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Research Article

Martelli Chaotic Properties of a Generalized Form of Zadeh's Extension Principle

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Let X denote a compact metric space and let $f : X \to X$ be a continuous map. It is known that a discrete dynamical system (X, f) naturally induces its fuzzified counterpart, that is, a discrete dynamical system on the space of fuzzy compact subsets of X. In 2011, a new generalized form of Zadeh's extension principle, so-called g-fuzzification, had been introduced by Kupka 2011. In this paper, we study the relations between Martelli's chaotic properties of the original and g-fuzzified system. More specifically, we study the transitivity, sensitivity, and stability of the orbits in system (X, f) and its connections with the same ones in its g-fuzzified system.

1. Introduction

The main goal of the theory of discrete dynamical system is to understand the asymptotic properties and topological structures of the orbits. In certain sense, the study of the orbits in discrete dynamical system is to investigate the movement of the points in the base space. In many cases, however, it is not sufficient to know how the points move, but it is necessary to understand the motion of the subsets of base space (e.g., in migration phenomenon), and this leads us to the problem of analyzing the dynamics of the set-valued discrete dynamical systems. In this direction, many elegant results have been obtained (please see [1–7] and the references cited therein).

As the complexity of research subjects increased, an accurate description for systems becomes more and more difficult, and the situation would become more complicated when the systems are affected by the uncertainty. In this case, the fuzzy system should be considered. It is well known that any given discrete dynamical system uniquely induces its fuzzified counterpart, that is, a discrete system on the space of fuzzy sets. It is natural to ask the following question: what is the relation between dynamical properties of the original and fuzzified systems?

Motivated by this question, the study of discrete fuzzy dynamical systems has recently become active [8–12]. As a partial response to the question above, in the case of Devaney chaos [13], the transitivity, periodic density, and sensitivity between two systems have been analyzed in 2008 [14]. In addition, by analyzing connections between the fuzzified dynamical systems related to the original one, the authors have pointed out that this kind of investigation should be useful in many real problems, such as in ecological modelling and demographic sciences. Some recent works along these lines appear. In 2011, Kupka proves that there exists a transitive fuzzification on the space of normal fuzzy sets, which contains the solution of the problem that was partially solved in [14]. Specifically, the author considers a symbolic dynamical system as the original system and then shows that Zadeh's extension of the shift map is transitive. As regards the periodic density, a concept of piecewise constant fuzzy set is introduced, and then period density equivalence of f and $\widehat{f}$ is proposed. Consequently, the question has been completely solved [15]. And then we discuss this issue by using the weakly mixing property [16].

Among the methods of fuzzification, Zadeh's extension [17] is often used, but it can lose information that is carried by the original system. Therefore, more general extension principles have been developed [18, 19]. Recently, a concept of g-fuzzification, which allows us to modify the membership grades of points in each iteration, has been introduced [19]. The use of g-fuzzification is quite natural but differs slightly from Zadeh's extension principle, and it can be useful in several situations. Take a fuzzy set "*old people*" for example; *old people* in ancient time are not considered as *old* at present since the average age of people is increasing. We also can find several examples to illustrate such kinds of fuzzy sets with variable membership grades. Zadeh's extension, however, does not reflect this fact. On the other hand, the situation becomes more complicated in fuzzy control. In [20], the authors show that a chaotic function on $\mathbb{R}^n$, for its fuzzification in the sense of Zadeh, is degenerate, because the iterates are asymptotically crisp and, ultimately, we obtain chaos of a mapping of ordinary sets rather than of fuzzy output. In this case, the usual fuzzification is inadequate to describe complexities which may arise in fuzzy control. Consequently, a concept of Γ-fuzzification has been developed in [18], which does not degenerate under chaotic iteration. Now, as a new generalized extension principle, g-fuzzification includes the usual fuzzification (Zadeh's extension) and Γ-fuzzification as two special cases of it, and the developed methods enable us to study the dynamics of discrete fuzzy systems in a more efficient way.

Chaotic dynamics has been hailed as the third great scientific revolution of the 20th century, along with relativity and quantum mechanics. But there is not a generally accepted definition of chaos yet. The different definitions of chaos being around have been designed to meet different purposes and they are based on very different backgrounds and levels of mathematical sophistication. To compare various kinds of definitions of chaos naturally attracts the interest of many researchers. In 2002, Huang and Ye showed that chaos in the sense of Devaney is stronger than that of Li-Yorke [21]. The conclusion stimulates the study of the relations between different definitions of chaos [22–24].

Among various definitions of chaos, Martelli's chaos is one of the definitions of chaos which are suitable for easy and reliable numerical verification [25]. The authors make comparison of different definitions of chaos and point out that Martelli's chaos embodies the essential features which all other definitions are trying to capture [26]. It is worth noting that although formulated in a different way, Martelli's chaos is practically equivalent to chaos in the sense of Wiggins [27]. There remains, however, a difference between the two definitions. Wiggins does not require sensitivity with respect to the base space, while Martelli requires instability with respect to the base space.

In this paper, we focus on relations between Martelli's chaotic properties of the original and g-fuzzified dynamical systems. Below, Section 2 gives basic notions and definitions. Section 3 discusses the relation between Martelli's chaotic properties of the original and g-fuzzified systems. A brief conclusion concludes the paper.

2. Preliminaries

In this section, we complete notations and recall some known definitions. Let $f : X \to X$ be a continuous map acting on a compact metric space (X, d). An orbit of a point $x_0 \in X$ is the set $\{f^n(x_0) : n \geq 0\}$, denoted by $\mathrm{orb}(x_0, f)$ or simply $\mathrm{orb}(x_0)$ when the function f is clearly specified. A point y is a *limit point* of $\mathrm{orb}(x_0)$ if a subsequence of $\mathrm{orb}(x_0)$ converges to y. The set of limit points of $\mathrm{orb}(x_0)$ is denoted by $L(x_0)$.

We say that f is *transitive* if for any pair of nonempty open sets U and V there exists $n \geq 1$ such that $f^n(U) \cap V \neq \emptyset$; f is *point transitive* if there exists a point $x_0 \in X$ such that the orbit of x_0 is dense in X; that is, $\overline{\mathrm{orb}(x_0)} = X$, and x_0 is called a *transitive point* of X.

We say that $\mathrm{orb}(x)$ is *unstable* if there exists δ_x such that, for any neighborhood U of x, there exist $y \in U$ and $n \geq 0$ such that $d(f^n(x), f^n(y)) > \delta_x$. An orbit which is not unstable is said to be *stable*.

We say that f has *sensitive dependence on initial conditions* if there is a constant $\delta > 0$ such that for every point x and every neighborhood U about x there are a $y \in U$ and a $k \geq 1$ such that $d(f^k(x), f^k(y)) \geq \delta$. Hence, every orbit $\mathrm{orb}(x)$ with $x \in X$ is unstable with the same constant δ. Consequently, sensitive dependence on initial conditions is stronger than instability.

Definition 1 (see [25]). Let (X, d) be a compact metric space and let $f : X \to X$ be continuous. Then, f is Martelli chaotic provided that there exists $x_0 \in X$ such that

(i) $L(x_0) = X$;

(ii) $\mathrm{orb}(x_0)$ is unstable.

In this research, we call a *Martelli chaotic* map *M-chaotic* for short.

Below, we present some definitions from fuzzy theory. Let $\mathscr{K}(X)$ be the class of all nonempty and compact subsets of X. If $A \in \mathscr{K}(X)$, we define the ε-neighbourhood of A as the set

$$N(A, \varepsilon) = \{x \in X \mid d(x, A) < \varepsilon\}, \tag{1}$$

where $d(x, A) = \inf_{a \in A} \|x - a\|$.

The Hausdorff separation $\rho(A, B)$ of $A, B \in \mathscr{K}(X)$ is defined by

$$\rho(A, B) = \inf\{\varepsilon > 0 \mid A \subseteq N(B, \varepsilon)\}. \tag{2}$$

The Hausdorff metric on $\mathscr{K}(X)$ is defined by letting

$$H(A, B) = \max\{\rho(A, B), \rho(B, A)\}. \tag{3}$$

Define $\mathscr{F}(X)$ as the class of all upper semicontinuous fuzzy sets $u : X \to [0, 1]$ such that $[u]_\alpha \in \mathscr{K}(X)$, where α-*cuts* and the *support* of u are defined by

$$\begin{aligned} [u]_\alpha &= \{x \in X \mid u(x) \geq \alpha\}, \quad \alpha \in [0, 1], \\ \mathrm{supp}(u) &= \overline{\{x \in X \mid u(x) > 0\}}, \end{aligned} \tag{4}$$

respectively.

Moreover, let $\mathscr{F}_0(X)$ denote the space of all nonempty fuzzy sets on X and let $\emptyset_X$ denote the *empty fuzzy set* ($\emptyset_X(x) = 0$ for all $x \in X$).

A *level-wise metric* d_∞ on $\mathscr{F}(X)$ is defined by

$$d_\infty(u, v) = \sup_{\alpha \in [0,1]} H([u]_\alpha, [v]_\alpha) \tag{5}$$

for all $u, v \in \mathscr{F}(X)$. It is well known that if (X, d) is complete, then $(\mathscr{F}(X), d_\infty)$ is also complete but is not compact and is not separable (see [19, 28, 29]).

Lemma 2 (see [9, 14]). *Let A be an open subset of X. Define $e(A) = \{u \in \mathscr{F}(X) : [u]_0 \subseteq A\}$, and then $e(A)$ is an open subset of $\mathscr{F}(X)$.*

Let $f : X \to X$ be continuous. A usual fuzzification (often called Zadeh's extension) $\widehat{f} : \mathbb{F}(X) \to \mathbb{F}(X)$ is defined by

$$\widehat{f}(u)(x) = \sup_{y \in f^{-1}(x)} u(y) \tag{6}$$

for any $u \in \mathbb{F}(X)$ and $x \in X$.

Now let us introduce g-fuzzification. Denote $D_m(I)$ as the set of all nondecreasing right continuous functions $g : I \to I$ for which $g(x) = x$ if $x = 0$ and $x = 1$. Let $C_m(I)$ be the set of all continuous maps from $D_m(I)$. For any $g \in D_m(I)$, a g-fuzzification $\widehat{f}_g : \mathbb{F}(X) \to \mathbb{F}(X)$ is defined by

$$\widehat{f}_g(u)(x) = \sup_{y \in f^{-1}(x)} \{g(u(y))\} \quad \text{for any } u \in \mathbb{F}(X), x \in X. \tag{7}$$

An α-cut $[u]^g_\alpha$ of a fuzzy set $u \in \mathbb{F}(X)$ with respect to $g \in D_m(I)$ is

$$[u]^g_\alpha = \{x \in \operatorname{supp}(u) \mid g(u(x)) \ge \alpha\} \quad \text{for } \alpha \in (0,1]. \tag{8}$$

Lemma 3 (see [19]). *Let $f : X \to X$ be continuous and let $\widehat{f}_g$ be g-fuzzification. Then,*

$$f([u]^g_\alpha) = [\widehat{f}_g(u)]_\alpha \tag{9}$$

holds for any $u \in \mathbb{F}_0(X)$, $g \in D_m(I)$, and $\alpha \in (0,1]$.

Lemma 4 (see [19]). *Let $g \in D_m(I)$, $u \in \mathbb{F}_0(X)$, and $\alpha \in (0,1]$. If $[u]^g_\alpha \ne \infty$. then there is $c \in (0,1]$ such that $[u]^g_\alpha = [u]_c$.*

3. M-Chaotic Relations between $\widehat{f}_g$ and f

In this section, we study the relations between Martelli's chaotic properties of the original system (X, f) and g-fuzzified system $(\mathbb{F}(X), \widehat{f}_g)$, where $\mathbb{F}(X)$ is equipped with the level-wise topology, that is, the metric topology induced by d_∞.

On the one hand, some conditions are discussed, under which $\widehat{f}_g$ M-chaotic implies f M-chaotic. On the other hand, several examples are presented to illustrate that, in general, f M-chaotic does not imply $\widehat{f}_g$ M-chaotic.

Proposition 5. *Define $[u]^g_0 = \{x \in [u]_0 \mid g(u(x)) > 0\}$. Then*

$$[u]^g_0 = [u]_0 \tag{10}$$

holds for every $u \in \mathbb{F}_0(X)$ and $g \in D_m(I)$.

Proof. The inclusion $[u]^g_0 \subseteq [u]_0$ follows directly from the definition of $[u]^g_0$. If $x \in [u]_0$, then, because g is nondecreasing, we have $g(u(x)) > 0$, which implies $x \in [u]^g_0$, and, consequently, $[u]^g_0 \supseteq [u]_0$ holds. □

Proposition 6. *Let U be subset of X and let $f : X \to X$ be continuous. Then, $\widehat{f}_g(e(U)) \subseteq e(f(U))$.*

Proof. If $u \in \widehat{f}_g(e(U))$, then there exists $u^* \in e(U)$ such that $u = \widehat{f}_g(u^*)$. Hence, due to Lemma 3 and Proposition 5, we have that $[u]_0 = [\widehat{f}_g(u^*)]_0 = f([u^*]^g_0) = f([u^*]_0)$, since $[u^*]_0 \subseteq U$ and $[u]_0 = f([u^*]_0) \subseteq f(U)$; thus $u \in e(f(U))$, and the inclusion follows. □

Proposition 7. *Let $u \in \mathbb{F}_0(X)$, $g \in D_m(I)$, and $\alpha_i \in (0,1]$, $i = 1, \dots, n, \dots$. Then, there exists an $\alpha_n \in (0,1]$ such that $[\widehat{f}^n_g(u)]_{\alpha_n} = [\widehat{f^n_g}(u)]_{\alpha_{2n-1}} = f^n([u]_{\alpha_{2n}})$.*

Proof. We do the proof by mathematical induction.

When $n = 1$, by Lemmas 3 and 4, the formula gives us $[\widehat{f}_g(u)]_{\alpha_1} = f([u]^g_{\alpha_1}) = f([u]_{\alpha_2})$; therefore, the statement holds for $n = 1$.

Assume that the statement is true for $n = k$; that is,

$$[\widehat{f}^k_g(u)]_{\alpha_k} = [\widehat{f^k_g}(u)]_{\alpha_{2k-1}} = f^k([u]_{\alpha_{2k}}). \tag{11}$$

Note that $[\widehat{f}^k_g(u)]_{\alpha_{k+2}} = f^k([u]_{\alpha_{2k+2}})$.

When $n = k+1$,

$$\begin{aligned}[\widehat{f}^{k+1}_g(u)]_{\alpha_{k+1}} &= [\widehat{f}_g\widehat{f}^k_g(u)]_{\alpha_{k+1}} \\ &= f\left([\widehat{f}^k_g(u)]^g_{\alpha_{k+1}}\right) = f\left([\widehat{f}^k_g(u)]_{\alpha_{k+2}}\right) \\ &= f\left(f^k([u]_{\alpha_{2k+2}})\right) = f^{k+1}([u]_{\alpha_{2(k+1)}}).\end{aligned} \tag{12}$$

On the other hand, $f^{k+1}([u]_{\alpha_{2(k+1)}}) = f^{k+1}([u]^g_{\alpha_{2(k+1)-1}}) = [\widehat{f^{k+1}_g}(u)]_{\alpha_{2k+1}}$.

This completes the proof. □

Remark 8. The proof of Proposition 7 can also be done as follows.

Due to Lemmas 3 and 4, we have that

$$\begin{aligned}[\widehat{f}^n_g(u)]_{\alpha_n} &= [\widehat{f}_g\widehat{f}^{n-1}_g(u)]_{\alpha_n} \\ &= f\left([\widehat{f}^{n-1}_g(u)]^g_{\alpha_n}\right) = f\left([\widehat{f}^{n-1}_g(u)]_{\alpha_{n+1}}\right) \\ &= f\left([\widehat{f}_g\widehat{f}^{n-2}_g(u)]_{\alpha_{n+1}}\right) = f^2\left([\widehat{f}^{n-2}_g(u)]^g_{\alpha_{n+1}}\right)\end{aligned}$$

$$= f^2\left(\left[\widehat{f}_g^{n-2}(u)\right]_{\alpha_{n+2}}\right)$$
$$\vdots$$
$$= f^n\left([u]^g_{\alpha_{2n-1}}\right) = f^n\left([u]_{\alpha_{2n}}\right). \tag{13}$$

On the other hand, by Lemma 3 again, we obtain $f^n([u]^g_{\alpha_{2n-1}}) = [\widehat{f^n_g}(u)]_{\alpha_{2n-1}}$, and, consequently, $[\widehat{f}^n_g(u)]_{\alpha_n} = [\widehat{f^n_g}(u)]_{\alpha_{2n-1}} = f^n([u]_{\alpha_{2n}})$ holds.

Theorem 9. *Let u_0 be a transitive point of $(\mathbb{F}(X), \widehat{f}_g)$. Then, every $x \in [u_0]^g_\alpha$ is a transitive point of (X, f) for $\alpha \in (0, 1]$.*

Proof. Since u_0 is a transitive point of $(\mathbb{F}(X), \widehat{f}_g)$, there exists $k \in \mathbb{N}$ such that $d_\infty(\widehat{f}^k_g(u_0), \nu) < \varepsilon$ for any $\nu \in \mathbb{F}(X)$ and $\varepsilon > 0$. By using Proposition 7 and Lemma 3, we obtain

$$\begin{aligned} d_\infty\left(\widehat{f}^k_g(u_0), \nu\right) &= \sup_{\beta\in[0,1]} H\left(\left[\widehat{f}^k_g(u_0)\right]_\beta, [\nu]_\beta\right) \\ &= \sup_{\alpha,\beta\in(0,1]} H\left(\left[\widehat{f^k_g}(u_0)\right]_\alpha, [\nu]_\beta\right) \\ &= \sup_{\alpha,\beta\in(0,1]} H\left(f^k[u_0]^g_\alpha, [\nu]_\beta\right) < \varepsilon, \end{aligned} \tag{14}$$

for some $\alpha, \beta \in [0, 1]$. Hence, for each $y \in [\nu]_\beta$, there exists $x \in [u_0]^g_\alpha$ such that $d(x, y) < \varepsilon$, which means that every $x \in [u_0]^g_\alpha$ is a transitive point of (X, f). □

Theorem 10. *If $L(u_0) = \mathbb{F}(X)$, then there exists $x_0 \in [u_0]^g_\alpha$ such that $L(x_0) = X$.*

Proof. It follows directly from Theorem 9. □

The following example shows that, in general, the converse of Theorem 10 is not true.

Example 11 (irrational rotation of circle). Let λ be an irrational number and $R_\lambda : S^1 \to S^1$ is defined by $R_\lambda(e^{i\theta}) = e^{i(\theta+2\pi\lambda)}$. It is well known that, for each $z \in S^1$, the orbit of z is dense in S^1 and, consequently, $L(z) = S^1$. Nevertheless, it is not necessary for some $\nu \in \mathbb{F}(S^1)$ to exist such that $L(\nu) = \mathbb{F}(S^1)$. In fact, assume that $u \in \mathbb{F}(S^1)$ and $\operatorname{diam}([u]^g_0) = 1$. Given that $0 < \varepsilon < 1/2$, let $U = B(\widehat{1}, \varepsilon/2)$ and $V = B(u, \varepsilon/2)$, and by Proposition 5, we obtain

$$\begin{aligned} \omega \in U = B\left(\widehat{1}, \frac{\varepsilon}{2}\right) &\Longrightarrow \operatorname{diam}\left([\omega]^g_0\right) = \operatorname{diam}\left([\omega]_0\right) \le \frac{\varepsilon}{2}, \\ \nu \in V = B\left(u, \frac{\varepsilon}{2}\right) &\Longrightarrow \operatorname{diam}\left([\nu]^g_0\right) = \operatorname{diam}\left([\nu]_0\right) \ge 1 - \varepsilon, \end{aligned} \tag{15}$$

since

$$\operatorname{diam}\left(\left[\widehat{R}^n_{\lambda_g}(\nu)\right]_0\right) = \operatorname{diam}\left(R^n_\lambda[\nu]_0\right) \ge 1 - \varepsilon \tag{16}$$

for $n \in \mathbb{N}$. Hence, $U \cap \widehat{R}^n_{\lambda_g}(V) = \emptyset$, which means that there exists no $\nu \in V$ such that $\widehat{R}^n_{\lambda_g}(\nu) = \omega$ for some $\omega \in \mathbb{F}(X)$, and, consequently, $L(\nu) \neq \mathbb{F}(X)$.

Theorem 12. *Let $f : X \to X$ be continuous, let $\widehat{f}_g$ be the g-extension of f, and let $u_0 \in \mathbb{F}(X)$. If the orbit of u_0 is unstable in $\mathbb{F}(X)$, then there exists $x_0 \in [u_0]^g_\beta$ such that the orbit of x_0 is unstable in X, where $\beta \in [0, 1]$.*

Proof. Let the assumptions be satisfied. Then, there exists δ_{u_0} such that for every $\epsilon > 0$ we can find $\nu \in \mathbb{F}(X)$ and $k \in \mathbb{N}$ satisfying $\nu \in B(u_0, \varepsilon)$ and

$$\begin{aligned} &d_\infty\left(\widehat{f}^k_g(u_0), \widehat{f}^k_g(\nu)\right) \\ &\quad = \sup_{\alpha\in[0,1]} H\left(\left[\widehat{f}^k_g(u_0)\right]_\alpha, \left[\widehat{f}^k_g(\nu)\right]_\alpha\right) \\ &\quad = \sup_{\beta,\gamma\in[0,1]} H\left(\left[\widehat{f^k_g}(u_0)\right]_\beta, \left[\widehat{f^k_g}(\nu)\right]_\gamma\right) \\ &\quad = \sup_{\beta,\gamma\in[0,1]} H\left(f^k[u_0]^g_\beta, f^k[\nu]^g_\gamma\right) > \delta_{u_0}. \end{aligned} \tag{17}$$

Thus, there exist $x_0 \in [u_0]^g_\beta$ and $y_0 \in [\nu]^g_\gamma$ such that $d(x_0, y_0) > \delta_{u_0}$. Since $\nu \in B(u_0, \varepsilon)$, we have $d(x_0, y_0) < \varepsilon$. This proves that there exists $x_0 \in [u_0]^g_\beta$ such that the orbit of x_0 is unstable in X with instable constant δ_{u_0}. □

Example 13. Consider the foregoing example (Example 11); because R_λ is isometric, it does not exhibit sensitive dependence on initial conditions and hence the orbit of each $z \in S^1$ is stable, which implies that, by Theorem 12, there exists no orbit of $u \in \mathbb{F}(S^1)$ that is unstable.

By combining Theorems 9, 10, and 12, we obtain the following theorem.

Theorem 14. *If $\widehat{f}_g$ is M-chaotic, then f is M-chaotic.*

We will need some notions from Denjoy map [13]. Recall that the circle S^1 can be considered as the quotient space $\mathbb{R}/\mathbb{Z}$, where $\mathbb{R}$ and $\mathbb{Z}$ are the sets of real numbers and integers, respectively. The irrational rotation of the circle $R_\lambda : S^1 \to S^1$ is then given by

$$R_\lambda(x) = x + \lambda \pmod 1, \tag{18}$$

where λ is irrational. Recall that a Denjoy map can be constructed as follows. Take any point $x_0 \in S^1$. We cut out each point $R^n_\lambda(x_0)$ on the orbit of x_0 and replace it with a small interval I_n. For $n \in \mathbb{N}$,

(a) $\mathscr{L}(I_0) = 1/4$, $\mathscr{L}(I_{n+1}) < \mathscr{L}(I_n)$, $\mathscr{L}(I_n) = \mathscr{L}(I_{-n})$, and $\sum_{n\in\mathbb{Z}} \mathscr{L}(I_n) = 1$, where $\mathscr{L}(I_n)$ denotes the length of the interval I_n;

(b) $\lim_{n\to\infty}(\mathscr{L}(I_{n+1})/\mathscr{L}(I_n)) = 1$.

Consequently, a new circle S^* has been constructed. The Denjoy homeomorphism $D_\lambda : S^* \to S^*$ is an orientation preserving homeomorphism of S^*. There exists a Cantor set $C_\lambda \subset S^*$ on which D_λ acts minimally. It is known that there exists a continuous surjection $h_\lambda : S^* \to S^1$ that semiconjugates D_λ with R_λ. In [30], the authors show that the system $(\mathbb{K}(C_\lambda), D_\lambda)$ is not sensitive.

Proposition 15. *Let $x \in C_\lambda$; then the orbit of x is unstable in (C_λ, D_λ) with the constant* $1/4$.

Proof. Suppose that $y \in B(x,\varepsilon)$ and $h_\lambda(x) \neq h_\lambda(y)$ for any $\varepsilon > 0$. Since the orbit of x_0 is dense in S^1, there exist some $k \in \mathbb{N}$ such that $R^k_{-\lambda}(x_0) \in [h_\lambda(x), h_\lambda(y)]$, where $[h_\lambda(x), h_\lambda(y)]$ is the closed arc in S^1. Thus, we have $x_0 \in R^k_\lambda([h_\lambda(x), h_\lambda(y)])$. Consequently, due to the construction of Denjoy map, we obtain $I_0 \subset [D^k_\lambda(x), D^k_\lambda(y)]$, which means that $d(D^k_\lambda(x), D^k_\lambda(y)) > 1/4$. □

The following proposition shows that the instability of the orbit in (C^λ, D_λ) cannot be inherited by its g-fuzzification. More specifically, there exist points arbitrarily close to $u \in \mathbb{F}(C_\lambda)$ which eventually also close to u under iteration of $\widehat{D_{\lambda}}_g$, although there exist some $x \in [u]_0$ such that the orbits of these points are unstable in (C^λ, D_λ). It should be mentioned that our approach was inspired by the idea in [8] where a continuous map i_λ was defined.

Define $i_\lambda : \mathbb{K}(C_\lambda) \to \mathbb{F}(C_\lambda)$ by $i_\lambda(K) = \lambda\chi_K$ for any $K \in \mathbb{K}(C_\lambda)$ and any $\lambda \in (0,1]$, where χ_K is the characteristic function of K (that is to say, $\chi_K(x) = 1$ if $x \in K$ and $\chi_K(x) = 0$ if $x \notin K$). Hence, $i_\lambda \circ \overline{D}_\lambda = \widehat{D_{\lambda}}_g \circ i_\lambda$. Note that i_λ is continuous.

Proposition 16. *Let $u \in \mathbb{F}(C_\lambda)$; then there exist some $v \in \mathbb{F}(C_\lambda)$ and $n > 0$ such that $d_\infty(\widehat{D_{\lambda}}_g^n(u), \widehat{D_{\lambda}}_g^n(v)) < \varepsilon$.*

Proof. Since $(\mathbb{K}(C_\lambda), \overline{D}_\lambda)$ is not sensitive, for $\varepsilon > 0$ and $\delta > 0$, there exist $M \in \mathbb{K}(C_\lambda)$ and $B(M,\delta)$ such that, for all $N \in B(M,\delta)$,

$$H\left(\overline{D}^n_\lambda(M), \overline{D}^n_\lambda(N)\right) < \varepsilon. \tag{19}$$

Suppose that $u \in e(M)$ (recall that $e(M) = \{u \in \mathbb{F}(C_\lambda) \mid [u]_0 \subseteq M\}$), and by continuity of i_λ and (19), we have

$$\begin{aligned} &H\left(\overline{D}^n_\lambda([u]_0), \overline{D}^n_\lambda(N)\right) < \varepsilon \\ &\Longrightarrow H\left(i_\lambda \circ \overline{D}^n_\lambda([u]_0), i_\lambda \circ \overline{D}^n_\lambda(N)\right) < \varepsilon \\ &\Longrightarrow H\left(\widehat{D_{\lambda}}_g^n \circ i_\lambda([u]_0), \widehat{D_{\lambda}}_g^n \circ i_\lambda(N)\right) \\ &= d_\infty\left(\widehat{D_{\lambda}}_g^n(u), \widehat{D_{\lambda}}_g^n(v)\right) < \varepsilon. \end{aligned} \tag{20}$$

Without loss of generality, assume that $v = i_\lambda(N) \in \mathbb{F}(C_\lambda)$. This completes the proof. □

Remark 17. Theorem 9 together with Theorem 10 shows that $\widehat{f}_g$ M-chaotic implies f M-chaotic, but generally speaking, the converse is not true, which has been discussed in Example 11 and Proposition 16.

4. Conclusions and Discussions

In this present investigation, we discuss relations between Martelli chaotic properties of the original and g-fuzzified dynamical systems. More specifically, we study stability of the orbits and transitivity and present several examples to illustrate the relations between two dynamical systems. We show that the dynamical properties of the original system and its fuzzy extension mutually inherit some global characteristics. The following main results are obtained.

(a) If $L(u_0) = \mathbb{F}(X)$, then there exists $x_0 \in [u_0]^g_\alpha$ such that $L(x_0) = X$ (Theorem 10).

(b) The instability of $\text{orb}(u, \widehat{f}_g)$ implies the instability of $\text{orb}(x, f)$, where $u \in \mathbb{F}(X)$, $x \in [u]^g_\beta$, and $\beta \in [0,1]$ (Theorem 12).

(c) $\widehat{f}_g$ M-chaotic implies f M-chaotic (Theorem 14).

(d) f M-chaotic does not imply $\widehat{f}_g$ M-chaotic (Example 11 and Proposition 16).

It is worth noting that any g-fuzzification is connected to a crisp discrete dynamical system in two different ways [19]. One way is to connect two systems via α-cut, and another approach is to consider g-fuzzified discrete dynamical system as a crisp system that is induced by a certain product map. We develop, in this present paper, the first method. It would be interesting to use the second approach to study the relations between dynamical properties of the original and g-fuzzified dynamical systems, and this will be one aspect of our future works.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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