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Research Article

A Cooperative Dual to the Nash Equilibrium for Two-Person Prescriptive Games

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An alternative to the Nash equilibrium (NE) is presented for two-person, one-shot prescriptive games in normal form, where the outcome is determined by an arbiter. The NE is the fundamental solution concept in noncooperative game theory. It is based on the assumption that players are completely selfish. However, NEs are often not played in practice, so we present a cooperative dual as an alternative solution concept by which an arbiter can assign the players' actions. In this dual equilibrium (DE), each player acts in the other's best interest. We formally define prescriptive games and the DE, then summarize the duality relationships between the NE and DE for two players. We also apply the DE to some prescriptive games and compare it to other outcomes.

1. Introduction

Game theory is the study of strategic interactions among agents called players. Ultimately it involves a solution concept to describe, predict, or prescribe the choices of these players [1]. Modern game theory [2, 3] is predominantly noncooperative and assumes that any joint rational actions by the players must be a Nash equilibrium (NE) [1–5]. In other words, rational players act in their individual self-interest. Each player's action maximizes his payoff for the actions of the other players. The result is that no player can improve his expected payoff by unilaterally changing his strategy. Various refinements of the NE [2, 3] have been proposed, yet players can often do better by cooperating. Social dilemmas such as the Prisoner's Dilemma, Snow drift, and Ultimatum games [6–9] illustrate that selfish behavior may conflict with group interests.

To address such issues, we consider here one-shot, two-person prescriptive games in normal form, where the outcome is determined by an arbiter. In this paper we provide the arbiter with an alternative approach to the NE for assigning the players' actions. Our framework is prescriptive because the assumptions of noncooperative games are often not met in practice and because outcomes are often influenced by external forces. An arbiter can assign reasonable actions to both players that would be precluded by selfish strategies chosen by the players themselves. Pure strategies are emphasized here. Mixed strategies are somewhat problematic to interpret [1, 10] for noncooperative games but even more so when an arbiter for a one-shot game must specify an action for each player.

In Section 2 we define prescriptive games and the dual equilibrium (DE) in which each player acts in the other's best interest. In Section 3 we describe how to obtain pure DEs and present some examples. In Section 4 we summarize the duality relationships between the DE and NE, which do not extend to one-shot prescriptive games with more than two players. In Section 5 we consider the special case of zero-sum games. In Section 6 we present conclusions and discuss future work.

2. The Dual Equilibrium

Let $\Gamma = \langle \mathbf{A}, \mathbf{B}, \alpha \rangle$ denote a two-person, one-shot prescriptive game in normal form, where $\mathbf{A}$ is the $n \times m$ payoff matrix $[a_{ij}]$

Table 1: Prisoner's Dilemma game matrices.

		Player II					
		t_1 Cooperate	t_2 Defect	t_1 Cooperate	t_2 Defect	t_1 Cooperate	t_2 Defect
Player I	s_1 Cooperate	(4, 4)	(0, 5)	(1, 1)	(2, 0)	(0, 0)	(4, 0)
	s_2 Defect	(5, 0)	(2, 2)	(0, 2)	(0, 0)	(0, 4)	(3, 3)

of von Neumann-Morgenstern (VNM) utilities for Player I when Player I plays pure strategy s_i and Player II plays pure strategy t_j. Similarly, $\mathbf{B}$ is the $n \times m$ payoff matrix $[b_{ij}]$ for Player II. The prescriptive mechanism α is an arbiter who assigns unique actions to the players for their one shot. The arbiter could be a person or group of people. It could, for example, be a licensing agreement for the licensees of a patent. The arbiter could also be a person selected to rule in a formal legal arbitration. It could be a computer algorithm for making real-time decisions on a website where the players have agreed to its terms and conditions, as well as a policy imposed by a governmental agency on some segment of the population. In this paper, the arbiter will assign pure DE strategies to the two players. Hence the arbiter could even be a tacit agreement between the two players based on social pressures that dictate that the players should cooperate unselfishly. In this case, their joint notion of rationality based on social pressure is incorporated in the DE. When α is implicit, as in such an agreement, Γ is simply referred to as the game $(\mathbf{A}, \mathbf{B})$. If there are multiple pure DEs, we assume that the arbiter selects a unique one. Regardless, a strategy pair (s_i, t_j) assigned by α is an equilibrium in the sense that the players cannot change the prescribed actions.

An NE and DE for $(\mathbf{A}, \mathbf{B})$ are next defined in terms of mixed strategies. Let $X = \{\mathbf{x} \mid \mathbf{x} = [x_1, \ldots, x_m]^T\}$ be the set of mixed strategies of Player I and $Y = \{\mathbf{y} \mid \mathbf{y} = [y_1, \ldots, y_n]^T\}$ the set of mixed strategies for Player II.

Definition 1 (NE). The mixed strategy pair $(\mathbf{x}^*, \mathbf{y}^*)$ is an NE for $(\mathbf{A}, \mathbf{B})$ if and only if

$$\max_{x \in X} \mathbf{x}^T \mathbf{A} \mathbf{y}^* = \mathbf{x}^{*T} \mathbf{A} \mathbf{y}^*, \qquad \max_{y \in Y} \mathbf{x}^{*T} \mathbf{B} \mathbf{y} = \mathbf{x}^{*T} \mathbf{B} \mathbf{y}^*. \quad (1)$$

Definition 2 (DE). The mixed strategy pair $(\mathbf{x}^*, \mathbf{y}^*)$ is a DE for $(\mathbf{A}, \mathbf{B})$ if and only if

$$\max_{y \in Y} \mathbf{x}^{*T} \mathbf{A} \mathbf{y} = \mathbf{x}^{*T} \mathbf{A} \mathbf{y}^*, \qquad \max_{x \in X} \mathbf{x}^T \mathbf{B} \mathbf{y}^* = \mathbf{x}^{*T} \mathbf{B} \mathbf{y}^*. \quad (2)$$

Definitions 1 and 2 depict one aspect of the duality between the NE and the DE, which is the players' opposing behaviors. In (1) each player selfishly responds to the NE strategy for the other player so as to maximize his own expected utility. In (2) each player unselfishly responds to the DE strategy for the other player so as to maximize the expected utility of the other player. In other words, in an NE no player can improve his payoff with a unilateral change in his strategy. In a DE a unilateral change in either player's strategy cannot improve the other player's payoff. A DE is a mutual-max outcome used in [11, page 1282] in defining a fairness equilibrium. A joint equilibrium (JE), which is both an NE and DE, incorporates selfishness and unselfishness in one outcome. It is a special case of the Rabin fairness equilibrium.

3. Computing Pure DEs

Pure NEs and DEs are easily obtained from the notions of regret and disappointment for a game $(\mathbf{A}, \mathbf{B})$. The regret function is a transformation of a player's VNM utilities for pure strategies to a loss function. For a fixed pure strategy t_j of Player II, Player I's regret for using pure strategy s_i is the regret function value $r_1(s_i, t_j) = \max_k a_{kj} - a_{ij}$. For Player II, $r_2(s_i, t_j) = \max_l b_{il} - b_{ij}$. The bimatrix $(\mathbf{A}, \mathbf{B})$ can thus be transformed into a regret bimatrix $\mathbf{R}(\mathbf{A}, \mathbf{B})$ that has the same NEs [5] as the bimatrix $(\mathbf{A}, \mathbf{B})$. In particular, a pure strategy pair is an NE if and only if $(0, 0)$ is the corresponding entry in $\mathbf{R}(\mathbf{A}, \mathbf{B})$. Likewise, the bimatrix $(\mathbf{A}, \mathbf{B})$ can be transformed into a disappointment matrix $\mathbf{D}(\mathbf{A}, \mathbf{B})$, where disappointment for a player may be interpreted as regret with respect to the other player. For a fixed pure strategy s_i of Player I, Player I's disappointment at Player II's using pure strategy t_j is Player I's disappointment function value $d_1(s_i, t_j) = \max_l a_{il} - a_{ij}$, while $d_2(s_i, t_j) = \max_k b_{kj} - b_{ij}$ for Player II. The proof of Proposition 3 below is similar to that in [5] showing that $\mathbf{R}(\mathbf{A}, \mathbf{B})$ has the same NEs as $(\mathbf{A}, \mathbf{B})$.

Proposition 3. *$\mathbf{D}(\mathbf{A}, \mathbf{B})$ has the same DEs as $(\mathbf{A}, \mathbf{B})$, and a pure strategy pair is a DE if and only if $(0, 0)$ is the corresponding entry in $\mathbf{D}(\mathbf{A}, \mathbf{B})$.*

Example 4 (Prisoner's Dilemma game). Table 1 shows the matrices $(\mathbf{A}, \mathbf{B})$, $\mathbf{R}$, and $\mathbf{D}$ from left to right for a Prisoner's Dilemma game [6], where the two players are arrested for a crime and held in separate rooms. To cooperate means to deny that either player had any part in the crime. To defect means to swear that the other player committed the crime alone. For each strategy pair, the VNM utility u denotes $5 - u$ years spent in jail. In a prescriptive version of the game, the arbiter α could be a lawyer who represents both players and tells them how to respond when interrogated. There is a pure NE (defect, defect) whose payoff $(2, 2)$ is dominated [12] by the payoff $(4, 4)$ of the pure DE (cooperate, cooperate). The maximin outcome [2], in which each player's action maximizes his minimum payoff resulting from the actions of the other players, is the NE (defect, defect).

Table 2: Snow Drift game matrices.

		Player II					
		t_1 Shovel	t_2 Refuse	t_1 Shovel	t_2 Refuse	t_1 Shovel	t_2 Refuse
Player I	s_1 Shovel	(200, 200)	(100, 300)	(100, 100)	(0, 0)	(0, 0)	(100, 0)
	s_2 Refuse	(300, 100)	(0, 0)	(0, 0)	(100, 100)	(0, 100)	(300, 300)

Table 3: Game with JE.

		Player II								
		t_1	t_2	t_3	t_1	t_2	t_3	t_1	t_2	t_3
Player I	s_1	(3, 3)	(2, 2)	(1, 1)	(0, 0)	(2, 1)	(6, 2)	(0, 0)	(1, 5)	(2, 4)
	s_2	(2, 3)	(3, 1)	(7, 2)	(1, 0)	(1, 2)	(0, 1)	(5, 0)	(4, 6)	(0, 3)
	s_3	(2, 1)	(4, 7)	(5, 5)	(1, 4)	(0, 0)	(2, 2)	(3, 0)	(1, 0)	(0, 0)

Example 5 (Snow Drift game). Table 2 gives $(\mathbf{A}, \mathbf{B})$, $\mathbf{R}$, and $\mathbf{D}$ for the Snow Drift game, which involves two drivers trapped on opposite sides of a snow drift blocking a road. Each has the option of staying in his car or shoveling snow to clear a path. The Snow Drift game has been said to more realistically reflect social situations that humans face than Prisoner's Dilemma [7]. In a prescriptive version of the game, the two drivers could be neighbors, and the arbiter α could be the social pressure to cooperate and preserve good will in the players' future interactions as neighbors. The pure NEs are (shovel, refuse) and (refuse, shovel). The pure DE is (shovel, shovel). The maximin outcome is the DE (shovel, shovel), in contrast to being the NE in Example 4. There is also a mixed NE and DE.

Example 6 (JE). Consider the matrices $(\mathbf{A}, \mathbf{B})$, $\mathbf{R}$, and $\mathbf{D}$ of Table 3. The strategy pair (s_1, t_1) is a JE, but $(3, 3)$ for (s_1, t_1) is dominated by the DE (s_3, t_3) with payoffs $(5, 5)$ and so is not a Pareto optimum [12] for $(\mathbf{A}, \mathbf{B})$. However, it is a Rabin fairness equilibrium. An arbiter prescribing a pure DE would assign (s_3, t_3) to the players. Any outcome in the s_2 or s_3 rows of Table 3 is a maximin outcome. That includes the DE but not the JE.

4. Duality Relationships

We now summarize the duality relationships between the NE and DE that exist for two-person games. The propositions below follow immediately from the definitions.

Definition 7. The two-person game $(\mathbf{B}, \mathbf{A})$ with Player I as the row player and Player II as the column player is the dual of the game $(\mathbf{A}, \mathbf{B})$ also with Player I as the row player and Player II as the column player.

Proposition 8. *The dual game of the dual game of* $(\mathbf{A}, \mathbf{B})$ *is* $(\mathbf{A}, \mathbf{B})$.

Definition 9. For the bimatrix $(\mathbf{A}, \mathbf{B})$, define its swap matrix as $(\mathbf{A}, \mathbf{B})^S = (\mathbf{B}, \mathbf{A})$. Denote the swap matrices of $\mathbf{R}(\mathbf{A}, \mathbf{B})$ and $\mathbf{D}(\mathbf{A}, \mathbf{B})$ by $\mathbf{R}(\mathbf{A}, \mathbf{B})^S$ and $\mathbf{D}(\mathbf{A}, \mathbf{B})^S$, respectively.

Proposition 10. $\mathbf{D}(\mathbf{A}, \mathbf{B}) = \mathbf{R}(\mathbf{B}, \mathbf{A})^S$ *and* $\mathbf{R}(\mathbf{A}, \mathbf{B}) = \mathbf{D}(\mathbf{B}, \mathbf{A})^S$. *Hence, the set of DEs for* $(\mathbf{A}, \mathbf{B})$ *is the set of NEs for* $(\mathbf{B}, \mathbf{A})$, *and the set of NEs for* $(\mathbf{A}, \mathbf{B})$ *is the set of DEs for* $(\mathbf{B}, \mathbf{A})$.

In the dual game of Definition 7 the players simply play for each other. Proposition 10 implies that any computational approaches and existence properties for two-player NEs are also valid for two-player DEs. In particular, a computational method for finding an NE for $(\mathbf{B}, \mathbf{A})$ can therefore be used to find a DE for $(\mathbf{A}, \mathbf{B})$. Moreover, a DE exists for $(\mathbf{A}, \mathbf{B})$ since an NE exists for $(\mathbf{B}, \mathbf{A})$ [2]. Games with more than two players, however, do not exhibit such duality.

5. Zero-Sum Games

To find DEs for the zero-sum game $(\mathbf{A}, -\mathbf{A})$ we need only consider the $\mathbf{A}$ matrix for Player I as in the case for zero-sum NEs. Proposition 11 states the standard NE version of the minimax theorem [13] for zero-sum games in (3) below for comparison with the DE version stated in (4). The proof of (4) follows immediately from (3) and Proposition 10.

Proposition 11. *Consider the zero-sum game* $(\mathbf{A}, -\mathbf{A})$. *Then there exists a value* v *such that for any NE* $(\mathbf{x}^*, \mathbf{y}^*)$

$$\max_{x\in X} \min_{y\in Y} \mathbf{x}^T \mathbf{A}\mathbf{y} = \mathbf{x}^{*T} \mathbf{A}\mathbf{y}^* = \min_{y\in Y} \max_{x\in X} \mathbf{x}^T \mathbf{A}\mathbf{y} = v. \quad (3)$$

In addition, there exists a value w *such that, for any DE* (x^{**}, y^{**}),

$$\min_{x\in X} \max_{y\in Y} \mathbf{x}^T \mathbf{A}\mathbf{y} = \mathbf{x}^{**T} \mathbf{A}\mathbf{y}^{**} = \max_{y\in Y} \min_{x\in X} \mathbf{x}^T \mathbf{A}\mathbf{y} = w. \quad (4)$$

Proposition 11 asserts that a pure DE is obtained for zero-sum games when the minimax value for row Player I equals the maximin value for column Player II. This situation is exactly the opposite of the standard approach for finding pure zero-sum NEs, which are also called saddle points. It should be noted that the value v in (3) may be larger, smaller, or equal to the value w in (4). In addition, it follows from Proposition 10 that the linear programs for finding mixed DE

Table 4: Zero-sum game.

		Player II			
		t_1	t_2	t_3	max
Player I	s_1	5	2	**6**	6
	s_2	3	6	7	7
	min	3	2	6	

strategies **x** and **y** for the zero-sum game $(\mathbf{A}, -\mathbf{A})$ are identical to the linear programs [2] for finding mixed NE strategies **x** and **y**, respectively, for the dual game $(-\mathbf{A}, \mathbf{A})$. In other words, the a_{ij} in the NE linear programs are replaced by $-a_{ij}$.

Example 12. Consider the zero-sum matrix game with **A** as in Table 4. There is no pure NE. The single mixed NE is $\mathbf{x} = (0.5, 0.5)^T$ and $\mathbf{y} = (0.67, 0.33, 0)^T$ with an expected payoff of 4 for Player I. On the other hand, the single DE occurs at (s_1, t_3) from the discussion following Proposition 11. At (s_1, t_3) the minimax payoff for Player I is 6, and the maximin payoff for Player II is therefore −6. In this example, the pure DE does not seem as good for Player II as Player I. The arbiter might well assign some outcome different from the DE.

6. Conclusions

In this paper we defined a two-person, one-shot prescriptive game, as well as a cooperative dual to the Nash equilibrium. Prescriptive games allow other factors than the players themselves to influence outcomes and also let nonselfish behavior be regarded as rational. In particular, the DE sometimes gives the players better payoffs than the NE and may thus be a better choice for an arbiter α assigning pure strategies to the players. Unfortunately there may be either no pure DE or none satisfactory to the arbiter. Future research should address these issues. One possibility is a scalar equilibrium as in [14] that gives a reasonable outcome in pure strategies. In addition, the DE should be studied for n-person games.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] R. Aumann, "What is game theory trying to accomplish?" in *Frontiers of Economics*, K. Arrow and S. Honkapohja, Eds., pp. 5–46, Basil Blackwell, Oxford, UK, 1985.

[2] M. Maschler, E. Solan, and S. Zamir, *Game Theory*, Cambridge University Press, Cambridge, UK, 2013.

[3] R. Myerson, *Game Theory: Analysis of Conflict*, Harvard University Press, Cambridge, Mass, USA, 1991.

[4] J. F. Nash, "Equilibrium points in N-person games," *Proceedings of the National Academy of Sciences*, vol. 36, no. 1, pp. 48–49, 1950.

[5] J. F. Nash, "Non-cooperative games," *The Annals of Mathematics*, vol. 54, no. 2, pp. 286–295, 1951.

[6] W. Poundstone, *Prisoner's Dilemma*, Random House, New York, NY, USA, 2011.

[7] R. Kümmerli, C. Colliard, N. Fiechter, B. Petitpierre, F. Russier, and L. Keller, "Human cooperation in social dilemmas: comparing the Snowdrift game with the Prisoner's Dilemma," *Proceedings of the Royal Society B*, vol. 274, no. 1628, pp. 2965–2970, 2007.

[8] M. A. Nowak, K. M. Page, and K. Sigmund, "Fairness versus reason in the ultimatum game," *Science*, vol. 289, no. 5485, pp. 1773–1775, 2000.

[9] M. Beckenkamp, "A game-theoretic taxonomy of social dilemmas," *Central European Journal of Operations Research*, vol. 14, no. 3, pp. 337–353, 2006.

[10] A. Rubinstein, "Comments on the interpretation of game theory," *Econometrica*, vol. 59, no. 4, pp. 909–924, 1991.

[11] M. Rabin, "Incorporating fairness into game theory and economics," *The American Economic Review*, vol. 83, no. 5, pp. 1281–1302, 1993.

[12] M. Ehrgott, *Multicriteria Optimization*, Springer, New York, NY, USA, 2nd edition, 2005.

[13] J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, NJ, USA, 1994.

[14] N. Engsuwan, *Scalar equilibria for n-person games [Ph.D. thesis]*, University of Texas at Arlington, Arlington, TX, USA, 2013.