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Research Article

Pricing Arithmetic Asian Options under Hybrid Stochastic and Local Volatility

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Recently, hybrid stochastic and local volatility models have become an industry standard for the pricing of derivatives and other problems in finance. In this study, we use a multiscale stochastic volatility model incorporated by the constant elasticity of variance to understand the price structure of continuous arithmetic average Asian options. The multiscale partial differential equation for the option price is approximated by a couple of single scale partial differential equations. In terms of the elasticity parameter governing the leverage effect, a correction to the stochastic volatility model is made for more efficient pricing and hedging of Asian options.

1. Introduction

Since the well-known work of Black and Scholes [1] on the classical vanilla European option, there has been concern about the pricing of more complicated exotic options. An exotic option is a derivative which has a payoff structure more complex than commonly traded vanilla options. They are usually traded in over-the-counter market or embedded in structured products. Also, the pricing of them tends to require more complex methods than the classical Black-Scholes approach.

This paper is concerned with one of the exotic options called an Asian option. This option is a path dependent option whose final payoff depends on the paths of its underlying asset. More precisely, the payoff is determined by the average value of underlying prices over some prescribed period of time. The name of "Asian" options is known to come from the fact that they were first priced in 1987 by David Spaughton and Mark Standish of Bankers Trust when they were working in Tokyo, Japan (cf. [2]). The main motivation of creating these options is that their averaging feature could reduce the risk of market manipulation of the underlying risky asset at maturity. Since Asian options reduce the volatility inherent in the option, the price of these options is usually lower than the price of classical European vanilla options. Note that there are two types of Asian options depending on the style of averaging: one is the arithmetic average Asian option and the other is the geometric average Asian option.

Since there is no general analytical formula for the price of Asian option, a variety of techniques have been developed to approximate the price of this option. Subsequently, there has been quite an amount of literature devoted to studying this option. For instance, Geman and Yor [3] computed the Laplace transform of the price of continuously sampled Asian options. However, there is a problem of slow convergence for low volatility or short time-to-maturity cases as indicated by Fu et al. [4]. Linetsky [5] derived a new integral formula for the price of Asian options but with the same convergence problem. Apart from this approximation technique, there are Monte Carlo simulation approach and partial differential equation (PDE) approach. Monte Carlo simulation methods typically require variance reduction techniques and also have to take into account the discretization error caused by discrete sampling. Refer to Kemna and Vorst [6] for a discussion of the pricing of Asian options with Monte Carlo methods. On the other hand, the PDE methods must deal with an extra state variable representing the running sum of the underlying process, which leads to an issue for reducing the dimension of the PDE. Refer to Ingersoll [7] and Rogers and Shi [8] and Vecer [9] for the PDE methods.

It is well known that the constant volatility assumption, on which a review of literature quoted above is based, for the underlying asset price is severely in contrast with many empirical studies which demonstrate the skew or smile effect of implied volatility, fat-tailed and asymmetric returns distributions, and the mean-reversion of volatility. Thus a number of alternative underlying models have been proposed. The constant elasticity of variance model by Cox [10], a stochastic volatility model by Heston [11] or Fouque et al. [12], and a Levy model by Carr et al. [13] are among those representative ones that can reproduce an empirically reasonable outcome.

So, it is desirable to study Asian options based on these alternative models. In fact, there are a number of recent studies along the lines of this type of extension. For instance, B. Peng and F. Peng [14] for the CEV model, Fouque and Han [15] for a stochastic volatility model, and Lemmens et al. [16] for a Levy type model are among those works extending the price for the Black-Scholes framework with constant volatility. As long as we understand, up to now, however, there has been no work for Asian options based on a hybrid stochastic and local volatility model. Recently, hybrid stochastic and local volatility models have become an industry standard for the pricing of derivatives and several financial institutions have incorporated those models into their systems [17]. Therefore, it is worth studying Asian options on the hybrid models.

This paper studies the pricing of an arithmetic Asian option under a hybrid stochastic and local volatility model which was introduced by Choi et al. [18], where the volatility is given by the product of a multiscale stochastic process and a power (the elasticity of variance) of the underlying's price. The hybrid nature of this volatility enables us to capture the leverage effect produced by the constant elasticity of variance (CEV) model as well as the smile effect of implied volatility, fat-tailed and asymmetric returns distributions, the tendency of the volatility process to revert towards a long-term mean at a certain rate, and a degree of correlation between the randomness of volatility and the randomness of underlying's price produced by a "pure" stochastic volatility (SV) model. This hybrid model is called the SVCEV model. So, this paper will generalize both [14, 15] into an approximation problem for the price of arithmetic average Asian options based on the SVCEV model.

This paper is structured as follows. In Section 2, we review the SVCEV model introduced by [18]. A multiscale partial differential equation for the price of an arithmetic average Asian option is obtained in Section 3. Section 4 is devoted to obtain an approximated option price under the condition of fast mean-reverting volatility. In Section 5, the effect of hybrid structure on the pricing of the Asian option is illustrated by a numerical study.

2. The SVCEV Model

In this section, we establish a partial differential equation for the price of Asian floating strike call option based on the SVCEV model.

As introduced by [18], the SVCEV model for the underlying's price is given by the stochastic differential equations (SDEs):

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma_t S_t^{1+\gamma} dW_t, \\ \sigma_t &:= f(Y_t), \\ dY_t &= \alpha(m - Y_t)\, dt + \beta dB_t, \end{aligned} \tag{1}$$

where μ and m are some constants, γ is greater than -1, α and β are positive constants, and W_t and B_t are correlated Brownian motions such that $d\langle W, B\rangle_t = \rho dt$ for some ρ. If $\rho > 0$, then S_t may fail to be a true martingale since Y_t may go infinite. Refer to [19]. Therefore, $\rho < 0$ is assumed here. It is observed in most financial markets that there is a minus correlation or leverage effect between stock price and volatility impact. However, some commodity markets show the opposite effect (the inverse leverage effect). So, the model under the negative correlation condition can apply to many financial markets but has limits for some commodity markets. Also, generally speaking, the correlation ρ may rely on time but it is assumed to be a constant for simplicity. In fact, in most real situations, it is taken to be such. We do not specify the function f but it has to satisfy a growth condition to avoid the nonexistence of moments of S_t. It is assumed in this paper that $0 < c_1 \le f \le c_2 < \infty$ for some constants c_1 and c_2.

From the I to formula, the solution Y of the second equation in (1) is an ergodic process given by the Ornstein-Uhlenbeck process:

$$Y_t = m + (Y_0 - m)\, e^{-\alpha t} + \beta \int_0^t e^{-\alpha(t-s)} dB_s \tag{2}$$

and so $Y_t \sim N(m+(Y_0-m)e^{-\alpha t}, (\beta^2/2\alpha)(1-e^{-2\alpha t}))$, which has an invariant distribution given by $N(m, \beta^2/2\alpha)$. Later, we will use notation $\langle\cdot\rangle$ for the average with respect to this invariant distribution; that is,

$$\langle g\rangle = \frac{1}{\sqrt{2\pi\nu^2}} \int_{-\infty}^{+\infty} g(y)\, e^{-(y-m)^2/2\nu^2} dy, \quad \nu^2 \equiv \frac{\beta^2}{2\alpha} \tag{3}$$

for arbitrary function g.

Now, we take the process Y_t to be a fast mean-reverting process as in Fouque et al. [20]. This means that we take into account a fast time scale volatility factor as a major driving force for the volatility movement of the underlying asset. This assumption can be justified by an empirical analysis of high-frequency S&P 500 index data. The analysis confirms that volatility is fast mean-reverting when looked at over the time scale of a derivative contract although it reverts slowly to its mean in comparison to the tick-by-tick fluctuations of the index value. See [20] for details. Also, this assumption will provide us with analytic tractability for an approximation for the price of the Asian option as also shown in [15]. So, in terms of a parameter, say ϵ, which is taken as positive and small, we assume that

$$\alpha = \frac{1}{\epsilon}, \qquad \beta = \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}, \tag{4}$$

where $\nu \sim \mathcal{O}(1)$ which implies that the long run magnitude of volatility fluctuations remains fixed as a constant. Then, under a risk-neutral probability measure P^*, we have

$$\begin{aligned} dS_t &= rS_t dt + f(Y_t) S_t^{1+\gamma} dW_t^*, \\ dY_t &= \left[\frac{1}{\epsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\Lambda(Y_t)\right] dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} dB_t^*, \end{aligned} \tag{5}$$

where W_t^* and B_t^* are two Brownian motions under the measure P^* whose correlation is given by $d\langle W_t^*, B_t^*\rangle = \rho dt$ and $\Lambda(y)$ denotes the combined market price of risk given by

$$\Lambda(y) = \rho\frac{\mu - r}{f(y)} + \lambda(y)\sqrt{1-\rho^2}. \tag{6}$$

Here, λ is the market price of volatility risk, which is assumed to be a bounded function depending on y.

3. A PDE for Option Price

In this section, we utilize the generalization of Vecer's dimension reduction technique given by Fouque and Han [15] to derive a two-space dimensional PDE representation for the option price.

In this paper, a payoff function for arithmetic average Asian options is given by

$$h\left(\frac{1}{T}\int_0^T S_t dt - K_1 S_T - K_2\right), \tag{7}$$

where h is a function satisfying

$$h(\alpha x) = \alpha h(x), \tag{8}$$

for any real number α. Note that, when $K_1 = 0$, it becomes a payoff for fixed strike Asian options, whereas, when $K_2 = 0$, it becomes a payoff for floating strike Asian options. So, the option price $P(t, x, y; T, K_1, K_2)$ is defined by

$$\begin{aligned} &P(t, s, y; T, K_1, K_2) \\ &= \mathbb{E}^*\left[e^{-r(T-t)} h\left(\frac{1}{T}\int_0^T S_u du - K_1 S_T - K_2\right) \mid S_t = s, Y_t = y\right] \end{aligned} \tag{9}$$

under a risk-neutral measure P^*.

First, we would like to replicate the averaged process $(1/t)\int_0^t S_u du$ with a portfolio

$$X_t = \alpha_t S_t + \beta_t e^{rt}. \tag{10}$$

Here, α_t is assumed to be a nonrandom function. By the self-financing strategy, we note that

$$\begin{aligned} dX_t &= \alpha_t dS_t + \beta_t d\left(e^{rt}\right) \\ &= \alpha_t dS_t + r(X_t - \alpha_t S_t) dt \\ &= rX_t dt + \alpha_t (dS_t - rS_t dt) \\ &= rX_t dt + \alpha_t f(Y_t) S_t^{1+\gamma} dW_t^*. \end{aligned} \tag{11}$$

The assumption of the nonrandomness of α_t yields

$$\begin{aligned} d\left(e^{r(T-t)}\alpha_t S_t\right) &= e^{r(T-t)}\alpha_t (dS_t - rS_t dt) \\ &\quad + e^{r(T-t)} S_t d\alpha_t. \end{aligned} \tag{12}$$

Thus, using (11) and (12), one can obtain

$$\begin{aligned} d\left(e^{r(T-t)}X_t\right) &= -re^{r(T-t)}X_t dt + e^{r(T-t)} dX_t \\ &= e^{r(T-t)}\alpha_t (dS_t - rS_t dt) \\ &= d\left(e^{r(T-t)}\alpha_t S_t\right) - e^{r(T-t)} S_t d\alpha_t. \end{aligned} \tag{13}$$

By integration of (13), we obtain

$$X_T - e^{rT}X_0 = \int_0^T d\left(\alpha_t e^{r(T-t)} S_t\right) - \int_0^T e^{r(T-t)} S_t d\alpha_t, \tag{14}$$

which can be written as

$$X_T = -\int_0^T e^{r(T-t)} S_t d\alpha_t + e^{rT}X_0 + \alpha_T S_T - \alpha_0 e^{rT} S_0. \tag{15}$$

For technical reason, if we choose the trading strategy α_t as

$$\alpha_t = \frac{1 - e^{-r(T-t)}}{rT} \tag{16}$$

and the initial portfolio price X_0 as

$$X_0 = x = \alpha_0 S_0 - e^{-rT} K_2, \tag{17}$$

then the final portfolio price X_T becomes $(1/T)\int_0^T S_t dt - K_2$ and so the general payoff function of arithmetic average Asian options given by (8) becomes $h(X_T - K_1 S_T)$. Refer to [9, 21] or [15] for details. In terms of the portfolio process X_t, the price of an arithmetic average Asian option at time $t = 0$ is given by

$$\begin{aligned} &P(0, s, y; T, K_1, K_2) \\ &\quad = \mathbb{E}^*\left[e^{-rT} h(X_T - K_1 S_T) \mid S_0 = s, Y_0 = y\right]. \end{aligned} \tag{18}$$

Next, we change the probability measure P^* into the probability measure $\widetilde{P}^*$ defined by

$$\begin{aligned} \frac{d\widetilde{P}^*}{dP^*} &= e^{-rT}\frac{S_T}{S_0} \\ &= \exp\left[\int_0^T f(Y_t) S_t^{\gamma} d\widetilde{W}_t^* - \frac{1}{2}\int_0^T f^2(Y_t) S_t^{2\gamma} dt\right]. \end{aligned} \tag{19}$$

Then we are ready to obtain the option price as the solution of a partial differential equation.

Theorem 1. *Let $\psi_t := X_t/S_t$. Then the Asian option price at time $t = 0$ can be expressed by*

$$\begin{aligned} &P(0, s, y; T, K_1, K_2) \\ &\quad = s\widetilde{\mathbb{E}}^*\left[h(\psi_T - K_1) \mid \psi_0 = \psi, Y_0 = y\right] \end{aligned} \tag{20}$$

under the measure $\widetilde{P}^*$, where

$$\psi = \frac{x}{s} = \frac{1-e^{-rT}}{rT} - \frac{K_2}{s}e^{-rT}. \tag{21}$$

If $u(t,\psi,y;T,K_1,K_2) := \widetilde{\mathbb{E}}^*[h(\psi_T - K_1) \mid \psi_t = \psi, Y_t = y]$, then u is given by the solution of the PDE:

$$\begin{aligned} &u_t + \frac{1}{2}(\alpha_t - \psi)^2 f^2(y) s^{2\gamma} u_{\psi\psi} \\ &\quad + \frac{\rho\nu\sqrt{2}}{\sqrt{\epsilon}}(\alpha_t - \psi) f(y) s^{\gamma} u_{\psi y} \\ &\quad + \left(\frac{1}{\epsilon}(m-y) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}(\Lambda(y) - \rho f(y))\right) u_y \\ &\quad + \frac{\nu^2}{\epsilon} u_{yy} = 0 \end{aligned} \tag{22}$$

with the terminal condition

$$u(T,\psi,y;T,K_1,K_2) = h(\psi - K_1). \tag{23}$$

Proof. By the Itô formula, we obtain the following two results:

$$\begin{aligned} d\left(S_t^{-1}\right) &= \frac{1}{S_t}\left[\left(f^2(Y_t) S_t^{2\gamma} - r\right) dt - f(Y_t) S_t^{\gamma} dW_t^*\right], \\ dX_t &= S_t\left[r\psi_t dt + \alpha_t f(Y_t) S_t^{\gamma} dW_t^*\right]. \end{aligned} \tag{24}$$

Then we have

$$\begin{aligned} d\psi_t &= X_t d\left(S_t^{-1}\right) + S_t^{-1} dX_t + dX_t d\left(S_t^{-1}\right) \\ &= \psi_t\left[f^2(Y_t) S_t^{2\gamma} dt - f(Y_t) S_t^{\gamma} dW_t^*\right] \\ &\quad - \alpha_t\left[f^2(Y_t) S_t^{2\gamma} dt - f(Y_t) S_t^{\gamma} dW_t^*\right] \\ &= f(Y_t) S^{\gamma}(\alpha_t - \psi_t)(dW_t^* - f(Y_t) S_t^{\gamma} dt) \\ &:= f(Y_t) S^{\gamma}(\alpha_t - \psi_t) d\widetilde{W}_t^*, \end{aligned} \tag{25}$$

where

$$\widetilde{W}_t^* := W_t^* - \int_0^t f(Y_u) S_u^{\gamma} du, \tag{26}$$

which is a Brownian motion under the measure $\widetilde{P}^*$ by the Girsanov theorem (cf. [22]).

From (8) and (18), we have

$$\begin{aligned} &P(0,s,y;T,K_1,K_2) \\ &\quad = \mathbb{E}^*\left[e^{-rT} h(X_T - K_1 S_T) \mid S_0 = s, Y_0 = y\right] \\ &\quad = s\mathbb{E}^*\left[e^{-rT}\frac{S_T}{S_0} h(\psi_T - K_1) \mid \psi_0 = \psi, Y_0 = y\right] \\ &\quad = s\widetilde{\mathbb{E}}^*\left[h(\psi_T - K_1) \mid \psi_0 = \psi, Y_0 = y\right], \end{aligned} \tag{27}$$

where

$$\psi = \frac{x}{s} = \frac{1-e^{-rT}}{rT} - \frac{K_2}{s}e^{-rT}. \tag{28}$$

If $u(t,\psi,y;T,K_1,K_2) := \widetilde{\mathbb{E}}^*[h(\psi_T - K_1) \mid \psi_t = \psi, Y_t = y]$, the Asian option price at time $t = 0$ is given by

$$P(0,s,y;T,K_1,K_2) = su(0,\psi,y;T,K_1,K_2). \tag{29}$$

Then, by the Feynman-Kac formula (cf. [22]) and (25), one can obtain (22). □

Once the solution $P(0,s,y;T,K_1,K_2)$ is determined, from the result of Fouque and Han [15], the price $P(t,s,y;T,K_1,K_2)$ is given by

$$\begin{aligned} P(t,s,y;T,K_1,K_2) &= \frac{T-t}{T} P(0,s,y;T,K_1,K_2) \\ &= \frac{T-t}{T} su(0,\psi,y;T,K_1,K_2), \end{aligned} \tag{30}$$

where u is the solution of the PDE (22) at time $t = 0$.

4. Multiscale Analysis

In this section, we are interested in the solution of the multiscale PDE (22) in the asymptotic form $u = u_0 + \sqrt{\epsilon}u_1 + \cdots + \epsilon^{i/2}u_i + \cdots$. Then one can come up with single scale PDEs much easier to solve than the PDE (22) itself as follows.

Substituting the asymptotic form of u into the PDE (22) yields

$$\begin{aligned} &\frac{1}{\epsilon}\mathscr{L}_0 u_0 + \frac{1}{\sqrt{\epsilon}}(\mathscr{L}_0 u_1 + \mathscr{L}_1 u_0) + (\mathscr{L}_0 u_2 + \mathscr{L}_1 u_1 + \mathscr{L}_2 u_0) \\ &\quad + \sqrt{\epsilon}(\mathscr{L}_0 u_3 + \mathscr{L}_1 u_2 + \mathscr{L}_2 u_1) + \cdots \\ &\quad + \epsilon^{i/2}(\mathscr{L}_0 u_{i+2} + \mathscr{L}_1 u_{i+1} + \mathscr{L}_2 u_i) + \cdots = 0 \end{aligned} \tag{31}$$

with the terminal condition

$$\sum_{i=0}^{\infty} \epsilon^{i/2} u_i = h(\psi - K_1), \tag{32}$$

where

$$\begin{aligned} \mathscr{L}_0 &:= (m-y)\frac{\partial}{\partial y} + \nu^2 \frac{\partial^2}{\partial y^2}, \\ \mathscr{L}_1(f(y)) &:= \rho\nu\sqrt{2}(\alpha_t - \psi) f(y) s^{\gamma} \frac{\partial^2}{\partial\psi\partial y} \\ &\quad + \sqrt{2}\nu(\Lambda(y) - \rho f(y)) \frac{\partial}{\partial y}, \\ \mathscr{L}_2(f(y)) &:= \frac{\partial}{\partial t} + \frac{1}{2}(\alpha_t - \psi)^2 f(y)^2 s^{2\gamma} \frac{\partial^2}{\partial\psi^2}. \end{aligned} \tag{33}$$

Note that $\mathscr{L}_0$ is the infinitesimal generator of the Ornstein-Uhlenbeck process Y_t. Then we have a hierarchy of PDEs as follows:

$$\begin{aligned}
&\frac{1}{\epsilon}-\text{term}: \mathscr{L}_0 u_0 = 0,\\
&\frac{1}{\sqrt{\epsilon}}-\text{term}: \mathscr{L}_0 u_1 + \mathscr{L}_1 u_0 = 0,\\
&\epsilon^{-(i+2)/2}-\text{term}: \mathscr{L}_0 u_{i+2} + \mathscr{L}_1 u_{i+1} + \mathscr{L}_2 u_i = 0, \quad \forall i \geq 0,
\end{aligned} \tag{34}$$

with the following terminal condition:

$$\begin{aligned}
u_0(T,\psi,y;T,K_1,K_2) &= h(\psi - K_1),\\
u_i(T,\psi,y;T,K_1,K_2) &= 0, \quad \forall i \geq 1.
\end{aligned} \tag{35}$$

In the following two theorems, we obtain PDEs for the leading order term u_0 as well as the first correction term u_1.

Theorem 2. *Assume that the term u_i does not grow as much as $u_i \sim e^{y^2/2}$ with respect to the variable y for $i = 0, 1$. Then the leading order term u_0 is independent of y and satisfies the PDE problem:*

$$\begin{aligned}
&\mathscr{L}_2(\overline{\sigma})\, u_0(t,\psi;T,K_1,K_2)\\
&\quad = \frac{\partial u_0}{\partial t} + \frac{1}{2}(\psi-\alpha_t)^2\overline{\sigma}^2 s^{2\gamma}\frac{\partial^2 u_0}{\partial \psi^2} = 0,\\
&u_0(T,\psi;T,K_1,K_2) = h(\psi - K_1),
\end{aligned} \tag{36}$$

where

$$\overline{\sigma} = \sqrt{\langle f^2\rangle} = \sqrt{\frac{1}{\sqrt{2\pi\nu^2}}\int_{-\infty}^{\infty} f^2(y)\, e^{-(y-m)^2/2\nu^2}\, dy}. \tag{37}$$

Proof. From the $\mathcal{O}(1/\epsilon)$ term in (34) and the fact that $\mathscr{L}_0$ is the generator of Y_t, the growth condition of u_0 with respect to the variable y leads to the fact that u_0 does not depend on y. So, it is represented as

$$u_0 = u_0(t,\psi;T,K_1,K_2) \tag{38}$$

without dependence on the variable y. From this fact, equation $\mathscr{L}_0 u_1 + \mathscr{L}_1 u_0 = 0$ becomes $\mathscr{L}_0 u_1 = 0$ and so, by the same reason as for u_0, u_1 also does not depend on the variable y:

$$u_1 = u_1(t,\psi;T,K_1,K_2). \tag{39}$$

Since $\mathscr{L}_1$ is an operator in which every term contains the derivative with respect to the variable y, $\mathscr{L}_0 u_2 + \mathscr{L}_1 u_1 + \mathscr{L}_2 u_0 = 0$ becomes $\mathscr{L}_0 u_2 + \mathscr{L}_2 u_0 = 0$. By applying the Fredholm alternative (cf. Ramm [23]) to this Poisson equation with respect to the operator $\mathscr{L}_0$, we have $\langle \mathscr{L}_2 u_0\rangle = 0$; that is,

$$\mathscr{L}_2(\overline{\sigma})\, u_0 = \frac{\partial u_0}{\partial t} + \frac{1}{2}(\psi-\alpha_t)^2\overline{\sigma}^2 s^{2\theta}\frac{\partial^2 u_0}{\partial \psi^2} = 0. \tag{40}$$

By adding the boundary condition (35) for u_0 to this PDE, we obtain the PDE problem (36) as desired. □

Next, we obtain a PDE for the first correction term u_1.

Theorem 3. *Assume that the term u_i does not grow as much as $u_i \sim e^{y^2/2}$ with respect to the variable y for $i = 0, 1$. Then the first order correction term u_1 is independent of y and satisfies the PDE problem:*

$$\begin{aligned}
\mathscr{L}_2(\overline{\sigma})\, u_1(t,\psi;T,K_1,K_2) &= V_2(\alpha_t-\psi)^2 s^{2\gamma}\frac{\partial^2 u_0}{\partial\psi^2}\\
&\quad + V_3(\alpha_t-\psi)^3 s^{2\gamma}\frac{\partial^3 u_0}{\partial\psi^3},\\
u_1(T,\psi;T,K_1,K_2) &= 0,
\end{aligned} \tag{41}$$

where V_2 and V_3 are constants, respectively, given by

$$\begin{aligned}
V_2 &= \frac{\nu}{\sqrt{2}}\left(\langle\Lambda\phi'\rangle - \rho\langle f\phi'\rangle\right),\\
V_3 &= -\frac{\rho\nu}{\sqrt{2}}\langle f\phi\rangle.
\end{aligned} \tag{42}$$

Here, ϕ is the solution of

$$\mathscr{L}_0\phi(y) = f(y)^2 - \overline{\sigma}^2. \tag{43}$$

Proof. Since $\langle\mathscr{L}_2 u_0\rangle = 0$ and $\mathscr{L}_0 u_2 + \mathscr{L}_2 u_0 = 0$, we obtain

$$\begin{aligned}
\mathscr{L}_0 u_2 &= -\mathscr{L}_2 u_0\\
&= -\left(\mathscr{L}_2 u_0 - \langle\mathscr{L}_2 u_0\rangle\right)\\
&= -\frac{1}{2}(\psi-\alpha_t)^2\left(f(y)^2 - \overline{\sigma}^2\right)s^{2\theta}\frac{\partial^2 u_0}{\partial\psi^2}.
\end{aligned} \tag{44}$$

Then, from the definition of ϕ, we obtain

$$\begin{aligned}
&u_2(t,\psi,y;T,K_1,K_2)\\
&\quad = -\frac{1}{2}(\psi-\alpha_t)^2\left(\phi(y) + c(t,\psi)\right)s^{2\gamma}\frac{\partial^2 u_0}{\partial\psi^2}
\end{aligned} \tag{45}$$

for some function $c(t,\psi)$ independent of y.

By applying the Fredholm alternative to the Poisson equation $\mathscr{L}_0 u_3 + \mathscr{L}_1 u_2 + \mathscr{L}_2 u_1 = 0$, we have equation $\langle\mathscr{L}_1 u_2 + \mathscr{L}_2 u_1\rangle = 0$; that is,

$$\mathscr{L}_2(\overline{\sigma})\, u_1 = -\langle\mathscr{L}_1(f(\cdot))\, u_2\rangle. \tag{46}$$

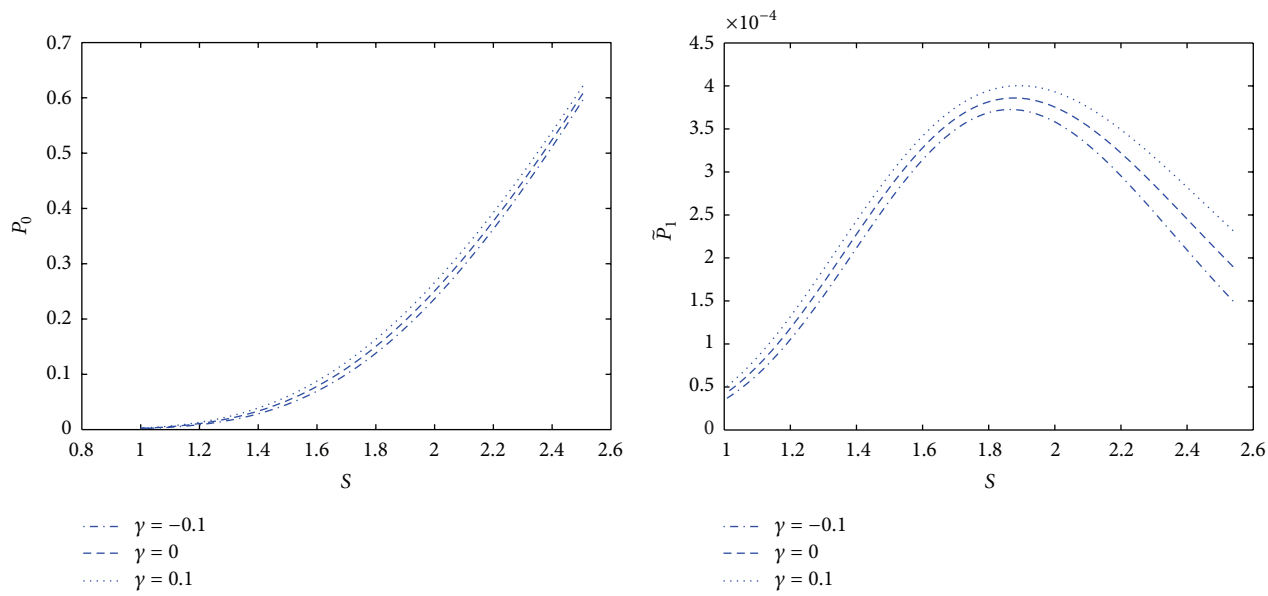
Combining (45) and (46), we obtain (41). □

5. Approximate Option Price

From the results in Section 4, we formally obtain the first order approximation:

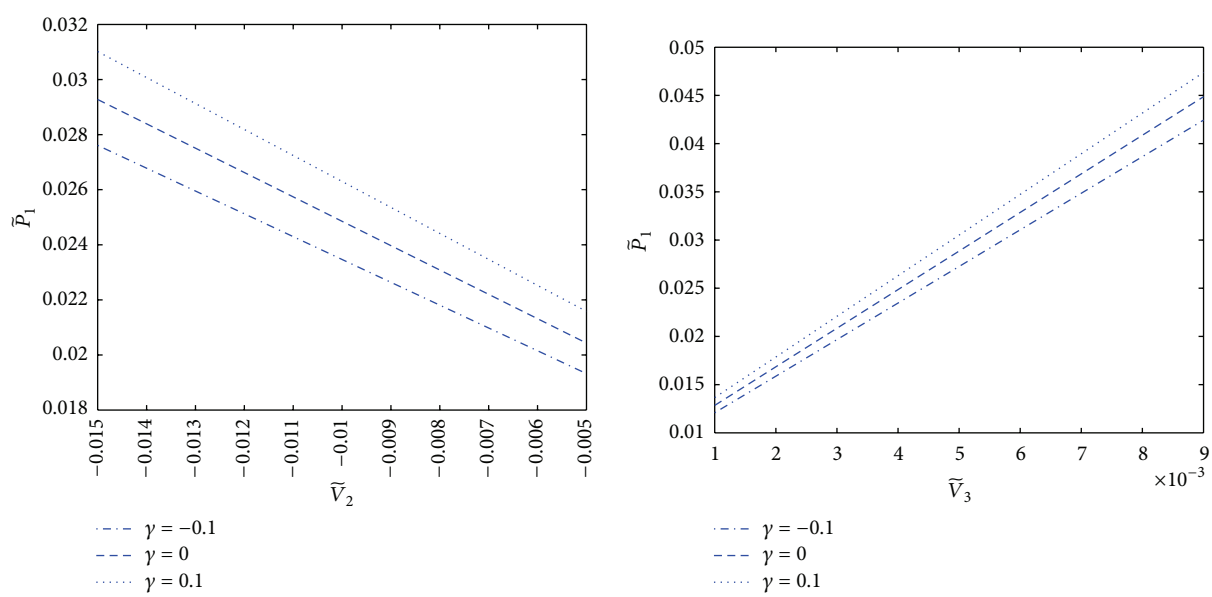
$$\begin{aligned}
&u(t,\psi,y;T,K_1,K_2)\\
&\quad \approx u_0(t,\psi;T,K_1,K_2) + \sqrt{\epsilon}\, u_1(t,\psi;T,K_1,K_2),
\end{aligned} \tag{47}$$

where u_0 and u_1 are independent of the unobservable variable y and they are given by the PDEs in Theorems 2 and 3, respectively. Note that u_0 solves a homogeneous



(a) P_0 (b) $\widetilde{P}_1$

FIGURE 1: The leading order term P_0 and the first order correction term $\widetilde{P}_1$ of the price of an arithmetic average Asian option are drawn for three different values of the elasticity of variance parameter γ, where $r = 0.06$, $\overline{\sigma} = 0.5$, $T = 1$, $K_1 = 0$, $K_2 = 2$, $\widetilde{V}_2 = -0.01$, and $\widetilde{V}_3 = 0.004$.



(a) $\widetilde{P}_1$ (b) $\widetilde{P}_1$

FIGURE 2: The correction term $\widetilde{P}_1$ with respect to the group parameters $\widetilde{V}_2$ and $\widetilde{V}_3$; $r = 0.06$, $\overline{\sigma} = 0.5$, $T = 1$, $S = 2$, $K_1 = 0$, and $K_2 = 2$.

equation with the nonzero final condition while u_1 solves a nonhomogeneous equation with the zero final condition. In terms of accuracy of the approximation (47), if the payoff function h is smooth enough, it follows straightforwardly from [20] that the approximation is of order ϵ in the pointwise convergent sense. Otherwise, it requires a regularization of the payoff as done for European vanilla options in [24]. Since the relevant argument (with an exact form of the density function for α_T) is omitted in this paper, we limit ourselves to the case of regularized payoffs here.

If we define P_0 and P_1 by

$$P_0(t, s; T, K_1, K_2) := \frac{T-t}{T} s u_0(0, \psi; T, K_1, K_2),$$
$$P_1(t, s; T, K_1, K_2) := \frac{T-t}{T} s u_1(0, \psi; T, K_1, K_2), \tag{48}$$

respectively, then, from (30), the option price $P(t, s, y; T, K_1, K_2)$ has an approximation given by

$$P(t, s, y; T, K_1, K_2) \approx P_0(t, s; T, K_1, K_2) + \sqrt{\epsilon} P_1(t, s; T, K_1, K_2). \tag{49}$$

In this section, we compute numerically the leading order price P_0 and the first correction term $\widetilde{P}_1 := \sqrt{\epsilon} P_1$ by using the finite difference method (the Crank-Nicolson method). The solution has the truncation error $\mathcal{O}((\Delta t)^2) + \mathcal{O}((\Delta \psi)^2)$, where $\Delta t = 0.005$ and $\Delta \psi = 0.0104$.

Figure 1 shows P_0 and $\widetilde{P}_1$ at time $t = 0$ for three different values of γ (the elasticity parameter). The parameter values used in this figure are $r = 0.06, \overline{\sigma} = 0.5, T = 1, K_1 = 0, K_2 = 2$, $\widetilde{V}_2 := \sqrt{\epsilon} V_2 = -0.01$, and $\widetilde{V}_3 := \sqrt{\epsilon} V_3 = 0.004$. Note that the parameter ϵ is absorbed by $\widetilde{V}_2$ and $\widetilde{V}_3$. We call the price $P_0 + \widetilde{P}_1$ the (approximate) SVCEV price for the arithmetic average Asian option. It can reduce to the price for the well-known models. In particular, the price P_0 with $\gamma = 0$ is the Black-Scholes price. The price $P_0 + \widetilde{P}_1$ with $\gamma = 0$ corresponds to the stochastic volatility model studied by Fouque and Han [15]. Let us call it the SV price. When the leverage effect ($\gamma < 0$) takes place, the SVCEV price becomes lower than the SV price. When the inverse leverage effect ($\gamma > 0$) takes place, the SVCEV price becomes higher than the SV price. Regardless of the value of γ, the first order stochastic volatility correction term $\widetilde{P}_1$ is positive and has the maximum value near $s = K_2$.

Figure 2 shows the correction term $\widetilde{P}_1$ with respect to the group parameters $\widetilde{V}_2$ and $\widetilde{V}_3$. This figure shows that the correction term $\widetilde{P}_1$ decreases as the parameter $\widetilde{V}_2$ increases whereas it increases as the $\widetilde{V}_3$ increases. Figure 2 also shows that as the parameter γ becomes larger, the correction term $\widetilde{P}_1$ becomes larger.

6. Conclusion

A frequent criticism of the stochastic volatility or local volatility models for path dependent options is that they do not produce deltas precise enough for hedging purposes. So, relevant industry experts recommend using a hybrid stochastic local volatility model of their own development for best pricing option products. See, for instance, [25]. In this paper, by transforming the path dependent problem for Asian options in the hybrid SVCEV model into a European vanilla style problem, we approximate the volatility dependent price by the nonvolatility dependent price. The approximate price provides not only a correction to the price of [15] for different values of the elasticity of variance parameter but also a correction to the CEV price based on the assumption of fast mean-reverting stochastic volatility. The elasticity of variance plays an important role in characterizing volatile markets as well as differentiating commodity markets from financial markets. Refer to [26]. The numerically solved Asian option prices show that both the leading order option price (the CEV price) and the corrected price (the approximate SVCEV price) increase as the elasticity of variance goes up. This result can provide more efficient risk hedging by choosing an appropriate elasticity parameter based on the observed market volatility or the chosen commodity.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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