

Research Article

New Exact Explicit Nonlinear Wave Solutions for the Broer-Kaup Equation

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We study the nonlinear wave solutions for the Broer-Kaup equation. Many exact explicit expressions of the nonlinear wave solutions for the equation are obtained by exploiting the bifurcation method and qualitative theory of dynamical systems. These solutions contain solitary wave solutions, singular solutions, periodic singular solutions, and kink-shaped solutions, most of which are new. Some previous results are extended.

1. Introduction

In 1975, Broer [1] obtained a dispersive equation as follows:

$$\begin{aligned}u_t + \eta_x + \frac{1}{2}(u^2)_x &= 0, \\ \eta_t + (u\eta + u + u_{xx})_x &= 0,\end{aligned}\tag{1}$$

which describes the evolution of horizontal velocity component $u(x, t)$ of water waves of height $\eta(x, t)$ propagating in both directions in an infinite narrow channel of finite constant depth. Equation (1) plays an important role in nonlinear physics and gains considerable attention [2–4]. The traveling wave solutions for (1) have been studied by many works, such as [5–10].

In this paper, we employ the bifurcation method and qualitative theory of dynamical systems [11–21] to investigate the nonlinear wave solutions for (1), and we obtain many exact explicit expressions of nonlinear wave solutions for (1). These nonlinear wave solutions contain solitary wave solutions, singular solutions, periodic singular solutions, and kink-shaped solutions, most of which, to our knowledge, are newly obtained.

The remainder of this paper is organized as follows. In Section 2, we show the bifurcation of phase portraits corresponding to (1). We state our main results and the theoretical derivation for the main results in Section 3. A short conclusion will be given in Section 4.

2. Bifurcation of Phase Portraits

In this section, we give the process of obtaining the bifurcation of phase portraits corresponding to (1).

For given constant c , substituting $u(x, t) = \varphi(\xi)$, $\eta(x, t) = \psi(\xi)$ with $\xi = x - ct$ into (1), it follows

$$\begin{aligned}-c\varphi' + \psi' + \frac{1}{2}(\varphi^2)' &= 0, \\ -c\psi' + (\psi\varphi + \varphi + \varphi'')' &= 0.\end{aligned}\tag{2}$$

Integrating (2) once leads to

$$\begin{aligned}-c\varphi + \psi + \frac{1}{2}\varphi^2 &= g, \\ -c\psi + \psi\varphi + \varphi + \varphi'' &= G,\end{aligned}\tag{3}$$

where both g and G are integral constants, respectively.

From the first equation of system (3), we obtain

$$\psi = g + c\varphi - \frac{1}{2}\varphi^2.\tag{4}$$

Substituting (4) into the second equation of system (3) leads to

$$\varphi'' = \frac{1}{2}\varphi^3 - \frac{3c}{2}\varphi^2 + (c^2 - g - 1)\varphi + cg + G.\tag{5}$$

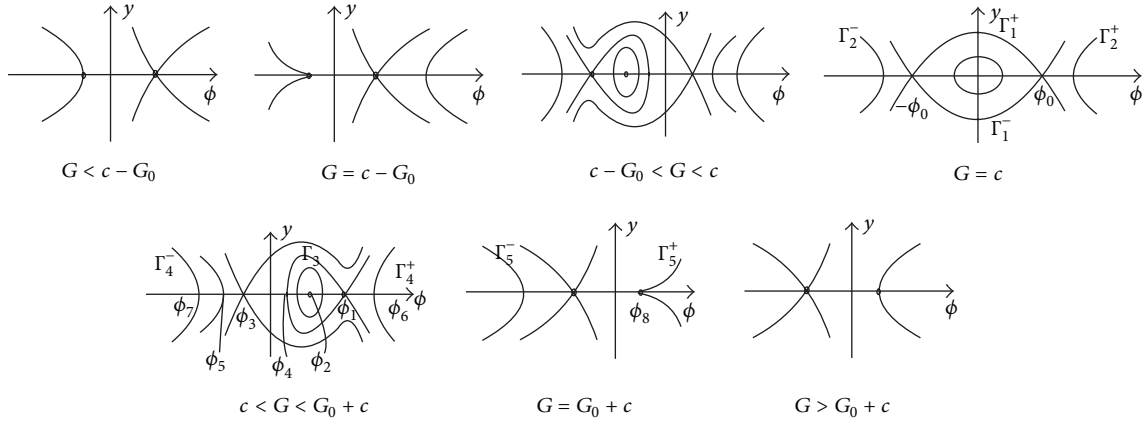


FIGURE 1: The phase portraits of system (7).

By setting $\varphi = \phi + c$, (5) becomes

$$\phi'' = \frac{1}{2}\phi^3 - \frac{1}{2}(c^2 + 2g + 2)\phi + G - c. \quad (6)$$

Letting $y = \phi'$, we obtain a planar system

$$\frac{d\phi}{d\xi} = y, \quad (7)$$

$$\frac{dy}{d\xi} = \frac{1}{2}\phi^3 - \frac{1}{2}(c^2 + 2g + 2)\phi + G - c,$$

with first integral

$$H(\phi, y) = \frac{1}{2}y^2 - \frac{1}{8}\phi^4 + \frac{1}{4}(c^2 + 2g + 2)\phi^2 - (G - c)\phi. \quad (8)$$

Now, we study the bifurcation of phase portraits of system (7). Set

$$f_0(\phi) = \frac{1}{2}\phi^3 - \frac{1}{2}(c^2 + 2g + 2)\phi, \quad (9)$$

$$f(\phi) = \frac{1}{2}\phi^3 - \frac{1}{2}(c^2 + 2g + 2)\phi + G - c.$$

Obviously, $f_0(\phi)$ has three zero points, which can be expressed as

$$\phi = 0, \pm\phi_0, \quad (10)$$

where $\phi_0 = \sqrt{c^2 + 2g + 2}$, when $c^2 + 2g + 2 > 0$.

Additionally, it is easy to obtain the two extreme points of $f(\phi)$ as follows:

$$\phi_{\pm}^* = \pm\sqrt{\frac{1}{3}(c^2 + 2g + 2)}. \quad (11)$$

Let

$$g_0 = |f_0(\phi_{\pm}^*)| = \sqrt{\frac{(c^2 + 2g + 2)^3}{27}}, \quad (12)$$

which is the absolute value of extreme values of $f_0(\phi)$.

Let $(\phi_i, 0)$ be one of the singular points of system (7). Then the characteristic values of the linearized system of system (7) at the singular point $(\phi_i, 0)$ are

$$\lambda_{\pm} = \pm\sqrt{f'(\phi_i)}. \quad (13)$$

From the qualitative theory of dynamical systems, we therefore know that

- (i) if $f'(\phi_i) > 0$, then $(\phi_i, 0)$ is a saddle point;
- (ii) if $f'(\phi_i) < 0$, then $(\phi_i, 0)$ is a center point;
- (iii) if $f'(\phi_i) = 0$, then $(\phi_i, 0)$ is a degenerate saddle point.

Therefore, based on the above analysis, we obtain the bifurcation of phase portraits of system (7) in Figure 1.

3. Main Results and the Theoretic Derivations of the Main Results

In this section, we state our results about solitary wave solutions, singular solutions, periodic singular solutions, and kink-shaped solutions for the first component of system (7). To relate conveniently, we omit $\varphi = \phi + c$ and the expression of the second component of system (7), that is, $\psi = g + c\varphi - (1/2)\varphi^2$, in the following theorem.

Theorem 1. For given constants c and α ($1 < \alpha < 3$), which will be given later, the Broer-Kaup equation (1) has the following exact explicit nonlinear wave solutions.

- (1) When $G = c$, one obtains two kink-shaped solutions

$$\phi_{1\pm}(x, t) = \pm\sqrt{c^2 + 2g + 2} \tanh\left(\frac{\sqrt{c^2 + 2g + 2}}{2}(x - ct)\right), \quad (14)$$

two singular solutions

$$\phi_{2^\pm}(x, t) = \pm \sqrt{c^2 + 2g + 2} \coth \left(\frac{\sqrt{c^2 + 2g + 2}}{2} (x - ct) \right), \tag{15}$$

and four periodic singular solutions

$$\begin{aligned} \phi_{3^\pm}(x, t) &= \pm \sqrt{2(c^2 + 2g + 2)} \\ &\times \sec \left(\sqrt{\frac{c^2 + 2g + 2}{2}} (x - ct) \right), \\ \phi_{4^\pm}(x, t) &= \pm \sqrt{2(c^2 + 2g + 2)} \\ &\times \csc \left(\sqrt{\frac{c^2 + 2g + 2}{2}} (x - ct) \right). \end{aligned} \tag{16}$$

(2) When $c < G < G_0 + c$, one gets two solitary wave solutions

$$\begin{aligned} \phi_{5^\pm}(x, t) &= \pm \sqrt{\frac{c^2 + 2g + 2}{\alpha}} \\ &\times \left(\sqrt{2\alpha - 2} \cosh \left(\sqrt{\frac{c^2 + 2g + 2(3 - \alpha)}{\alpha}} (x - ct) \right) \right. \\ &\quad \left. + 2\alpha - 4 \right) \\ &\times \left(\sqrt{2\alpha - 2} \cosh \left(\sqrt{\frac{c^2 + 2g + 2(3 - \alpha)}{\alpha}} (x - ct) \right) \right. \\ &\quad \left. \times (x - ct) \right)^{-1}, \end{aligned} \tag{17}$$

two singular solutions

$$\begin{aligned} \phi_{6^\pm}(x, t) &= \pm \sqrt{\frac{c^2 + 2g + 2}{\alpha}} \\ &\times \left(2 \cosh \left(\sqrt{\frac{c^2 + 2g + 2(3 - \alpha)}{\alpha}} (x - ct) \right) \right. \end{aligned}$$

$$\begin{aligned} &\left. + \sqrt{6 - 2\alpha} \sinh \left(\sqrt{\frac{c^2 + 2g + 2(3 - \alpha)}{\alpha}} (x - ct) \right) \right) \\ &\times |x - ct| + 4 - 2\alpha \end{aligned}$$

$$\begin{aligned} &\times \left(2 \cosh \left(\sqrt{\frac{c^2 + 2g + 2(3 - \alpha)}{\alpha}} (x - ct) \right) \right. \\ &\left. + \sqrt{6 - 2\alpha} \sinh \left(\sqrt{\frac{c^2 + 2g + 2(3 - \alpha)}{\alpha}} (x - ct) \right) \right) \end{aligned}$$

$$\begin{aligned} &\times |x - ct| - 2 \Big)^{-1}, \end{aligned} \tag{18}$$

and two periodic singular solutions

$$\begin{aligned} \phi_{7^\pm}(x, t) &= \pm \left(\phi_2 (\phi_6 - \phi_7) \sin \left(\arcsin \left(\frac{2\phi_2 - \phi_6 - \phi_7}{\phi_6 - \phi_7} \right) \right. \right. \\ &\quad \left. \left. - \frac{\sqrt{-(\phi_2 - \phi_6)(\phi_2 - \phi_7)}}{2} \right) \right. \\ &\quad \left. \times |x - ct| \right) \\ &\quad + 2\phi_6\phi_7 - \phi_2\phi_6 - \phi_2\phi_7 \\ &\times \left((\phi_6 - \phi_7) \sin \left(\arcsin \left(\frac{2\phi_2 - \phi_6 - \phi_7}{\phi_6 - \phi_7} \right) \right. \right. \\ &\quad \left. \left. - \frac{\sqrt{-(\phi_2 - \phi_6)(\phi_2 - \phi_7)}}{2} \right) \right. \\ &\quad \left. \times |x - ct| \right) - 4\phi_2 \Big)^{-1}, \end{aligned} \tag{19}$$

where ϕ_2, ϕ_6 , and ϕ_7 will be given in the proof of the theorem.

(3) When $G = G_0 + c$, one obtains four singular solutions as follows:

$$\begin{aligned} \phi_{8^\pm}(x, t) &= \pm \sqrt{\frac{c^2 + 2g + 2}{3}} \\ &\times \frac{\left(1 + \sqrt{(c^2 + 2g + 2)/3} |x - ct| \right)^2 + 3}{\left(1 + \sqrt{(c^2 + 2g + 2)/3} |x - ct| \right)^2 - 1}, \end{aligned}$$

$$\begin{aligned} \phi_{9\pm}(x, t) &= \pm \sqrt{\frac{c^2 + 2g + 2}{3}} \\ &\times \frac{(c^2 + 2g + 2)(x - ct)^2 + 9}{(c^2 + 2g + 2)(x - ct)^2 - 3}. \end{aligned} \tag{20}$$

Proof. (1) When $G = c$, we consider the following two kinds of orbits. \square

(i) First, we see that there are two heteroclinic orbits Γ_1^\pm connected as two saddle points $(\phi_0, 0)$ and $(-\phi_0, 0)$ from Figure 1. In (ϕ, y) -plane, from (8), the expressions of the heteroclinic orbits are given as

$$y = \pm \frac{1}{2} (\phi_0^2 - \phi^2). \tag{21}$$

Substituting (21) into the first equation of system (7) and integrating along the heteroclinic orbits, it follows that

$$\begin{aligned} \int_0^\phi \frac{ds}{(\phi_0^2 - s^2)} &= \frac{1}{2} |\xi|, \\ \int_\phi^{+\infty} \frac{ds}{(s^2 - \phi_0^2)} &= \frac{1}{2} |\xi|. \end{aligned} \tag{22}$$

From (22), we have

$$\begin{aligned} \phi &= \pm \sqrt{c^2 + 2g + 2} \tanh\left(\frac{\sqrt{c^2 + 2g + 2}}{2} \xi\right), \\ \phi &= \pm \sqrt{c^2 + 2g + 2} \coth\left(\frac{\sqrt{c^2 + 2g + 2}}{2} \xi\right). \end{aligned} \tag{23}$$

Noting that $\phi = \phi(\xi)$ and $\xi = x - ct$, we get two kink-shaped solutions $\phi_{1\pm}(x, t)$ and two singular solutions $\phi_{2\pm}(x, t)$ as (14) and (15).

(ii) Second, from the phase portrait in Figure 1, we note that there are two special orbits Γ_2^\pm , which have the same Hamiltonian as that of the center point $(0, 0)$. In (ϕ, y) -plane, from (8), the expressions of these two orbits are given as

$$y = \pm \frac{1}{2} \phi \sqrt{\phi^2 - 2\phi_0^2}. \tag{24}$$

Substituting (24) into the first equation of system (7) and integrating along the two orbits Γ_2^\pm , it follows that

$$\int_\phi^{+\infty} \frac{ds}{s\sqrt{s^2 - 2\phi_0^2}} = \frac{1}{2} |\xi|. \tag{25}$$

From (25), we have

$$\phi = \pm \sqrt{2(c^2 + 2g + 2)} \operatorname{sec}\left(\sqrt{\frac{c^2 + 2g + 2}{2}} \xi\right). \tag{26}$$

At the same time, we note that if $\phi = \phi(\xi)$ is a solution of system, then $\phi = \phi(\xi + \gamma)$ is also a solution of system. Specially, we take $\gamma = \pi/2$; we obtain another two solutions

$$\phi = \pm \sqrt{2(c^2 + 2g + 2)} \operatorname{csc}\left(\sqrt{\frac{c^2 + 2g + 2}{2}} \xi\right). \tag{27}$$

Noting that $\phi = \phi(\xi)$ and $\xi = x - ct$, we get four periodic singular solutions $\phi_{3\pm}(x, t)$ and $\phi_{4\pm}(x, t)$ as (16).

(2) When $c < G < G_0 + c$, we set the largest solution of $f(\phi) = 0$ to be $\phi_1 = \sqrt{(c^2 + 2g + 2)/\alpha}$ ($1 < \alpha < 3$), and then we can obtain another two solutions of $f(\phi) = 0$ as follows:

$$\begin{aligned} \phi_2 &= \frac{1}{2} \sqrt{\frac{c^2 + 2g + 2}{\alpha}} (\sqrt{4\alpha - 3} - 1), \\ \phi_3 &= -\frac{1}{2} \sqrt{\frac{c^2 + 2g + 2}{\alpha}} (\sqrt{4\alpha - 3} + 1). \end{aligned} \tag{28}$$

(i) First, we see that there is a homoclinic orbit Γ_3 , which passes the saddle point $(\phi_1, 0)$. In (ϕ, y) -plane, from (8), the expressions of the homoclinic orbit are given as

$$y = \pm \frac{1}{2} \sqrt{(\phi_1 - \phi)^2 (\phi - \phi_4) (\phi - \phi_5)}, \tag{29}$$

where

$$\begin{aligned} \phi_4 &= \sqrt{\frac{c^2 + 2g + 2}{\alpha}} (\sqrt{2\alpha - 2} - 1), \\ \phi_5 &= -\sqrt{\frac{c^2 + 2g + 2}{\alpha}} (\sqrt{2\alpha - 2} + 1). \end{aligned} \tag{30}$$

Substituting (29) into the first equation of system (7) and integrating along the homoclinic orbit, it follows that

$$\begin{aligned} \int_{\phi_4}^\phi \frac{ds}{(\phi_1 - s) \sqrt{(s - \phi_4)(s - \phi_5)}} &= \frac{1}{2} |\xi|, \\ \int_\phi^{+\infty} \frac{ds}{(s - \phi_1) \sqrt{(s - \phi_4)(s - \phi_5)}} &= \frac{1}{2} |\xi|. \end{aligned} \tag{31}$$

From (31), we have

$$\begin{aligned} \phi &= \pm \sqrt{\frac{c^2 + 2g + 2}{\alpha}} \\ &\times \left(\sqrt{2\alpha - 2} \cosh\left(\sqrt{\frac{c^2 + 2g + 2}{\alpha}} \frac{(3 - \alpha)}{2} \xi\right) \right. \\ &\quad \left. + 2\alpha - 4 \right) \end{aligned}$$

$$\begin{aligned} & \times \left(\sqrt{2\alpha - 2} \cosh \left(\sqrt{\frac{c^2 + 2g + 2(3 - \alpha)}{\alpha}} \frac{\xi}{2} \right) + 2 \right)^{-1}, \\ \phi = & \pm \sqrt{\frac{c^2 + 2g + 2}{\alpha}} \\ & \times \left(2 \cosh \left(\sqrt{\frac{c^2 + 2g + 2(3 - \alpha)}{\alpha}} \frac{\xi}{2} \right) \right. \\ & + \sqrt{6 - 2\alpha} \sinh \left(\sqrt{\frac{c^2 + 2g + 2(3 - \alpha)}{\alpha}} \frac{|\xi|}{2} \right) \\ & \left. + 4 - 2\alpha \right) \\ & \times \left(2 \cosh \left(\sqrt{\frac{c^2 + 2g + 2(3 - \alpha)}{\alpha}} \frac{\xi}{2} \right) \right. \\ & + \sqrt{6 - 2\alpha} \sinh \left(\sqrt{\frac{c^2 + 2g + 2(3 - \alpha)}{\alpha}} \frac{|\xi|}{2} \right) \\ & \left. - 2 \right)^{-1}. \end{aligned} \tag{32}$$

Noting that $\phi = \phi(\xi)$ and $\xi = x - ct$, we get two solitary solutions $\phi_{5\pm}(x, t)$ and two singular solutions $\phi_{6\pm}(x, t)$ as (17) and (18).

(ii) Second, from the phase portrait in Figure 1, we note that there are another two special orbits Γ_4^\pm , which have the same Hamiltonian as that of the center point $(\phi_2, 0)$. In (ϕ, y) -plane, from (8), the expressions of these two orbits are given as

$$y = \pm \frac{1}{2} \sqrt{(\phi - \phi_2)^2 (\phi - \phi_6) (\phi - \phi_7)}, \tag{33}$$

where

$$\begin{aligned} \phi_6 &= -\phi_2 + \sqrt{2(c^2 + 2g + 2) - 2\phi_2^2}, \\ \phi_7 &= -\phi_2 - \sqrt{2(c^2 + 2g + 2) - 2\phi_2^2}. \end{aligned} \tag{34}$$

Substituting (33) into the first equation of system (7) and integrating along these two special orbits Γ_4^\pm , it follows that

$$\int_\phi^{+\infty} \frac{ds}{(s - \phi_2) \sqrt{(s - \phi_6)(s - \phi_7)}} = \frac{1}{2} |\xi|. \tag{35}$$

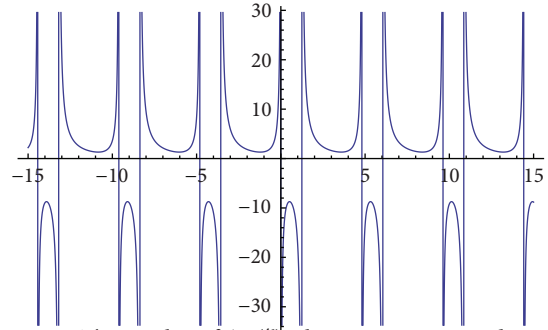


FIGURE 2: The graphic of $\phi_{8^+}(\xi)$ when $c = 2, g = 1,$ and $\alpha = 2.$

From (35), we have

$$\begin{aligned} \phi = & \pm \left(\phi_2 (\phi_6 - \phi_7) \sin \left(\arcsin \left(\frac{2\phi_2 - \phi_6 - \phi_7}{\phi_6 - \phi_7} \right) \right. \right. \\ & \left. \left. - \frac{\sqrt{-(\phi_2 - \phi_6)(\phi_2 - \phi_7)}}{2} |\xi| \right) \right. \\ & \left. + 2\phi_6\phi_7 - \phi_2\phi_6 - \phi_2\phi_7 \right) \\ & \times \left((\phi_6 - \phi_7) \sin \left(\arcsin \left(\frac{2\phi_2 - \phi_6 - \phi_7}{\phi_6 - \phi_7} \right) \right. \right. \\ & \left. \left. - \frac{\sqrt{-(\phi_2 - \phi_6)(\phi_2 - \phi_7)}}{2} |\xi| \right) \right. \\ & \left. - 4\phi_2 \right)^{-1}. \end{aligned} \tag{36}$$

Noting that $\phi = \phi(\xi)$ and $\xi = x - ct$, we get two periodic singular solutions $\phi_{7\pm}(x, t)$ as (19).

To illustrate, we give the graphic of $\phi_{8^+}(\xi)$ in Figure 2 by taking $c = 2, g = 1,$ and $\alpha = 2.$

(3) When $G = G_0 + c$, from the phase portrait in Figure 1, we note that there are two orbits Γ_5^\pm , which have the same Hamiltonian as the degenerate saddle point $(\phi_8, 0)$. In (ϕ, y) -plane, from (8), the expressions of these two orbits are given as

$$y = \pm \frac{1}{2} \sqrt{(\phi - \phi_8)^3 (\phi + 3\phi_8)}, \tag{37}$$

where

$$\phi_8 = \sqrt{\frac{1}{3}(c^2 + 2g + 2)}. \tag{38}$$

Substituting (37) into the first equation of system (7) and integrating along these two orbits Γ_5^\pm , it follows that

$$\int_\phi^{+\infty} \frac{ds}{(s - \phi_8) \sqrt{(s - \phi_8)(s + 3\phi_8)}} = \frac{1}{2} |\xi|,$$

$$\int_{\phi}^{-3\phi_8} \frac{ds}{(\phi_8 - s) \sqrt{(\phi_8 - s)(-3\phi_8 - s)}} = \frac{1}{2} |\xi|. \tag{39}$$

From (3), we have

$$\phi = \pm \sqrt{\frac{c^2 + 2g + 2}{3} \frac{\left(1 + \sqrt{(c^2 + 2g + 2)/3} |\xi|\right)^2 + 3}{\left(1 + \sqrt{(c^2 + 2g + 2)/3} |\xi|\right)^2 - 1}}, \tag{40}$$

$$\phi = \pm \sqrt{\frac{c^2 + 2g + 2}{3} \frac{(c^2 + 2g + 2)(x - ct)^2 + 9}{(c^2 + 2g + 2)(x - ct)^2 - 3}}.$$

Noting that $\phi = \phi(\xi)$ and $\xi = x - ct$, we get four singular solutions $\phi_{8\pm}(x, t)$ and $\phi_{9\pm}(x, t)$ as (20).

Remark 2. One may find that we only consider some special orbits in Figure 1 when $G \geq c$. In fact, we may obtain exactly the same results when $G < c$.

Remark 3. We employ the software Mathematica to check the correctness of the above nonlinear wave solutions. To illustrate, we show the commands of verifying $\phi_{8\pm}(x, t)$,

$$\begin{aligned} \phi_2 &= \frac{1}{2} \sqrt{\frac{c^2 + 2g + 2}{\alpha}} (\sqrt{4\alpha - 3} - 1), \\ \phi_6 &= -\phi_2 + \sqrt{2(c^2 + 2g + 2) - 2(\phi_2)^2}, \\ \phi_7 &= -\phi_2 - \sqrt{2(c^2 + 2g + 2) - 2(\phi_2)^2}, \\ \phi &= \left(\phi_2 (\phi_6 - \phi_7) \sin \left[\text{ArcSin} \left[\frac{2\phi_2 - \phi_6 - \phi_7}{\phi_6 - \phi_7} \right] \right. \right. \\ &\quad \left. \left. - \frac{\sqrt{-(\phi_2 - \phi_6)(\phi_2 - \phi_7)}}{2} (x - ct) \right] \right. \\ &\quad \left. + 2\phi_6\phi_7 - \phi_2\phi_6 - \phi_2\phi_7 \right) \\ &\times \left((\phi_6 - \phi_7) \sin \left[\text{ArcSin} \left[\frac{2\phi_2 - \phi_6 - \phi_7}{\phi_6 - \phi_7} \right] \right. \right. \\ &\quad \left. \left. - \frac{\sqrt{-(\phi_2 - \phi_6)(\phi_2 - \phi_7)}}{2} (x - ct) \right] \right. \\ &\quad \left. - 4\phi_2 \right)^{-1}, \end{aligned}$$

$$\varphi = \phi + c,$$

$$\psi = g + c\varphi - \frac{1}{2}\varphi^2,$$

$$\text{Simplify} \left[\partial_t \varphi + \partial_x \psi + \frac{1}{2} \partial_x (\varphi^2) \right],$$

$$\text{Simplify} \left[\partial_t \psi + \partial_x (\varphi \psi + \varphi + \partial_{x,x} \varphi) \right]. \tag{41}$$

4. Conclusions

In this paper, by employing the bifurcation method and qualitative theory of dynamical systems, we study the nonlinear wave solutions for the Broer-Kaup equation (1) and obtain exact explicit expressions of the various kinds of nonlinear wave solutions, which include solitary wave solutions, singular solutions, periodic singular solutions, and kink-shaped solutions. To the best of our knowledge, most of the nonlinear wave solutions are newly obtained.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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