_0 \in \mathbb{N}$ such that

$$\left|\left(F^n\right)^{(m)}(x)\right| \leq L\lambda^n, \qquad \left|\left(F_i^n\right)^{(m)}(x)\right| \leq L\lambda^n, \tag{9}$$

$$\left|F_i^n(x) - F^n(x)\right| \leq \left\{\left(1+\frac{L\lambda}{\mu-\lambda}\right)\mu^{n-1} - \frac{L}{\mu-\lambda}\lambda^n\right\}\|F_i - F\|_1 \tag{10}$$

for all $i \geq N_0$, $n \in \mathbb{N}$, and $0 \leq m \leq r$ and for all $x \in I_\varepsilon := [0,\varepsilon]$.

Proof. Let $K := 2\|F\|_r$ and choose a sufficiently small $\varepsilon > 0$ such that $\lambda + K\varepsilon < 1$ and $\sup_{x\in I_\varepsilon}|F'(x)| < \mu$. Then by the mean value theorem,

$$\begin{aligned}|F(x)| &= |F(x) - F(0)| = \left|F'(\xi_x)\right||x| \\ &\le \left|F'(0)\right||x| + \left|F'(\xi_x) - F'(0)\right||x| \\ &\le \lambda|x| + K|x|^2\end{aligned} \tag{11}$$

for all $x \in I$. In particular, $|F^{n-1}(x)| \le (\lambda+K\varepsilon)^{n-1}$ for all $n \ge 1$ and $x \in I_\varepsilon$. It follows that

$$\begin{aligned}\left|F^n(x)\right| &\le \lambda\left|F^{n-1}(x)\right| + K\left|F^{n-1}(x)\right|^2 \\ &\le \left\{\lambda + K(\lambda+K\varepsilon)^{n-1}\right\}\left|F^{n-1}(x)\right| \\ &\le \prod_{s=0}^{n-1}\left\{\lambda + K(\lambda+K\varepsilon)^s\right\}|x| \\ &= \lambda^n\prod_{s=0}^{n-1}\left\{1 + K\lambda^{-1}(\lambda+K\varepsilon)^s\right\}|x| \\ &\le L_0\lambda^n, \quad \forall x \in I_\varepsilon,\end{aligned} \tag{12}$$

where $L_0 := \prod_{s=0}^{+\infty}\{1 + K\lambda^{-1}(\lambda+\eta\varepsilon)^s\} \in (1, +\infty)$.

Then we give the proof of the first inequality given in (9) by induction on m greater than 1. From (7), write

$$\begin{aligned}\left(F^n\right)^{(m)}(x) &= F'\left(F^{n-1}(x)\right)\left(F^{n-1}\right)^{(m)}(x) \\ &\quad + \sum_{\substack{p\ge2\\ \omega\in\Omega(m)}} c_\omega F^{(p)}\left(F^{n-1}(x)\right)\prod_{q=1}^{p}\left(F^{n-1}\right)^{(j_q)}(x).\end{aligned} \tag{13}$$

When $m = 1$, the second term is absent. Applying (12) and by induction, we have

$$\begin{aligned}\left|\left(F^n\right)'(x)\right| &= \left|F'\left(F^{n-1}(x)\right)\left(F^{n-1}\right)'(x)\right| \\ &\le \left\{\left|F'(0)\right| + \left|F'\left(F^{n-1}(x)\right) - F'(0)\right|\right\}\left|\left(F^{n-1}\right)'(x)\right| \\ &\le \left\{\lambda + KL_0\lambda^{n-1}\right\}\left|\left(F^{n-1}\right)'(x)\right| \\ &\le \prod_{s=0}^{n-1}\left\{\lambda + KL_0\lambda^s\right\}|x| \\ &= \lambda^n\prod_{s=0}^{n-1}\left\{1 + KL_0\lambda^{s-1}\right\}|x| \le L_1\lambda^n, \quad \forall x \in I_\varepsilon,\end{aligned} \tag{14}$$

where $L_1 := \prod_{s=0}^{+\infty}\{1 + KL_0\lambda^{s-1}\} \in (1, +\infty)$. Further, assume that the first inequality in (9) holds for all $m \le \ell < r$. Let $\Theta := \sum_{p\ge2,\,\omega\in\Omega(\ell+1)} c_\omega K\prod_{q=1}^{\ell}L_q$. Noting that $\lambda \in (0,1)$ and

$$\begin{aligned}\left|F'\left(F^{n-1}(x)\right)\right| &\le \left|F'(0)\right| + \left|F'\left(F^{n-1}(x)\right) - F'(0)\right| \\ &\le \lambda + KL_0\lambda^{n-1}, \quad \forall x \in I_\varepsilon,\end{aligned} \tag{15}$$

we get

$$\begin{aligned}&\left|\left(F^n\right)^{(\ell+1)}(x)\right| \\ &\quad = \left|F'\left(F^{n-1}(x)\right)\left(F^{n-1}\right)^{(\ell+1)}(x)\right. \\ &\qquad \left. + \sum_{\substack{p\ge2\\ \omega\in\Omega(\ell+1)}} c_\omega F^{(p)}\left(F^{n-1}(x)\right)\prod_{q=1}^{p}\left(F^{n-1}\right)^{(j_q)}(x)\right| \\ &\quad \le \left(\lambda + KL_0\lambda^{n-1}\right)\left|\left(F^{n-1}\right)^{(\ell+1)}(x)\right| + \Theta\lambda^{2(n-1)} \\ &\quad \le \lambda^n\prod_{s=-1}^{n-2}\left(1 + KL_0\lambda^s\right) + \Theta\sum_{t=0}^{n-1}\lambda^{2t}\prod_{s=t+1}^{n-1}\left(\lambda + KL_0\lambda^s\right) \\ &\quad = \lambda^n\prod_{s=-1}^{n-2}\left(1 + KL_0\lambda^s\right) + \Theta\lambda^n\sum_{t=0}^{n-1}\lambda^{t-1}\prod_{s=t}^{n-2}\left(1 + KL_0\lambda^s\right) \\ &\quad \le L_{\ell+1}\lambda^n\end{aligned} \tag{16}$$

inductively, where

$$\begin{aligned}L_{\ell+1} := \Bigg\{&\prod_{s=-1}^{+\infty}\left(1 + L_0K\lambda^s\right) \\ &+ \Theta\sum_{t=0}^{+\infty}\lambda^{t-1}\prod_{s=0}^{+\infty}\left(1 + KL_0\lambda^s\right)\Bigg\} \in (1, +\infty).\end{aligned} \tag{17}$$

Put $L := \max\{L_0, L_1, \ldots, L_r\}$. Then the first inequality given in (9) is proved.

It follows from (8) that there is an $N_0 \in \mathbb{N}$ such that if $i \ge N_0$, then $\|F_i\|_r \le 2\|F\|_r = K$ and $\sup_{x\in I_\varepsilon}|F_i'(x)| < \mu$. Thus, by the same procedure as before, we can prove the second inequality given in (9).

In the following, we are going to prove inequality (10). It is clear that (10) holds when $n = 1$. Further, assume that (10) holds for $n = \ell \in \mathbb{N}$. It follows that

$$\begin{aligned}&\left|F_i^{\ell+1}(x) - F^{\ell+1}(x)\right| \\ &\quad = \left|F_i\left(F_i^{\ell}(x)\right) - F\left(F^{\ell}(x)\right)\right| \\ &\quad \le \left|F_i\left(F_i^{\ell}(x)\right) - F_i\left(F^{\ell}(x)\right)\right| \\ &\qquad + \left|F_i\left(F^{\ell}(x)\right) - F\left(F^{\ell}(x)\right)\right| \\ &\quad \le \sup_{\xi\in I_\varepsilon}\left|F_i'(\xi)\right|\cdot\left|F_i^{\ell}(x) - F^{\ell}(x)\right| \\ &\qquad + \sup_{\xi\in I_\varepsilon}\left|\left(F_i - F\right)'(\xi)\right|\cdot\left|F^{\ell}(x)\right| \\ &\quad \le \mu\left|F_i^{\ell}(x) - F^{\ell}(x)\right| + L\lambda^{\ell}\|F_i - F\|_1\end{aligned}$$

$$\begin{aligned}
&\le \mu\left\{\left(1+\frac{L\lambda}{\mu-\lambda}\right)\mu^{\ell-1}-\frac{L}{\mu-\lambda}\lambda^{\ell}\right\}\\
&\quad\times\|F_i-F\|_1+L\lambda^{\ell}\|F_i-F\|_1\\
&\le\left\{\left(1+\frac{L\lambda}{\mu-\lambda}\right)\mu^{\ell}-\frac{L}{\mu-\lambda}\lambda^{\ell+1}\right\}\|F_i-F\|_1
\end{aligned}\tag{18}$$

for all $i \ge N_0$ and for all $x \in I_\varepsilon$. Thus, we can obtain (10) by induction. This completes the proof. □

In Lemma 1 we gave a better estimate for F^n and F_i^n and their derivatives than that given in [25, Lemma 2.1]. In Lemma 1 the growth rate on n is much lower in the sense that the constant $L\lambda^n$ given in (9) tends to 0 as $n \to +\infty$ faster than the constant μ_0^n given in (2.4) of [25].

3. The Main Result

Our aim of this section is to prove the following stability result.

Theorem 2. *Given integers $k, r \ge 2$, let $F \in \mathscr{H}_-^{r+1}(\lambda)$ with some $\lambda \in (0,1)$ and let (F_i) be a sequence of functions in $\mathscr{H}_-^{r+1}(\lambda)$. If*

$$\begin{gathered}
\lim_{i\to+\infty}\|F_i-F\|_{r+1}=0,\\
F^{(m)}(0)=F_i^{(m)}(0),\quad \forall m=2,\dots,r,
\end{gathered}\tag{19}$$

then

$$\lim_{i\to+\infty}\|f_i-f\|_r=0,\tag{20}$$

where f and f_i are unique kth order C^r iterative roots of F and F_i, respectively, defined on I.

In order to prove Theorem 2 we need the following lemma on C^r stability of Schröder's equation.

Lemma 3. *Given an integer $r \ge 2$, let $F \in \mathscr{H}_-^{r+1}(\lambda)$ with some $\lambda \in (0,1)$ and let (F_i) be a sequence of functions in $\mathscr{H}_-^{r+1}(\lambda)$ satisfying condition (19). Then*

$$\lim_{i\to+\infty}\|\varphi_i-\varphi\|_r=0,\tag{21}$$

where $\varphi : I \to \mathbb{R}$ and $\varphi_i : I \to \mathbb{R}$ are the principal solutions of Schröder's equations $\varphi(F(x)) = \lambda\varphi(x)$ and $\varphi_i(F_i(x)) = \lambda\varphi_i(x)$, respectively.

Proof. From (6) we can see that

$$\begin{gathered}
\varphi^{(m)}(x)=\lim_{n\to+\infty}\lambda^{-n}\left(F^n\right)^{(m)}(x),\\
\varphi_i^{(m)}(x)=\lim_{n\to+\infty}\lambda^{-n}\left(F_i^n\right)^{(m)}(x)
\end{gathered}\tag{22}$$

for all $m \in [0,r] \cap \mathbb{Z}$ and $x \in I$. In what follows we intend to discuss our results in a sufficiently small interval I_ε first and extend them to the whole interval I, where ε is given in Lemma 1.

In order to prove the convergence of the sequence (φ_i) in I_ε, we claim that there exists a constant M_m, which is independent of n, such that

$$\lambda^{-n}\left|\left(F_i^n\right)^{(m)}(x)-\left(F^n\right)^{(m)}(x)\right|\le M_m\|F_i-F\|_{r+1},\quad \forall x\in I_\varepsilon,\tag{23}$$

where $m = 0, 1, \dots, r$. If (23) is true, then we get

$$\begin{aligned}
\left|\varphi_i^{(m)}(x)-\varphi^{(m)}(x)\right| &= \lim_{n\to+\infty}\lambda^{-n}\left|\left(F_i^n\right)^{(m)}(x)-\left(F^n\right)^{(m)}(x)\right|\\
&\le M_m\|F_i-F\|_{r+1},\quad \forall x\in I_\varepsilon,
\end{aligned}\tag{24}$$

implying the stability in I_ε. Next, we extend the result (24) from I_ε to the whole interval I. As indicated in [25], we have

$$\lim_{i\to+\infty}\|F_i^n-F^n\|_{r+1}=0,\quad \forall n\in\mathbb{N},\tag{25}$$

by (8) because the composition operator is continuous by Example 4.4.5 in [27]. Moreover, since 0 is the unique stable fixed point of F in I and by (8), there is an integer $N \in \mathbb{N}$ such that $F^N(x), F_i^N(x) \in I_\varepsilon$ for all $i \in \mathbb{N}$ and $x \in I$. Then, according to Schröder's equation, we can obtain the formulae

$$\varphi(x)=\lambda^{-N}\widetilde{\varphi}\left(F^N(x)\right),\qquad \varphi_i(x)=\lambda^{-N}\widetilde{\varphi}_i\left(F_i^N(x)\right),\quad \forall x\in I,\tag{26}$$

where $\widetilde{\varphi} := \varphi|_{I_\varepsilon}$ and $\widetilde{\varphi}_i := \varphi_i|_{I_\varepsilon}$. Then by Lemma 1 and from (24), (26), and the uniform continuity of $\varphi^{(m)}$, we get $\lim_{i\to+\infty}\|\varphi_i-\varphi\|_0 = 0$ and

$$\begin{aligned}
&\left|\varphi_i^{(m)}(x)-\varphi^{(m)}(x)\right|\\
&\quad=\lambda^{-N}\left|\sum_{\omega\in\Omega(m)}c_\omega\widetilde{\varphi}_i^{(p)}\left(F_i^N(x)\right)\prod_{q=1}^{p}\left(F_i^N\right)^{(j_q)}(x)\right.\\
&\qquad\left.-\sum_{\omega\in\Omega(m)}c_\omega\widetilde{\varphi}^{(p)}\left(F^N(x)\right)\prod_{q=1}^{p}\left(F^N\right)^{(j_q)}(x)\right|\\
&\quad\le\lambda^{-N}\sum_{\omega\in\Omega(m)}c_\omega\left|\widetilde{\varphi}_i^{(p)}\left(F_i^N(x)\right)\prod_{q=1}^{p}\left(F_i^N\right)^{(j_q)}(x)\right.\\
&\qquad\left.-\widetilde{\varphi}^{(p)}\left(F^N(x)\right)\prod_{q=1}^{p}\left(F^N\right)^{(j_q)}(x)\right|\\
&\quad\le\lambda^{-N}\sum_{\omega\in\Omega(m)}c_\omega\left\{\left|\widetilde{\varphi}_i^{(p)}\left(F_i^N(x)\right)-\widetilde{\varphi}^{(p)}\left(F^N(x)\right)\right|\right.\\
&\qquad\times\prod_{q=1}^{p}\left|\left(F_i^N\right)^{(j_q)}(x)\right|+\left|\widetilde{\varphi}^{(p)}\left(F^N(x)\right)\right|
\end{aligned}$$

$$\times \left| \prod_{q=1}^{p} \left(F_i^N\right)^{(j_q)}(x) - \prod_{q=1}^{p} \left(F^N\right)^{(j_q)}(x) \right| \Bigg\}$$

$$\leq \lambda^{-N} \sum_{\omega \in \Omega(m)} c_\omega \Big\{ L^p \lambda^{pN} \times \left(\left| \widetilde{\varphi}_i^{(p)}\left(F_i^N(x)\right) - \widetilde{\varphi}^{(p)}\left(F_i^N(x)\right) \right| + \left| \widetilde{\varphi}^{(p)}\left(F_i^N(x)\right) - \widetilde{\varphi}^{(p)}\left(F^N(x)\right) \right| \right) + L^{p-1}\lambda^{(p-1)N} \left| \widetilde{\varphi}^{(p)}\left(F^N(x)\right) \right| \times \left\| F_i^N - F^N \right\|_{r+1} \Big\}$$

$$\leq \lambda^{-N} \sum_{\omega \in \Omega(m)} c_\omega \Big\{ L^p \lambda^{pN} \times \left(M_p \left\| F_i - F \right\|_{r+1} + \left| \widetilde{\varphi}^{(p)}\left(F_i^N(x)\right) - \widetilde{\varphi}^{(p)}\left(F^N(x)\right) \right| \right) + L^{p-1}\lambda^{(p-1)N} \left| \widetilde{\varphi}^{(p)}\left(F^N(x)\right) \right| \times \left\| F_i^N - F^N \right\|_{r+1} \Big\} \longrightarrow 0 \tag{27}$$

as $i \to +\infty$ for all $m = 1, \ldots, r$. Hence, (21) is proved and the proof is completed.

In the following, we will prove the claimed (23) by induction on m. Clearly, (2.11) in [25] and what is indicated above the proof of Theorem 2.1 in [25] show that (23) holds for $m = 1$ and $m = 0$, respectively. Then we suppose that (23) is also satisfied for all $m \in [1, \ell]$, where $2 \leq \ell < r$, and we will prove (23) for $m = \ell + 1$. Our strategy is to prove that, for given $s \in \mathbb{N}$, there exists $M_{\ell+1}(s)$ such that

$$\lambda^{-s} \left| \left(F_i^s\right)^{(\ell+1)}(x) - \left(F^s\right)^{(\ell+1)}(x) \right| \leq M_{\ell+1}(s) \left\| F_i - F \right\|_{r+1} \tag{28}$$

and $(M_{\ell+1}(s))$ is a bounded sequence whose upper bound satisfies (23). Clearly, for $s = 0$ we can find constant $M_{\ell+1}(0)$ satisfying (28). Then, assume that there exists $M_{\ell+1}(s)$ such that (28) holds for any $s = 0, \ldots, n-1$. Applying Lemma 1 and the inductive hypothesis, we have

$$\lambda^{-n} \left| \left(F_i^n\right)^{(\ell+1)}(x) - \left(F^n\right)^{(\ell+1)}(x) \right|$$

$$= \lambda^{-n} \Bigg| F_i'\left(F_i^{n-1}(x)\right)\left(F_i^{n-1}\right)^{(\ell+1)}(x) + \sum_{\substack{p \geq 2 \\ \omega \in \Omega(\ell+1)}} c_\omega F_i^{(p)}\left(F_i^{n-1}(x)\right) \prod_{q=1}^{p} \left(F_i^{n-1}\right)^{(j_q)}(x) - F'\left(F^{n-1}(x)\right)\left(F^{n-1}\right)^{(\ell+1)}(x) - \sum_{\substack{p \geq 2 \\ \omega \in \Omega(\ell+1)}} c_\omega F^{(p)}\left(F^{n-1}(x)\right) \prod_{q=1}^{p} \left(F^{n-1}\right)^{(j_q)}(x) \Bigg|$$

$$\leq \lambda^{-n} \Bigg\{ \left| F_i'\left(F_i^{n-1}(x)\right)\left(F_i^{n-1}\right)^{(\ell+1)}(x) - F'\left(F^{n-1}(x)\right)\left(F^{n-1}\right)^{(\ell+1)}(x) \right| + \sum_{\substack{p \geq 2 \\ \omega \in \Omega(\ell+1)}} c_\omega \Bigg| F_i^{(p)}\left(F_i^{n-1}(x)\right) \prod_{q=1}^{p} \left(F_i^{n-1}\right)^{(j_q)}(x) - F^{(p)}\left(F^{n-1}(x)\right) \prod_{q=1}^{p} \left(F^{n-1}\right)^{(j_q)}(x) \Bigg| \Bigg\}$$

$$\leq \lambda^{-1} M_{\ell+1}(n-1) \left| F_i'\left(F_i^{n-1}(x)\right) \right| \left\| F_i - F \right\|_{r+1} + L\lambda^{-1} A_{1,n-1}(x) + \sum_{\substack{p \geq 2 \\ \omega \in \Omega(\ell+1)}} c_\omega \Big\{ L^p \lambda^{(p-1)n-p} A_{p,n-1}(x) + \sum_{q=1}^{p} M_{j_q} L^{p-1} \lambda^{(p-1)n-p} \times \left| F^{(p)}\left(F^{n-1}(x)\right) \right| \left\| F_i - F \right\|_{r+1} \Big\}, \tag{29}$$

where $A_{p,n}(x) := |F_i^{(p)}(F_i^n(x)) - F^{(p)}(F^n(x))|$. By Lemma 1 and condition (19), we obtain that $|F_i'(F_i^{n-1}(x))| \leq |F_i'(0)| + |F_i'(F_i^{n-1}(x)) - F_i'(0)| \leq \lambda + L\eta\lambda^{n-1}$,

$$\left| F^{(p)}\left(F^{n-1}(x)\right) \right| \leq \left| F^{(p)}(0) \right| + \left| F^{(p)}\left(F^{n-1}(x)\right) - F^{(p)}(0) \right| \leq \eta + L\eta\lambda^{n-1},$$

$$A_{p,n}(x) \leq \left| F_i^{(p)}\left(F_i^n(x)\right) - F^{(p)}\left(F_i^n(x)\right) - F_i^{(p)}(0) + F^{(p)}(0) \right| + \left| F^{(p)}\left(F_i^n(x)\right) - F^{(p)}\left(F^n(x)\right) \right|$$

$$\leq L\lambda^n \left\| F_i - F \right\|_{r+1} + \eta \left| F_i^n(x) - F^n(x) \right|$$

$$\leq L\lambda^n \left\| F_i - F \right\|_{r+1} + \eta \left\{ \left(1 + \frac{L\lambda}{\mu - \lambda}\right)\mu^{n-1} - \frac{L}{\mu - \lambda}\lambda^n \right\} \times \left\| F_i - F \right\|_1$$

$$= \left\{ \left(1 + \frac{L\lambda}{\mu - \lambda}\right)\eta\mu^{n-1} + \left(1 - \frac{\eta}{\mu - \lambda}\right)L\lambda^n \right\} \left\| F_i - F \right\|_{r+1}, \tag{30}$$

where $\eta := 2\|F\|_{r+1}$ and μ is given in Lemma 1, for all $x \in I_\varepsilon$, $0 < p \le r$, $i \ge N_0$, and $n \in \mathbb{N}$. It follows that

$$\begin{aligned}
&\lambda^{-n}\left|\left(F_i^n\right)^{(\ell+1)}(x) - \left(F^n\right)^{(\ell+1)}(x)\right| \\
&\quad\le \left\{\left(1+L\eta\lambda^{n-2}\right)M_{\ell+1}(n-1) + \left(1+\frac{L\lambda}{\mu-\lambda}\right)L\eta\mu^{n-2}\lambda^{-1}\right. \\
&\qquad \left. + \left(1-\frac{\eta}{\mu-\lambda}\right)L^2\lambda^{n-2}\right\}\|F_i - F\|_{r+1} \\
&\qquad + \sum_{\substack{p\ge 2\\ \omega\in\Omega(\ell+1)}} c_\omega \left\{\left(1+\frac{L\lambda}{\mu-\lambda}\right)L^p\eta\mu^{n-2}\lambda^{(p-1)n-p}\right. \\
&\qquad\qquad + \left(1-\frac{\eta}{\mu-\lambda}\right)L^{p+1}\lambda^{pn-p-1} \\
&\qquad\qquad \left. + \left(1+L\lambda^{n-1}\right)L^{p-1}\eta\lambda^{(p-1)n-p}\sum_{q=1}^{p}M_{j_q}\right\} \\
&\qquad \times \|F_i - F\|_{r+1} \\
&\quad\le \left\{\left(1+L\eta\lambda^{n-2}\right)M_{\ell+1}(n-1) + \left(1+\frac{L\lambda}{\mu-\lambda}\right)L\eta\mu^{n-2}\lambda^{-1}\right. \\
&\qquad + \left(1-\frac{\eta}{\mu-\lambda}\right)L^2\mu^{n-2} \\
&\qquad + \sum_{\substack{p\ge 2\\ \omega\in\Omega(\ell+1)}} c_\omega \left\{\left(1+\frac{L\lambda}{\mu-\lambda}\right)L^p\eta\mu^{2n-4}\right. \\
&\qquad\qquad + \left(1-\frac{\eta}{\mu-\lambda}\right)L^{p+1}\mu^{2n-3} \\
&\qquad\qquad \left.\left. + (1+L)L^{p-1}\eta\mu^{n-2}\sum_{q=1}^{p}M_{j_q}\right\}\right\} \\
&\qquad \times \|F_i - F\|_{r+1} \\
&\quad\le \left\{\left(1+L\eta\mu^{n-2}\right)M_{\ell+1}(n-1) + \Gamma\mu^n\right\}\|F_i - F\|_{r+1}
\end{aligned} \tag{31}$$

because $\lambda < \mu \in (0,1)$, where

$$\begin{aligned}
\Gamma := \mu^4 &\left\{\left(1+\frac{L\lambda}{\mu-\lambda}\right)L\eta\lambda^{-1} + \left(1-\frac{\eta}{\mu-\lambda}\right)L^2\right. \\
&+ \sum_{\substack{p\ge 2\\ \omega\in\Omega(\ell+1)}} c_\omega \left\{\left(1+\frac{L\lambda}{\mu-\lambda}\right)L^p\eta\right. \\
&\qquad + \left(1-\frac{\eta}{\mu-\lambda}\right)L^{p+1} \\
&\qquad \left.\left. + (1+L)L^{p-1}\eta\sum_{q=1}^{p}M_{j_q}\right\}\right\}.
\end{aligned} \tag{32}$$

Let

$$\begin{aligned}
M_{\ell+1}(n) &:= \left(1+L\eta\mu^{n-2}\right)M_{\ell+1}(n-1) + \Gamma\mu^n \\
&= \prod_{j=1}^{n-2}\left(1+L\eta\mu^j\right)M_{\ell+1}(2) + \Gamma\sum_{j=3}^{n}\mu^j\prod_{t=j-1}^{n-2}\left(1+L\eta\mu^t\right).
\end{aligned} \tag{33}$$

It is easy to check that

$$M_{\ell+1}(n) \le \prod_{j=1}^{+\infty}\left(1+L\eta\mu^j\right)M_{\ell+1}(2) + \Gamma\sum_{j=3}^{+\infty}\mu^j\prod_{t=0}^{+\infty}\left(1+L\eta\mu^t\right) \tag{34}$$

for all $n \in \mathbb{N}$. By putting $M_{\ell+1} := \prod_{j=1}^{+\infty}(1+L\eta\mu^j)M_{\ell+1}(2) + \Gamma\sum_{j=3}^{+\infty}\mu^j\prod_{t=0}^{+\infty}(1+L\eta\mu^t)$, we can prove (23) for $m = \ell+1$ and (23) is proved by induction. □

Having this preparation, we can give a proof to the main result of this paper.

Proof of Theorem 2. By (4) the C^r iterative roots f and f_i for each $i \in \mathbb{N}$ can be presented by

$$f(x) = \varphi^{-1}\left(\lambda^{1/k}\varphi(x)\right), \qquad f_i(x) = \varphi_i^{-1}\left(\lambda^{1/k}\varphi_i(x)\right), \qquad \forall x \in I, \tag{35}$$

respectively. In order to prove the convergence of (f_i) in I, note that, for sufficiently large $i \in \mathbb{N}$ such that $\lambda^{1/k}\varphi_i(x) \in \varphi(I)$ for all $x \in I$, we have

$$\begin{aligned}
&\left|\left(\varphi_i^{-1}\right)^{(p)}\left(\lambda^{1/k}\varphi_i(x)\right)\prod_{q=1}^{p}\varphi_i^{(j_q)}(x)\right. \\
&\qquad \left. - \left(\varphi^{-1}\right)^{(p)}\left(\lambda^{1/k}\varphi(x)\right)\prod_{q=1}^{p}\varphi^{(j_q)}(x)\right| \\
&\quad\le \left|\left(\varphi_i^{-1}\right)^{(p)}\left(\lambda^{1/k}\varphi_i(x)\right) - \left(\varphi^{-1}\right)^{(p)}\left(\lambda^{1/k}\varphi(x)\right)\right| \\
&\qquad \times \prod_{q=1}^{p}\left|\varphi_i^{(j_q)}(x)\right| + \left|\left(\varphi^{-1}\right)^{(p)}\left(\lambda^{1/k}\varphi(x)\right)\right| \\
&\qquad \times \left|\prod_{q=1}^{p}\varphi_i^{(j_q)}(x) - \prod_{q=1}^{p}\varphi^{(j_q)}(x)\right| \\
&\quad\le 2^p(\|\varphi\|_r)^p\left\{\left\|\varphi_i^{-1} - \varphi^{-1}\right\|_r + \left|\left(\varphi^{-1}\right)^{(p)}\left(\lambda^{1/k}\varphi_i(x)\right)\right.\right. \\
&\qquad\qquad \left.\left. - \left(\varphi^{-1}\right)^{(p)}\left(\lambda^{1/k}\varphi(x)\right)\right|\right\} \\
&\qquad + p2^{p-1}(\|\varphi\|_r)^{p-1}\left\|\varphi^{-1}\right\|_r\|\varphi_i - \varphi\|_r \longrightarrow 0
\end{aligned} \tag{36}$$

as $i \to +\infty$ for all integers $p \in [1, r]$ by (21) and the fact that the inverse operator is continuous and $(\varphi^{-1})^{(p)}$ is uniformly continuous. Hence, $\lim_{i\to+\infty}\|f_i - f\|_0 = 0$ and

$$\begin{aligned}
&\left|(f_i)^{(m)}(x) - f^{(m)}(x)\right| \\
&= \left|\sum_{\omega\in\Omega(m)} c_\omega \lambda^{p/k}\left(\varphi_i^{-1}\right)^{(p)}\left(\lambda^{1/k}\varphi_i(x)\right)\prod_{q=1}^{p}\varphi_i^{(j_q)}(x)\right. \\
&\qquad \left. - \sum_{\omega\in\Omega(m)} c_\omega \lambda^{p/k}\left(\varphi^{-1}\right)^{(p)}\left(\lambda^{1/k}\varphi(x)\right)\prod_{q=1}^{p}\varphi^{(j_q)}(x)\right| \\
&\le \sum_{\omega\in\Omega(m)} c_\omega \lambda^{p/k}\left|\left(\varphi_i^{-1}\right)^{(p)}\left(\lambda^{1/k}\varphi_i(x)\right)\prod_{q=1}^{p}\varphi_i^{(j_q)}(x)\right. \\
&\qquad \left. - \left(\varphi^{-1}\right)^{(p)}\left(\lambda^{1/k}\varphi(x)\right)\prod_{q=1}^{p}\varphi^{(j_q)}(x)\right| \longrightarrow 0
\end{aligned} \tag{37}$$

as $i \to +\infty$ for all $m = 1, \ldots, r$, which implies that $\lim_{i\to+\infty}\|f_i - f\|_r = 0$ and completes the proof. □

Theorem 2 is also valid for $r = 1$, which is the same as Theorem 2.1 in [25] for $r = 1$. However, it is hard to use the estimates, for example, (2.4) and (2.5) in [25], to generalize the result to the general r parallel. In fact, we cannot use those estimates to give a uniform constant Γ with respect to n, the order of iteration, in (33). Using those estimates, corresponding to Γ given in (32), we obtain the quantity

$$\begin{aligned}
\Gamma^*(n) := \mu_0^4 \Bigg\{ & L^2 + (n-1) L\eta\lambda^{-1} \\
& + \sum_{\substack{p\ge 2 \\ \omega\in\Omega(\ell+1)}} c_\omega \Bigg\{ L^{p+1} + (n-1) L^p \eta \\
& \qquad + (1+L) L^{p-1}\eta \sum_{q=1}^{p} M_{j_q} \Bigg\} \Bigg\},
\end{aligned} \tag{38}$$

which tends to $+\infty$ as $n \to +\infty$. For this reason it is hard to prove the boundedness of $M_{\ell+1}(n)$. As remarked after the proof of Lemma 1, our estimation in (9) and (10) enables us to give the boundedness of $M_{\ell+1}(n)$ and complete the proof of (23).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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