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Research Article

Analysis of an M/M/c Queueing System with Impatient Customers and Synchronous Vacations

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We consider an M/M/c queueing system with impatient customers and a synchronous vacation policy, where customer impatience is due to the servers' vacation. Whenever a system becomes empty, all the servers take a vacation. If the system is still empty, when the vacation ends, all the servers take another vacation; otherwise, they return to serve the queue. We develop the balance equations for the steady-state probabilities and solve the equations by using the probability generating function method. We obtain explicit expressions of some important performance measures by means of the two indexes. Based on these, we obtain some results about limiting behavior for some performance measures. We derive closed-form expressions of some important performance measures for two special cases. Finally, some numerical results are also presented.

1. Introduction

Queueing systems with vacations have been developed for wide range of applications in flexible manufacturing and computer communication systems over more than two decades. Literatures relating to this topic can be found in several excellent surveys by Doshi [1], Takagi [2], and a monograph by Tian and Zhang [3]. There is now growing interest in the analysis of queueing systems with impatient customers. This is due to their potential applications in communication systems, call centers, production-inventory systems, and many other related areas; see for instance [4–6] and the references therein.

The M/M/c queue with exponentially distributed vacation times was first studied by Levy and Yechiali [7], where each server takes vacations independently when the server finishes a service and finds no customers waiting in the queue. In the literature, this type of vacation policy is called asynchronous vacation policy. Vinod [8] studied this model by using QBD process and obtained the matrix-geometric solution. Chao and Zhao [9] investigated the multiserver vacation models of both synchronous (servers taking the same vacation together) and asynchronous types and provided some algorithms for computing the stationary probability distributions and expected performance measures. Tian et al. [10] and Zhang and Tian [11] have established the conditional stochastic decomposition properties on queue length and waiting time and provided the stationary distributions for the queue length and the waiting time in the Markovian multiserver synchronous vacation queueing systems.

Recently, Altman and Yechiali [12] presented a comprehensive analysis for M/M/c queueing models with server vacations and customer impatience, where customers became impatient only when the servers were on vacation. They considered asynchronous vacation policy for both the single and the multiple vacation cases by using the probability generating function method. However, they did not obtain the detailed results for the stationary probabilities and some expected performance measures such as the mean queue length and expected waiting time. For other works on the vacation queueing models with impatient customers, we refer to Altman and Yechiali [13], Yue et al. [14], Perel and Yechiali [15], and Economou and Kapodistria [16].

To the best of our knowledge, there is no work on multiserver synchronous vacation queueing models with impatient customers. In this paper, we consider an M/M/c queueing system with impatient customers and synchronous vacations. When a server finishes serving a customer and finds the system empty, all servers immediately leave for a vacation. If all servers return from a vacation to find an empty queue, they immediately leave for another vacation; otherwise, all servers return to serve the queue. This type of vacation is called a multiple synchronous vacation policy. Customers became impatient only when the servers were on vacation. That is, an arriving customer who finds that all servers are on vacation activates an “impatience timer.” If the customer’s service has not been completed before the customer’s impatience timer expires, the customer abandons the queue and never returns.

The model considered in this paper has potential applications in practical systems. For example, consider a production-inventory system with impatient customers, where a single product is produced at a multipurpose facility. The major job of the facility is to produce products to fulfill customer orders. In addition, the facility may perform other optional jobs utilizing the time between subsequent productions. The production facility can produce ahead of the demand in a make-to-stock fashion. However, the system manager does not want to keep higher level of inventory of items because more items in inventory result in the increasing of the holding costs. Therefore, whenever the last order is completed and no order occurs, the manager may decide to stop the major production and perform optional jobs for a period of time. Upon completion of each optional job the manager checks the orders and decides whether or not to restart major production. If no orders occur at this moment, a decision is made to perform other optional jobs next. The optional jobs can be referred to as a sequence of finite maintenance policy (such as inspection, replacement, or preventive maintenance for machines in the facility) or secondary jobs. Upon arrival, an order is either fulfilled from the inventory if any is available or back-ordered. Customers whose orders are back-ordered may become impatient and decide to cancel their orders if the customers’ waiting time exceeds a customer’s level of patience. This is especially likely when the facility performs optional jobs. This system can be modeled by our model developed in this paper. For other similar examples in communication networks, we refer to Ke [17] and Ke and Chang [18].

The rest of paper is organized as follows. In Section 2, we describe the model. In Section 3, we first develop the differential equations for the probability generating functions of the steady-state probabilities. Then, we obtain the explicit expressions for some performance measures in terms of two indexes. Based on these, we obtain some results about limiting behavior for some performance measures. The closed-form expressions of some performance measures are obtained for two special cases. In Section 4, we present some numerical results. Conclusions are given in Section 5.

2. Model Description

We consider an M/M/c queueing system with impatient customers and a synchronous vacation policy. Customers arrive according to a Poisson process at rate λ. The service is provided by c servers, who serve the customers on a first-come first-served (FCFS) basis. The service time of each customer is exponentially distributed with mean $1/\mu$.

The multiple synchronous vacation policy is described as follows. When the server finishes serving a customer and finds the system empty, all servers immediately leave for a vacation. If servers return from a vacation to find an empty queue, they immediately leave for another vacation; otherwise, they return to serve the queue. The duration of a vacation is exponentially distributed with mean $1/\gamma$.

During the vacation, customers are impatient. That is, an arriving customer who finds that all servers are on vacation activates an “impatience timer” T, which is exponentially distributed with mean $1/\xi$. If the customer’s service has not been completed before the customer’s timer expires, the customer abandons the queue and never returns.

3. Stationary Analysis

In this section, we present a stationary analysis for the model described in the last section.

3.1. Balance Equations. Let $L(t)$ denote the number of customers in the system at time t and let $J(t)$ denote the status of the server at time t, which is defined as follows: $J(t) = 0$ denotes that all c servers are taking a vacation at time t and $J(t) = 1$ denotes that some servers are busy serving customers at time t. Then, the process $\{(L(t), J(t)), t \geq 0\}$ defines a continuous-time Markov process with state space $\Omega = \{(0,0)\} \cup \{(n, j) : n \geq 1, j = 0, 1\}$.

Let

$$P_{nj} = \lim_{t\to\infty} P\{L(t) = n, J(t) = j\}, \quad (n, j) \in \Omega, \tag{1}$$

denote the steady-state probabilities of the process $\{(L(t), J(t)), t \geq 0\}$. Then, the set of balance equations is given as follows:

$$\lambda P_{00} = \mu P_{11} + \xi P_{10}, \tag{2}$$

$$(\lambda + n\xi + \gamma) P_{n0} = \lambda P_{(n-1)0} + (n+1)\xi P_{(n+1)0}, \quad n \geq 1, \tag{3}$$

$$(\lambda + \mu) P_{11} = 2\mu P_{21} + \gamma P_{10}, \tag{4}$$

$$(\lambda + n\mu) P_{n1} = \lambda P_{(n-1)1} + (n+1)\mu P_{(n+1)1} + \gamma P_{n0}, \quad 2 \leq n \leq c-1, \tag{5}$$

$$(\lambda + c\mu) P_{n1} = \lambda P_{(n-1)1} + c\mu P_{(n+1)1} + \gamma P_{n0}, \quad n \geq c, \tag{6}$$

and the normalizing condition is as follows:

$$\sum_{n=0}^{\infty} P_{n0} + \sum_{n=1}^{\infty} P_{n1} = 1. \tag{7}$$

3.2. Generating Functions. We define the probability generating functions as follows:

$$G_0(z)=\sum_{n=0}^{\infty}P_{n0}z^n,\qquad G_1(z)=\sum_{n=1}^{\infty}P_{n1}z^n,\quad 0\le z\le 1. \tag{8}$$

Then, multiplying (3) by z^n, summing all possible values of n, and using (2), we get

$$\xi(1-z)\frac{d}{dz}G_0(z)=[\lambda(1-z)+\gamma]G_0(z)-(\gamma P_{00}+\mu P_{11}). \tag{9}$$

Similarly, multiplying (5) and (6) by z^n, then summing all possible values of n, and using (4), we get

$$(1-z)(\lambda z-\mu)G_1(z)=\gamma zG_0(z)-(\gamma P_{00}+\mu P_{11})z+\mu(1-z)\sum_{n=1}^{c}(n-c)P_{n1}z^n. \tag{10}$$

The differential equation (9) is the same as (2.4) in the paper written by Altman and Yechiali [12], where they solve this differential equation and obtain $G_0(z)$ as follows:

$$G_0(z)=e^{(\lambda/\xi)z}(1-z)^{-\gamma/\xi}\times\left[G_0(0)-\frac{H}{\xi}\int_0^z(1-x)^{(\gamma/\xi)-1}e^{-(\lambda/\xi)x}dx\right], \tag{11}$$

where

$$H=\gamma P_{00}+\mu P_{11}. \tag{12}$$

See Altman and Yechiali [12, Equation (2.11)]. Then

$$G_0(1)=e^{(\lambda/\xi)z}\left[G_0(0)-\frac{H}{\xi}\int_0^1(1-x)^{(\gamma/\xi)-1}e^{-(\lambda/\xi)x}dx\right]\times\lim_{z\to1}(1-z)^{-\gamma/\xi}. \tag{13}$$

Since $G_0(1)=\sum_{n=0}^{\infty}P_{n0}>0$ and $\lim_{z\to1}(1-z)^{-\gamma/\xi}=\infty$, we must have that

$$P_{00}=G_0(0)=\frac{H}{\xi}K, \tag{14}$$

where

$$K=\int_0^1 e^{-(\lambda/\xi)x}(1-x)^{(\gamma/\xi)-1}dx. \tag{15}$$

Substituting (14) into (11) and noting that $G_0(0)=P_{00}$, we have

$$G_0(z)=\frac{e^{(\lambda/\xi)z}}{(1-z)^{\gamma/\xi}}\left[1-\frac{1}{K}\int_0^z(1-x)^{(\gamma/\xi)-1}e^{-(\lambda/\xi)x}dx\right]P_{00}. \tag{16}$$

Equation (10) can be written as follows:

$$G_1(z)=\frac{[\gamma G_0(z)-H]z}{(\lambda z-\mu)(1-z)}-\frac{\mu}{\lambda z-\mu}Q(z), \tag{17}$$

where $Q(z)=\sum_{n=1}^{c}(c-n)P_{n1}z^n$.

Equation (16) shows that $G_0(z)$ can be expressed in terms of P_{00}. Equation (17) shows that $G_1(z)$ can be expressed in terms of $G_0(z)$, H, and $Q(z)$. In other words, once P_{00} and P_{j1} ($j=1,2,\ldots,c-1$) are obtained, $G_0(z)$ and $G_1(z)$ are completely determined.

3.3. Performance Measures. In this subsection, we derive some performance measures of the system by using the expressions of the PGF we obtained in last section. Furthermore, we consider the limiting behavior for some performance measures.

3.3.1. Expected System Sizes. Let L_0 and L_1 be the system size when all servers are on vacation and not on vacation, respectively. Then, the expected system sizes $E[L_0]$ and $E[L_1]$ are defined by

$$E[L_j]=\sum_{n=1}^{\infty}nP_{nj},\quad j=0,1. \tag{18}$$

Let

$$G_0'(1)=\frac{d}{dz}G_0(z)\Big|_{z=1},\qquad G_0''(1)=\frac{d^2}{dz^2}G_0(z)\Big|_{z=1}. \tag{19}$$

Then, $G_0'(1)=E[L_0]$ and $G_0''(1)=E[L_0(L_0-1)]$.

First, we derive $E[L_0]$. Using L'Hôpital rule, we have from (16) that

$$G_0(1)=\frac{\xi}{\gamma K}P_{00}. \tag{20}$$

Substituting (14) into (20), we get

$$\gamma G_0(1)=H. \tag{21}$$

From (17), using L'Hôpital rule, we get

$$G_1(1)=\lim_{z\to1}G_1(z)=\frac{[\gamma G_0(1)-H]+\gamma G_0'(1)}{\mu-\lambda}+\frac{\mu}{\mu-\lambda}Q(1), \tag{22}$$

where $Q(1)=\sum_{j=1}^{c}(c-j)P_{j1}$, and substituting (21) into (22), we get

$$G_1(1)=\frac{\gamma}{\mu-\lambda}E[L_0]+\frac{\mu}{\mu-\lambda}Q(1). \tag{23}$$

From (9), we get

$$\begin{aligned} E[L_0] &= \lim_{z\to 1} G_0'(z) \\ &= \lim_{z\to 1} \frac{[\lambda(1-z)+\gamma]G_0(z)-H}{\xi(1-z)} \\ &= \frac{-\lambda G_0(1)+\gamma G_0'(1)}{-\xi} \\ &= \frac{-\lambda G_0(1)+\gamma E[L_0]}{-\xi}, \end{aligned} \tag{24}$$

where the third equality follows by using L'Hôpital rule. Thus, we have

$$G_0(1) = \frac{\gamma+\xi}{\lambda} E[L_0]. \tag{25}$$

Using (22) and (25) and noting that $G_0(1)+G_1(1)=1$, we obtain

$$E[L_0] = \frac{\lambda(1-\rho)}{\gamma+\xi(1-\rho)} - \frac{\lambda}{\gamma+\xi(1-\rho)} Q(1), \tag{26}$$

where $\rho = \lambda/\mu$.

Now, we derive $E[L_1]$. From (17), we have

$$\begin{aligned} E[L_1] &= \lim_{z\to 1} G_1'(z) \\ &= \frac{\gamma}{2(\mu-\lambda)^2}\lim_{z\to 1}(\mu-\lambda z)\left[zG_0''(z)+2G_0'(z)\right] \\ &\quad - \frac{\lambda}{(\mu-\lambda)^2}\lim_{z\to 1}\frac{z[\gamma G_0(z)-H]}{(1-z)} \\ &\quad + \mu\lim_{z\to 1}\frac{\left[(\mu-\lambda z)Q'(z)+\lambda Q(z)\right]}{(\mu-\lambda z)^2} \\ &= \frac{\gamma(\mu-\lambda)G_0''(1)+2\mu\gamma G_0'(1)}{2(\mu-\lambda)^2} \\ &\quad + \frac{\mu\left[(\mu-\lambda)Q'(1)+\lambda Q(1)\right]}{(\mu-\lambda)^2} \\ &= \frac{(\mu-\lambda)\gamma E[L_0(L_0-1)]+2\mu\gamma E[L_0]}{2(\mu-\lambda)^2} \\ &\quad + \frac{1}{1-\rho}Q'(1) + \frac{\rho}{(1-\rho)^2}Q(1), \end{aligned} \tag{27}$$

where

$$Q'(1) = \left.\frac{dQ(z)}{dz}\right|_{z=1} = \sum_{j=1}^{c-1} j(c-j)P_{j1}. \tag{28}$$

In order to obtain $G_0''(1)$, we take derivatives for two times on both sides of (9); then we get

$$\xi(1-z)G_0'''(z) + 2\lambda G_0'(z) = [\lambda(1-z)+\gamma+2\xi]G_0''(z), \tag{29}$$

where $G_0'''(z) = d^3/dz^3 G_0(z)$. Letting $z=1$ in (29), we get

$$G_0''(1) = \frac{2\lambda}{\gamma+2\xi}G_0'(1) \tag{30}$$

or, equivalently,

$$E[L_0(L_0-1)] = \frac{2\lambda}{\gamma+2\xi}E[L_0]. \tag{31}$$

Substituting (31) into (27), we get

$$\begin{aligned} E[L_1] &= \frac{\rho}{(1-\rho)}\left(\frac{\gamma}{\gamma+2\xi}+\frac{1}{\lambda(1-\rho)}\right)E[L_0] \\ &\quad + \frac{1}{1-\rho}Q'(1) + \frac{\rho}{(1-\rho)^2}Q(1). \end{aligned} \tag{32}$$

Let L be the number of customers in the system. Then, the expected system size can be calculated by $E[L] = E[L_0] + E[L_1]$, where $E[L_0]$ and $E[L_1]$ are given by (26) and (32).

Remark 1. If we take derivatives for $(k+1)$th times on both sides of (9) and use the same method as getting (31), then we get

$$\left.\frac{d^k G_0(z)}{dz^k}\right|_{z=1} = \frac{k\lambda}{\gamma+k\xi} \times \left.\frac{d^{k-1}G_0(z)}{dz^{k-1}}\right|_{z=1} \tag{33}$$

or, equivalently,

$$\begin{aligned} &E[L_0(L_0-1)\cdots(L_0-k)] \\ &\quad = \frac{k\lambda}{\gamma+k\xi}E[L_0(L_0-1)\cdots(L_0-k+1)]. \end{aligned} \tag{34}$$

Using this iteration, we get

$$E[L_0(L_0-1)\cdots(L_0-k)] = \prod_{i=2}^{k+1}\frac{i\lambda}{\gamma+i\xi}E[L_0], \tag{35}$$

for $k = 1, 2, \ldots$.

3.3.2. Other Performance Measures. Now, we derive some other performance measures such as the probability when servers are on vacation (or not on vacation), the proportion of customers served per unit of time, and the average rate of abandonment due to impatience.

Let $P_{.0} = \sum_{n=0}^{\infty} P_{n0}$ and $P_{.1} = \sum_{n=1}^{\infty} P_{n1}$. Clearly, $P_{.0} = G_0(1)$ is the probability when servers are on vacation and $P_{.1} = G_1(1) = 1 - P_{.0}$ is the probability when servers are not on vacation.

Substituting (26) into (23) and (25), we get

$$P_{.0} = \frac{(1-\rho)(\gamma+\xi)}{\gamma+\xi(1-\rho)} - \frac{\gamma+\xi}{\gamma+\xi(1-\rho)}Q(1), \quad (36)$$

$$P_{.1} = \frac{\rho\gamma}{\gamma+\xi(1-\rho)} + \frac{\gamma+\xi}{\gamma+\xi(1-\rho)}Q(1). \quad (37)$$

When the system is in state $(n, 1)$, the service rates of the servers are $n\mu$ for $n = 1, 2, \ldots, c$ and $c\mu$ for $n = c+1, c+2, \ldots$, respectively. Thus, the expected number of customers served per unit of time is given by

$$N_s = \sum_{n=1}^{c} \mu n P_{n1} + \sum_{n=c+1}^{\infty} c\mu P_{n1} = \mu\left[cP_{.1} - Q(1)\right], \quad (38)$$

implying that the proportion of customers served per unit of time is given by

$$P_s = \frac{N_{\text{served}}}{\lambda} = \frac{1}{\rho}\left[cP_{.1} - Q(1)\right], \quad (39)$$

where $P_{.1}$ is given by (37).

When the system is in state $(n, 0)$, $n \geq 1$, the rate of abandonment of a customer due to impatience is $n\xi$. Thus, the average rate of abandonment due to impatience is given by

$$R_a = \sum_{n=1}^{\infty} n\xi P_{n0} = \xi E\left[L_0\right], \quad (40)$$

where $E[L_0]$ is given by (26).

All the performance measures of the system we derived in this subsection are expressed in terms of $Q(1)$ or/and $Q'(1)$. In the next subsection, we consider the calculation of these two indexes $Q(1)$ and $Q'(1)$.

Remark 2. Let $P_I = \sum_{j=1}^{c-1} P_{j1}$ be the probability that there are idle servers when all the servers are not on vacation. Clearly, $Q(1)/P_I$ is the conditional expected number of idle servers provided that there are idle servers when all servers are not on vacation and $Q'(1)/P_I$ is the conditional expectation of the product of the number of the busy servers and the number of the idle servers provided that there are idle servers when all servers are not on vacation.

3.3.3. Limiting Behavior. We consider the limiting behavior for some performance measures when $\rho \to 1$.

Since $P_{.0} \geq 0$, from (36), it is easy to see that $0 \leq Q(1) \leq 1 - \rho$, which implies that

$$\lim_{\rho\to 1} Q(1) = 0. \quad (41)$$

Since $Q(1) = \sum_{j=1}^{c}(c-j)P_{j1}$, it is easy to see that $\lim_{\rho\to 1} P_{j1} = 0$ for $j = 1, 2, \ldots, c-1$. This implies that

$$\lim_{\rho\to 1} Q'(1) = \lim_{\rho\to 1} \sum_{j=1}^{c} j(c-j)P_{j1} = 0. \quad (42)$$

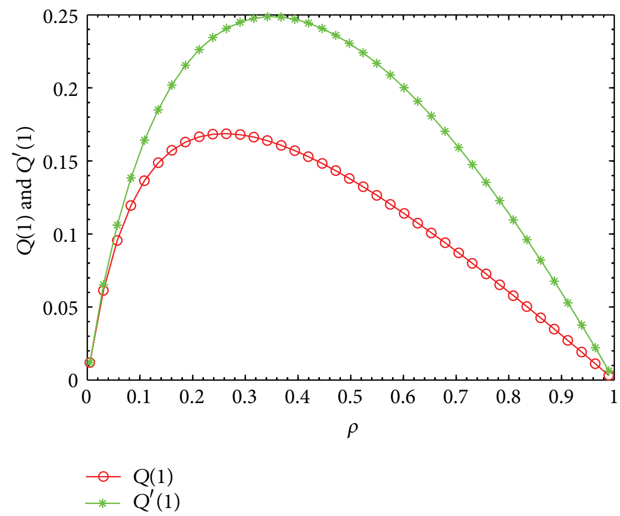

FIGURE 1: Effects of ρ on $Q(1)$ and $Q'(1)$.

The two curves of $Q(1)$ and $Q'(1)$ with respect to ρ are shown in Figure 1, where the parameters are given as follows: $\xi = 1$, $\gamma = 0.5$, $\mu = 2$, and $c = 5$. From Figure 1, it is observed that both $Q(1)$ and $Q'(1)$ go to zero when $\rho \to 1$. This observation coincides with the results given by (41) and (42). Furthermore, Figure 1 shows that the two curves are very close when ρ is very small.

Noting (41), we get from (36) and (37) that

$$\lim_{\rho\to 1} P_{.0} = 0, \qquad \lim_{\rho\to 1} P_{.1} = 1. \quad (43)$$

Further, we get from (39) that

$$\lim_{\rho\to 1} P_s = c. \quad (44)$$

Remark 3. Noting that $\lim_{\rho\to 1} P_{j1} = 0$ for $j = 1, 2, \ldots, c-1$ and using (43), we get $\lim_{\rho\to 1} \sum_{j=c}^{\infty} P_{j1} = 1$. This implies that if $\rho \to 1$, then all servers in the system will be busy. This explains the result given by (44).

In Figures 2 and 3, we investigate the effect of ρ on P_0 and P_s, where the parameters are given as follows: $\xi = 0.3$, $\gamma = 0.5$, and $\mu = 2$.

From Figure 2, it is observed that $P_{.0}$ is a decreasing function of ρ and it has its limit at 0 when $\rho \to 1$ for all three values of c. This agrees with (43). The three curves show that $P_{.0}$ has its limit at 1 when $\rho \to 0$. This agrees with the intuitive expectation. In addition, we observe that $P_{.0}$ is a concave function of ρ for $c = 1$, but it is not a concave function of ρ for other values of $c = 5$ and $c = 10$.

From Figure 3, it is observed that P_s is an increasing function of ρ for $c = 1$ which agrees with the analytical result given by Altman and Yechiali [12, page 270, Equation (3.32)], but it is not an increasing function of ρ for other values of $c = 5$ and $c = 10$. All the three curves show that P_s has its limit at c when $\rho \to 1$. This agrees with (44).

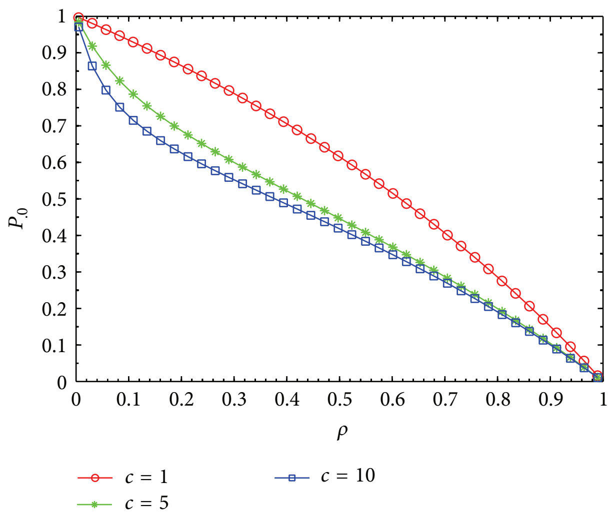

FIGURE 2: Effect of ρ on $P_{.0}$.

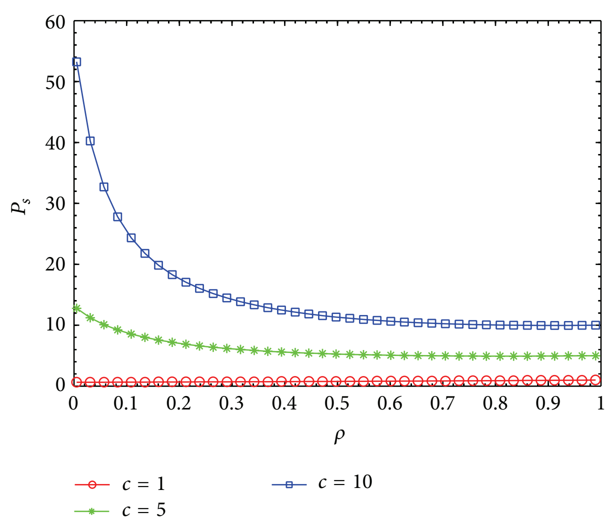

FIGURE 3: Effect of ρ on P_s.

Noting (41), we have from (26) that

$$\lim_{\rho \to 1} \frac{1}{\lambda} E[L_0] = \lim_{\rho \to 1} \frac{1}{\mu} E[L_0] = 0. \tag{45}$$

Further, we have from (32) that

$$\lim_{\rho \to 1} E[L_1] = \lim_{\rho \to 1} \frac{\gamma}{\mu(1-\rho)} \left(\frac{\lambda}{\gamma + 2\xi} + \frac{1}{1-\rho} \right) E[L_0]. \tag{46}$$

Thus, when $\rho \to 1$ we have an approximation for $E[L_1]$ as follows:

$$E[L_1] \cong \frac{\gamma}{\mu(1-\rho)} \left(\frac{\lambda}{\gamma + 2\xi} + \frac{1}{1-\rho} \right) E[L_0]. \tag{47}$$

Remark 4. The right hand side of (47) is the same as the corresponding expected system size for the single server model (see Altman and Yechiali [12, page 264]).

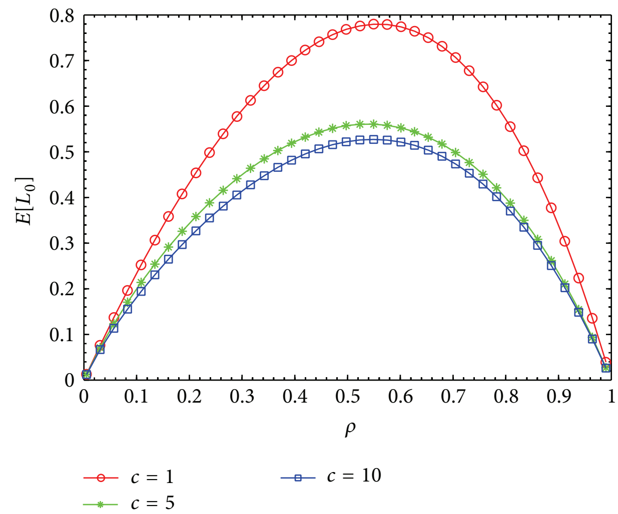

FIGURE 4: Effect of ρ on $E[L_0]$.

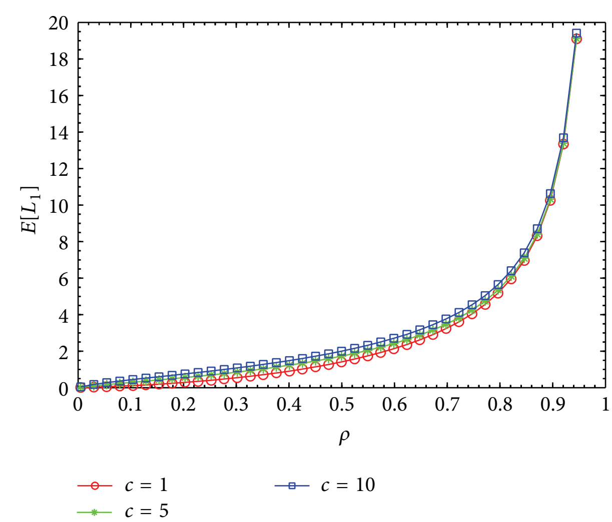

FIGURE 5: Effect of ρ on $E[L_1]$.

In Figures 4 and 5, we investigate the effect of ρ on $E[L_0]$ and $E[L_1]$, where the parameters are given as follows: $\xi = 0.3$, $\gamma = 0.5$, and $\mu = 2$.

From Figure 4, it is observed that $E[L_0]$ is a concave function of ρ for all three values of c. It is insensitive to the variables of c when ρ is very small or ρ is close to 1. From Figure 5, it is observed that $E[L_1]$ is an increasing convex function of ρ. Figure 5 shows that $E[L_1]$ increases slowly when ρ is less than 0.9, but it increases significantly when ρ is larger than 0.9. In addition, we observe that all three curves almost coincide when $\rho \to 1$. This agrees with (47) (see Remark 4).

3.4. Calculation of $Q(1)$ and $Q'(1)$. In order to compute $Q(1)$ and $Q'(1)$, we need to compute P_{j1} for $j = 1, 2, \ldots, c-1$. From (2), (3), (4), and (5), the $2c - 1$ unknown probabilities P_{j1} for

$j = 1, 2, \ldots, c-1$ and P_{j0} for $j = 0, 1, \ldots, c-1$ satisfy the following $2c-3$ linear equations:

$$\lambda P_{00} = \mu P_{11} + \xi P_{10}, \tag{48}$$

$$\begin{aligned}(\lambda + n\xi + \gamma) P_{n0} &= \lambda P_{(n-1)0} + (n+1)\xi P_{(n+1)0}, \\ &\qquad 1 \le n \le c-2,\end{aligned} \tag{49}$$

$$(\lambda + \mu) P_{11} = 2\mu P_{21} + \gamma P_{10}, \tag{50}$$

$$\begin{aligned}(\lambda + n\mu) P_{n1} &= \lambda P_{(n-1)1} + (n+1)\mu P_{(n+1)1} + \gamma P_{n0}, \\ &\qquad 2 \le n \le c-2.\end{aligned} \tag{51}$$

Therefore, we need another two independent equations to calculate all $2c-1$ unknowns.

We show that P_{11} can be expressed by P_{00}. Equation (14) can be written as

$$G_0(0) = P_{00} = \frac{\gamma P_{00} + \mu P_{11}}{\xi} K \tag{52}$$

which yields that

$$P_{11} = \delta P_{00}, \tag{53}$$

where

$$\delta = \frac{\xi - \gamma K}{K\mu}. \tag{54}$$

Remark 5. It is easy to confirm that $\xi - \gamma K > 0$ (see Altman and Yechiali [12, page 263]).

Substituting (36) into (21), we get

$$P_{00} + \frac{\mu}{\gamma} P_{11} = \frac{(\mu - \lambda)(\gamma + \xi)}{\mu\gamma + \xi(\mu - \lambda)} - \frac{\mu(\gamma + \xi)}{\mu\gamma + \xi(\mu - \lambda)} Q(1). \tag{55}$$

Equations (53) and (55) give another two independent equations. All in all, we have $2c-1$ independent equations to solve for the $2c-1$ unknowns. In the following, we solve these equations analytically.

Substituting (53) into (48) and (50), we have

$$(\lambda - \mu\delta) P_{00} = \xi P_{10}, \tag{56}$$

$$(\lambda + \mu)\delta P_{00} = 2\mu P_{21} + \gamma P_{10}. \tag{57}$$

Thus, P_{j0}, $j = 1, 2, \ldots, c-1$, and P_{j1}, $j = 2, 3, \ldots, c-1$, satisfy (49), (51), (56), and (57). These equations can be written as equations in matrix form.

For this, we define the following column vectors:

$$P_0 = (P_{10}, P_{20}, \ldots, P_{(c-1)0})^T, \qquad P_1 = (P_{21}, P_{31}, \ldots, P_{(c-1)1})^T, \tag{58}$$

Then, we have

$$\begin{aligned} AP_0 &= DP_{00}, \\ P_0 + CP_1 &= EP_{00}, \end{aligned} \tag{59}$$

where A, B, and C are matrices given as follows:

$$A = \begin{pmatrix} \xi & 0 & 0 & \cdots & 0 & 0 & 0 \\ -a_1 & 2\xi & 0 & \cdots & 0 & 0 & 0 \\ \lambda & -a_2 & 3\xi & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & -a_{c-2} & (c-1)\xi \end{pmatrix},$$

$$B = \begin{pmatrix} \gamma & 0 & \cdots & 0 & 0 \\ 0 & \gamma & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \gamma & 0 \end{pmatrix}, \tag{60}$$

$$C = \begin{pmatrix} 2\mu & 0 & 0 & \cdots & 0 & 0 & 0 \\ -b_2 & 3\mu & 0 & \cdots & 0 & 0 & 0 \\ \lambda & -b_3 & 4\mu & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & -b_{c-2} & (c-1)\mu \end{pmatrix},$$

where $a_n = \lambda + n\xi + \gamma$ and $b_n = \lambda + n\mu$ for $n = 1, 2, \ldots, c-2$ and D and E are two column vectors given as follows:

$$D = (\lambda - \mu\delta, -\lambda, 0, \ldots, 0)^T, \qquad E = ((\lambda + \mu)\delta, -\lambda\delta.0, \ldots, 0)^T. \tag{61}$$

Clearly, matrices A and C are inverse matrices. Thus, from (59), we have

$$\begin{aligned} P_0 &= A^{-1} D P_{00}, \\ P_1 &= C^{-1}\left(E - BA^{-1}D\right) P_{00}, \end{aligned} \tag{62}$$

where A^{-1} and C^{-1} are inverses of matrices A and C, respectively.

Let e_0 be a vector with $c-1$ elements all to be one and let e_1 be a column vector with $c-2$ elements all to be one. Using (53) and (62), $Q(1)$ can be written by

$$Q(1) = (c-1)\delta P_{00} + FC^{-1}\left(E - BA^{-1}D\right) P_{00}, \tag{63}$$

where

$$F = (c-2, c-3, \ldots, 1) \tag{64}$$

is a vector. Submitting (53) and (63) into (55), we can obtain P_{00}.

The matrices A^{-1} and C^{-1} can be computed iteratively. Let x_{ij} denote the elements of matrix A^{-1} and let y_{ij} denote the elements of matrix C^{-1}. Then, we have

$$\begin{aligned}
x_{ij} &= 0, \quad i < j, \; j = 2,3,\ldots,c-1,\\
x_{jj} &= \frac{1}{j\xi}, \quad j = 1,2,\ldots,c-1,\\
x_{ij} &= \frac{1}{i\xi}\left(a_{i-1}x_{(i-1)j} - \lambda x_{(i-2)j}\right), \quad i > j, \; j = 1,2,\ldots,c-1.
\end{aligned} \tag{65}$$

See the appendix for the proof of (65).

Since the matrix C has the same structure as the matrix A, using (65), it is easy to get the elements of C^{-1} as follows:

$$\begin{aligned}
y_{ij} &= 0, \quad i < j, \; j = 2,3,\ldots,c-2,\\
y_{jj} &= \frac{1}{(j+1)\mu}, \quad j = 1,2,\ldots,c-2,\\
y_{ij} &= \frac{1}{(i+1)\mu}\left(b_i y_{(i-1)j} - \lambda y_{(i-2)j}\right), \quad i > j, \; j = 1,2,\ldots,c-2.
\end{aligned} \tag{66}$$

Using (65) and (66), it is easy to get from (62) that

$$P_{10} = (\lambda - \mu\delta)\, x_{11} P_{00}, \tag{67}$$

$$P_{j0} = \left[(\lambda - \mu\delta)\, x_{j1} - \lambda x_{j2}\right] P_{00}, \quad j = 2,3,\ldots,c-1, \tag{68}$$

$$P_{(j+1)1} = \left(b_1 y_{j1} - \lambda y_{j2}\right)\delta P_{00} - \gamma\sum_{k=1}^{j} y_{jk} P_{k0}, \quad j = 1,2,\ldots,c-2. \tag{69}$$

Define

$$\begin{aligned}
\phi_0 &= c - 1 + \sum_{j=1}^{c-2}(c-j-1)\left(b_1 y_{j1} + \lambda y_{j2}\right),\\
\phi_k &= \sum_{j=k}^{c-2}(c-j-1)\, y_{jk}, \quad k = 1,2,\ldots,c-2.
\end{aligned} \tag{70}$$

Then, using (69), we have

$$Q(1) = \sum_{j=1}^{c-1}(c-j)\, P_{j1} = \Delta(\phi)\, P_{00}, \tag{71}$$

where

$$\Delta(\phi) = \delta\phi_0 - \gamma\sum_{k=1}^{c-2}\phi_k\left[(\lambda - \mu\delta)\, x_{k1} - \lambda x_{k2}\right]. \tag{72}$$

Substituting (53) and (71) into (55), we get

$$P_{00} = \frac{K\gamma(\gamma+\xi)(\mu-\lambda)}{\xi\left[\mu\gamma + \xi(\mu-\lambda)\right] + K\gamma\mu(\mu+\xi)\Delta(\phi)}, \tag{73}$$

where K is defined by (15).

Define

$$\begin{aligned}
\psi_0 &= c - 1 + \sum_{j=1}^{c-2}(j+1)(c-j-1)\left(b_1 y_{j1} + \lambda y_{j2}\right),\\
\psi_k &= \sum_{j=k}^{c-2}(j+1)(c-j-1)\, y_{jk} \quad , k = 1,2,\ldots,c-2.
\end{aligned} \tag{74}$$

Then, using (69), we have

$$Q'(1) = \sum_{j=1}^{c-1} j(c-j)\, P_{j1} = \Delta(\psi)\, P_{00}, \tag{75}$$

where

$$\Delta(\psi) = \delta\psi_0 - \gamma\sum_{k=1}^{c-2}\psi_k\left[(\lambda - \mu\delta)\, x_{k1} - \lambda x_{k2}\right]. \tag{76}$$

3.5. Special Cases. In this subsection, we consider two special cases: $c = 1$ and $c = 2$. The performance measures of the system for these two cases are distinguished by using the superscripts M/M/1 and M/M/2, respectively.

Case 1 (single server model). If $c = 1$, then $Q(1) = 0$ and $Q'(1) = 0$. Thus, from (36) and (37), we have

$$P_{.0}^{\mathrm{M/M/1}} = \frac{(1-\rho)(\gamma+\xi)}{\gamma + \xi(1-\rho)}, \qquad P_{.1}^{\mathrm{M/M/1}} = \frac{\rho\gamma}{\gamma + \xi(1-\rho)}. \tag{77}$$

These results agree with the results given by Altman and Yechiali [12, Equation (2.17), page 264]. From (26) and (27), we get

$$E[L_0]^{\mathrm{M/M/1}} = \frac{\lambda(1-\rho)}{\gamma + \xi(1-\rho)}, \tag{78}$$

$$\begin{aligned}
E[L_1]^{\mathrm{M/M/1}} = {} & \Big((\mu-\lambda)\gamma E[L_0(L_0-1)]^{\mathrm{M/M/1}}\\
& + 2\mu\gamma E[L_0]^{\mathrm{M/M/1}}\Big) \times \left(2(\mu-\lambda)^2\right)^{-1}.
\end{aligned} \tag{79}$$

These results agree with the results given by Altman and Yechiali [12, pages 264-265].

From (39), we get

$$P_s^{\mathrm{M/M/1}} = \frac{\gamma}{\gamma + \xi(1-\rho)}. \tag{80}$$

This agrees with the result given by Altman and Yechiali [12, Equation (3.32), page 270].

From (31) and (40), we get

$$E[L_0(L_0-1)]^{\mathrm{M/M/1}} = \frac{2\lambda}{\gamma+2\xi} E[L_0]^{\mathrm{M/M/1}}, \qquad R_a^{\mathrm{M/M/1}} = \xi E[L_0]^{\mathrm{M/M/1}}, \tag{81}$$

where $E[L_0]^{\mathrm{M/M/1}}$ is given by (78).

Case 2 (two-server model). If $c = 2$, then $Q(1) = Q'(1) = P_{11}$. Using (53) and substituting $Q(1) = P_{11}$ into (55), we obtain

$$P_{00}^{\mathrm{M/M/2}} = \frac{(\gamma+\xi)(1-\rho)}{\left(1+\frac{\mu}{\gamma}\delta\right)[\gamma+\xi(1-\rho)]+\delta(\gamma+\xi)}. \tag{82}$$

Thus, from (36) and (37), we get

$$P_{.0}^{\mathrm{M/M/2}} = \frac{(1-\rho)(\gamma+\xi)}{\gamma+\xi(1-\rho)} - \frac{\delta(\gamma+\xi)}{\gamma+\xi(1-\rho)} P_{00}^{\mathrm{M/M/2}}, \tag{83}$$

$$P_{.1}^{\mathrm{M/M/2}} = \frac{\rho\gamma}{\gamma+\xi(1-\rho)} + \frac{\delta(\gamma+\xi)}{\gamma+\xi(1-\rho)} P_{00}^{\mathrm{M/M/2}}, \tag{84}$$

where $P_{00}^{\mathrm{M/M/2}}$ is given by (82).

From (26) and (32), we get

$$E[L_0]^{\mathrm{M/M/2}} = \frac{\lambda(1-\rho)}{\gamma+\xi(1-\rho)} - \frac{\lambda\delta}{\gamma+\xi(1-\rho)} P_{00}^{\mathrm{M/M/2}}, \tag{85}$$

$$E[L_1]^{\mathrm{M/M/2}} = \frac{\rho}{(1-\rho)}\left(\frac{\gamma}{\gamma+2\xi}+\frac{1}{\lambda(1-\rho)}\right)E[L_0]^{\mathrm{M/M/2}} + \frac{\delta}{(1-\rho)^2} P_{00}^{\mathrm{M/M/2}}, \tag{86}$$

where $P_{00}^{\mathrm{M/M/2}}$ is given by (82) and $E[L_0]^{\mathrm{M/M/2}}$ is given by (85).

Substituting (84) and $Q(1) = P_{11} = \delta P_{00}$ into (39), we get

$$P_s^{\mathrm{M/M/2}} = \frac{2\gamma}{\gamma+\xi(1-\rho)} + \frac{\delta[\gamma+\xi(1+\rho)]}{\rho[\gamma+\xi(1-\rho)]} P_{00}^{\mathrm{M/M/2}}, \tag{87}$$

where $P_{00}^{\mathrm{M/M/2}}$ is given by (82).

From (31) and (40), we get

$$E[L_0(L_0-1)]^{\mathrm{M/M/2}} = \frac{2\lambda}{\gamma+2\xi} E[L_0]^{\mathrm{M/M/2}}, \qquad R_a = \xi E[L_0]^{\mathrm{M/M/2}}, \tag{88}$$

where $E[L_0]^{\mathrm{M/M/2}}$ is given by (85).

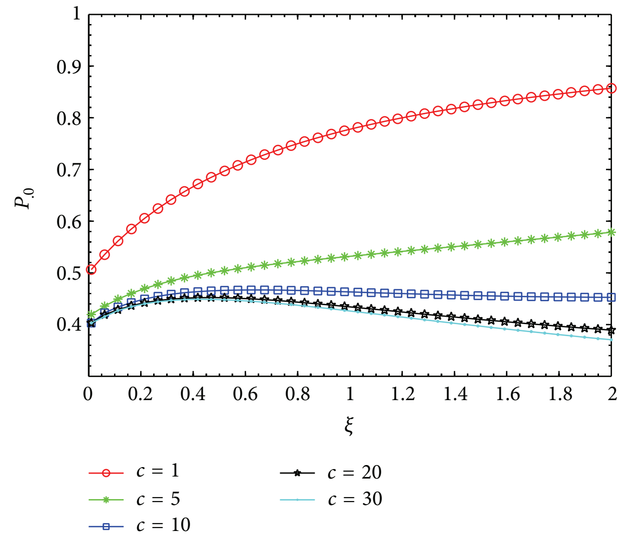

Figure 6: Effect of ξ on $P_{.0}$.

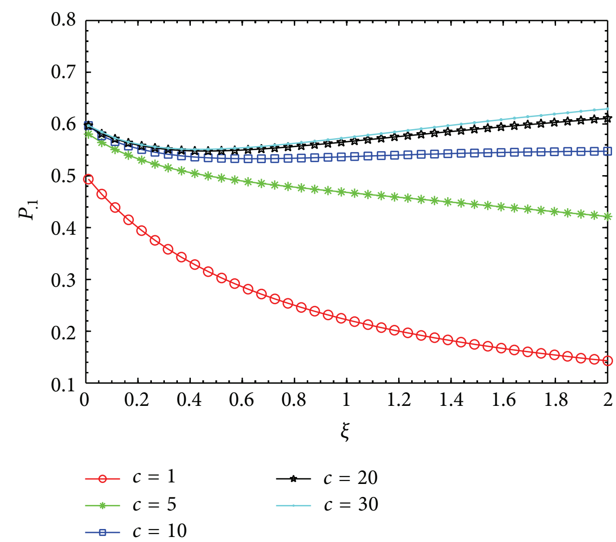

Figure 7: Effect of ξ on $P_{.1}$.

4. Numerical Results

In this section, we investigate numerically the effects of the parameter ξ and the number c of servers on some performance measures.

For M/M/1 queueing model with impatient customers and multiple vacation policy, Altman and Yechiali [12] show that the probability $P_{.0}[P_{.1}]$ is an increasing (decreasing) concave (convex) function of ξ, having its limits at $1-(\lambda/\mu)$ [at (λ/μ)]. Also, they show that $E[L_0]$ behaves similar to $P_{.1}$. In Figures 6 and 7, we consider whether the above property holds for our multiserver model, where the parameters are given as follows: $\lambda = 1$, $\mu = 2$, and $\gamma = 0.4$.

From Figures 6 and 7, it is observed that $P_{.0}$ is an increasing function of ξ for $c = 1$ and $P_{.1}$ is a decreasing function of ξ for $c = 1$. However, this property may not hold

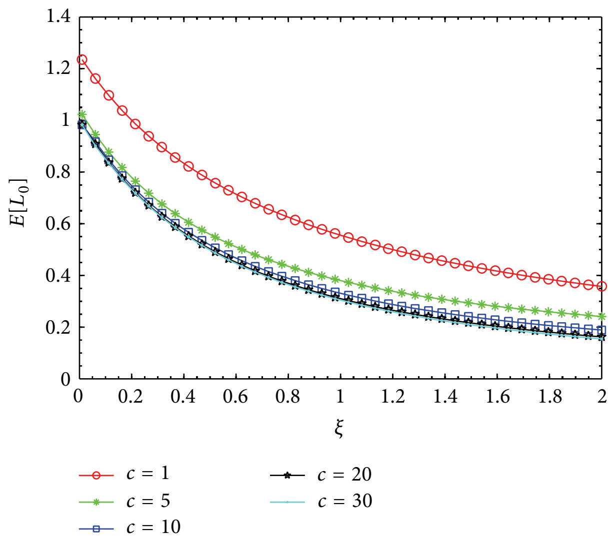

Figure 8: Effect of ξ on $E[L_0]$.

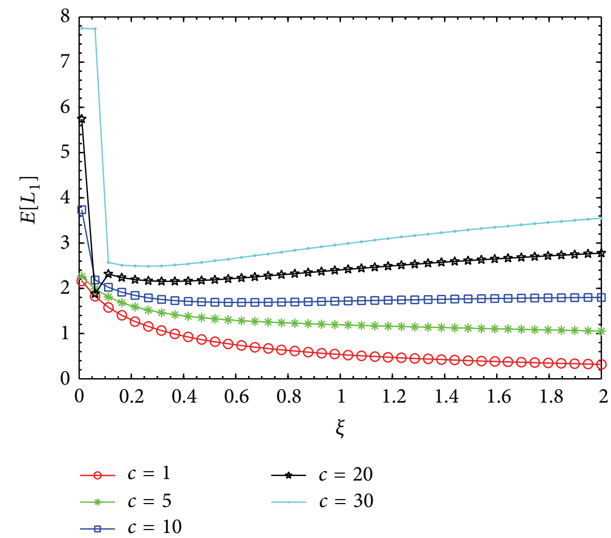

Figure 9: Effect of ξ on $E[L_1]$.

for our multiserver model. This can be observed from the other three curves for $c = 10, 20, 30$ in Figure 6 and Figure 7.

Altman and Yechiali [12] show that the probability $P_{.0}$ and $P_{.1}$ have their limits when $\xi \rightarrow 0$. We observe from Figures 6 and 7 that this property may also hold for our multiserver model. Further, we observe that their limits for the cases of $c = 5, 10, 20, 30$ when $\xi \rightarrow 0$ are very close.

From Figure 8, it is observed that $E[L_0]$ is a decreasing function of ξ for both the case of single server $c = 1$ and cases of multiserver $c = 5, 10, 20, 30$. This is because that the increasing of ξ leads to the increasing of the number of reneging customers, which results in the decreasing of the mean system size $E[L_0]$. Further, we observe that the four curves for $c = 5, 10, 20, 30$ are very close. This means that when c is large enough $E[L_0]$ is insensitive to the number c of servers.

From Figure 9, we observe that $E[L_1]$ is a decreasing function of ξ for the case of single server $c = 1$. This can be verified analytically from (32) by letting $c = 1$. However, $E[L_1]$ may not be a decreasing function of ξ for the case of multiserver. This can be observed from the two curves of $c = 20$ and $c = 30$. Further, we observe that $E[L_1]$ varies significantly with the increasing of c. This means that it is sensitive to the number c of servers.

5. Conclusions

In this paper, we have studied an M/M/c queueing system with impatient customers and a synchronous vacation policy, where the customer impatience is due to the server being on vacation. We have derived some performance measures for the system in terms of two indexes $Q(1)$ and $Q'(1)$. Based on these results, we have obtained some results about limiting behavior for some performance measures. In addition, we have obtained the iterative formulas for calculating these two indexes $Q(1)$ and $Q'(1)$. Furthermore, we have derived closed-form expressions of some important performance measures for two special cases. The effects of the reneging rate and the number of servers on the some performance measures have been investigated numerically. The results obtained in this paper may have potential applications in production-inventory systems and communication networks.

Appendix

Proof of (65). Let $X_j = (x_{1j}, x_{2j}, \ldots, x_{(c-1)j})^T$, $j = 1, 2, \ldots, c-1$, be the jth column vector of the inverse matrix A^{-1} and let $\varepsilon_j = (0, \ldots, 1, \ldots, 0)^T$ be the jth unit column vector; then we have

$$AX_j = \varepsilon_j, \quad j = 1, 2, \ldots, c-1. \tag{A.1}$$

For $j = 1, 2, \ldots, c-1$, (A.1) can be rewritten as the following set of equations:

$$\lambda x_{i-2j} - a_{i-1} x_{i-1j} + i\xi c_{ij} = 0, \quad i \neq j, \ i = 1, 2, \ldots, c-1, \tag{A.2}$$

$$\lambda x_{i-2j} - a_{i-1} x_{i-1j} + i\xi x_{ij} = 1, \quad i = j, \ i = 1, 2, \ldots, c-1, \tag{A.3}$$

where x_{0j} and x_{-1j} are defined to be zero. Repeating the use of (A.2) gives

$$x_{ij} = 0, \quad i = 1, 2, \ldots, j-1. \tag{A.4}$$

Substituting (A.4) into (A.3) yields

$$x_{jj} = \frac{1}{j\xi}. \tag{A.5}$$

From (A.2), we have

$$x_{ij} = \frac{1}{i\xi}\left(a_{i-1} x_{i-1j} - \lambda x_{i-2j}\right), \quad i = j+1, \ j+2, \ldots, c-1. \tag{A.6}$$

This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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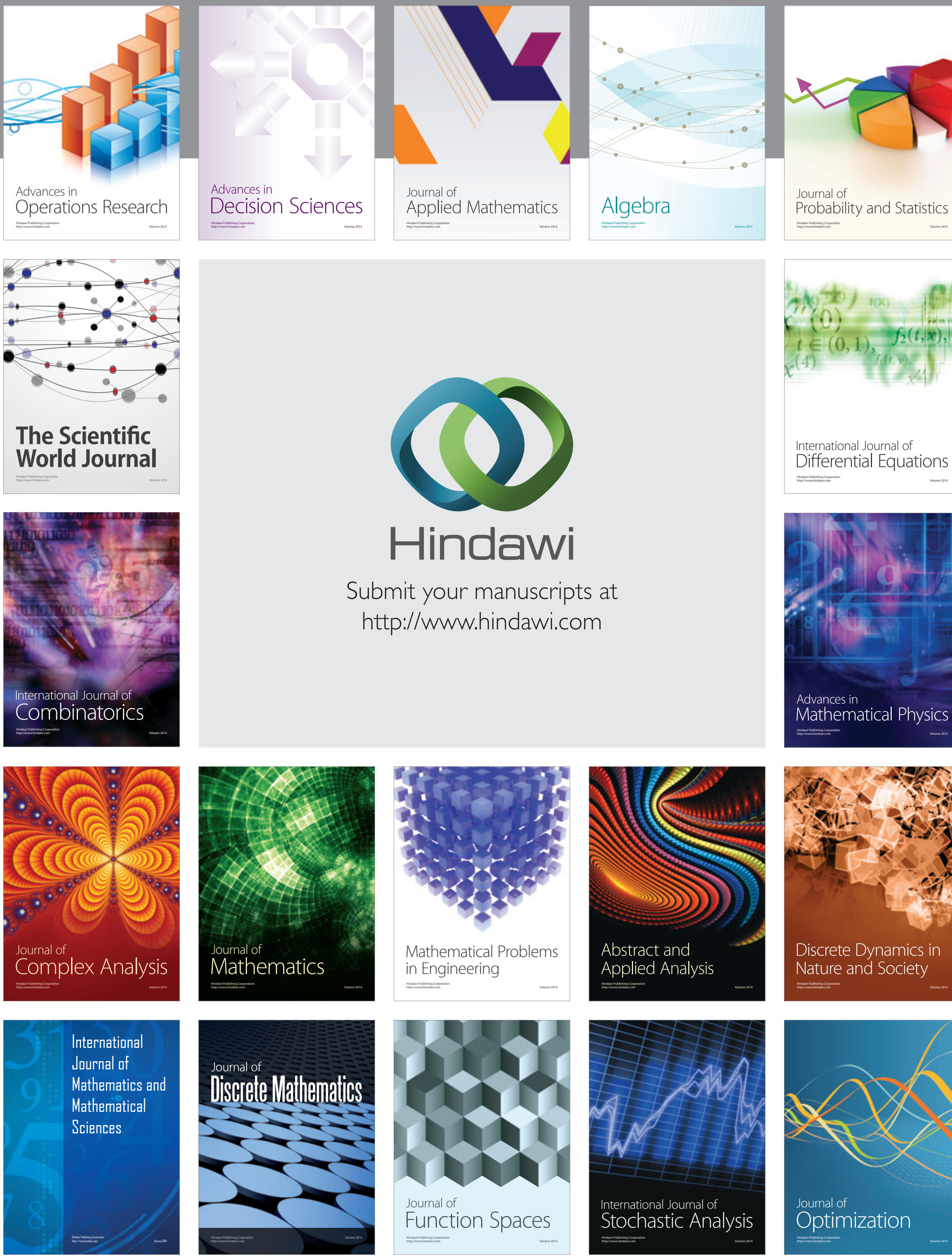
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