Supplementary material for: A family of trigonometrically-fitted Enright second derivative methods for stiff and oscillatory initial value problems

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Appendix 1

1.A: Defining variables in equation (17)

$N_{\beta 0} = -(\csc(\frac{u}{2})^4(696 + 54u^2 + 114(-8 + u^2)\cos u - 6(-44 + 5u^2)\cos(2u) - 48\cos(3u) + 6u^2\cos(3u) - 433u\sin u + 116u\sin(2u) - 29u\sin(3u)))$

$N_{\beta 1} = \csc(\frac{u}{2})^4(2544 + 222u^2 + 48(-65 + 9u^2)\cos u + (576 - 84u^2)\cos(2u) + 48\cos(3u) - 48\cos(4u) + 6u^2\cos(4u) - 1716u\sin u + 370u\sin(2u) - 20u\sin(3u) - 29u\sin(4u))$

$N_{\beta 2} = -(\csc(\frac{u}{2})^4(1548 + 177u^2 + 6(-258 + 47u^2)\cos u - 288\cos(2u) + 396\cos(3u) - 42u^2\cos(3u) - 108\cos(4u) + 15u^2\cos(4u) - 1232u\sin u + 104u\sin(2u) + 144u\sin(3u) - 68u\sin(4u)))$

$N_{\beta 3} = \csc(\frac{u}{2})^4(624 + 330u^2 + 1104\cos u + 12(-272 + 47u^2)\cos(2u) + 1968\cos(3u) - 432u^2\cos(3u) - 432\cos(4u) + 114u^2\cos(4u) - 348u\sin u - 1682u\sin(2u) + 1348u\sin(3u) - 371u\sin(4u))$

$N_{\beta 4} = \csc(\frac{u}{2})^4(624 - 6(332 + 55u^2)\cos u + 6(396 + 59u^2)\cos(2u) - 1272\cos(3u) - 222u^2\cos(3u) + 264\cos(4u) + 54u^2\cos(4u) + 2143u\sin u - 1484u\sin(2u) + 371u\sin(3u))$

$N_{\gamma n+4} = -3\sin(\frac{u}{2})(9u\cos(\frac{u}{2}) + 3u\cos(\frac{3u}{2}) - 8\sin(\frac{3u}{2}))$

1.B: Defining variables in equation (18)

$N_{\hat{\beta}_{0,0}} = (\csc(\frac{u}{2})^4(-144 - 18u^2 + 2(88 + 45u^2)\cos u - 2(44 + 27u^2)\cos(2u) + 144\cos(3u) + 126u^2\cos(3u) - 88\cos(4u) - 261u\sin u + 420u\sin(2u) - 225u\sin(3u) - 48u\sin(4u)))$

$N_{\hat{\beta}_{1,0}} = -(\csc(\frac{u}{2})^4(-496 + 54u^2 - 48(-11 + 3u^2)\cos u + 12(-16 + 45u^2)\cos(2u) + 496\cos(3u) - 336\cos(4u) + 126u^2\cos(4u) + 1596u\sin u - 1062u\sin(2u) + 348u\sin(3u) - 417u\sin(4u)))$

$N_{\hat{\beta}_{2,0}} = (3\csc(\frac{u}{2})^4(-84 - 9u^2 + (44 + 54u^2)\cos u + 32\cos(2u) + 84\cos(3u) + 90u^2\cos(3u) - 76\cos(4u) + 9u^2\cos(4u) - 48u\sin u + 216u\sin(2u) - 144u\sin(3u) - 60u\sin(4u)))$

$N_{\hat{\beta}_{3,0}} = -(\csc(\frac{u}{2})^4(144 + 18u^2 - 880\cos u + 4(272 + 81u^2)\cos(2u) - 144\cos(3u) + 144u^2\cos(3u) - 208\cos(4u) + 90u^2\cos(4u) + 1140u\sin u - 378u\sin(2u) - 108u\sin(3u) - 303u\sin(4u)))$

$N_{\hat{\beta}_{4,0}} = (\csc(\frac{u}{2})^4(296 + 18(-44 + u^2)\cos u + 18(44 + 3u^2)\cos(2u) - 296\cos(3u) + 54u^2\cos(3u) + 18u^2\cos(4u) + 309u\sin u - 36u\sin(2u) - 111u\sin(3u) - 48u\sin(4u)))$

$N_{\hat{\gamma}_{n+4,0}} = -3\sin(\frac{u}{2})(9u\cos(\frac{u}{2}) + 3u\cos(\frac{3u}{2}) - 8\sin(\frac{3u}{2}))$

1.C: Defining variables in equation (19)

$N_{\hat{\beta}_{0,1}} = (\csc(\frac{u}{2})^4(-48 + (33 + 6u^2)\cos u + 24(2 + u^2)\cos(2u) - 33\cos(3u) + 6u^2\cos(3u) + 71u\sin u - 28u\sin(2u) - 29u\sin(3u)))$

$N_{\hat{\beta}_{1,1}} = -(\csc(\frac{u}{2})^4(-192 + 6u^2 + 132\cos u + 12(16 + 11u^2)\cos(2u) - 132\cos(3u) + 6u^2\cos(4u) + 408u\sin u - 242u\sin(2u) - 56u\sin(3u) - 11u\sin(4u)))$

$N_{\hat{\beta}_{2,1}} = (\csc(\frac{u}{2})^4(-144 - 12u^2 + 33(3 + 2u^2)\cos u + 144\cos(2u) - 99\cos(3u) + 66u^2\cos(3u) - 12u^2\cos(4u) - 35u\sin u + 176u\sin(2u) - 207u\sin(3u) + 22u\sin(4u)))$

$N_{\hat{\beta}_{3,1}} = -(\csc(\frac{u}{2})^4(-192 + 6u^2 + 132\cos u + 12(16 + 11u^2)\cos(2u) - 132\cos(3u) + 6u^2\cos(4u) + 408u\sin u - 242u\sin(2u) - 56u\sin(3u) - 11u\sin(4u)))$

$N_{\hat{\beta}_{4,1}} = (\csc(\frac{u}{2})^4(-48 + (33 + 6u^2)\cos u + 24(2 + u^2)\cos(2u) - 33\cos(3u) + 6u^2\cos(3u) + 71u\sin u - 28u\sin(2u) - 29u\sin(3u)))$

1.D: Defining variables in equation (20)

$N_{\hat{\beta}_{0,2}} = -(\csc(\frac{u}{2})^4(648 + 6u^2 - 6(160 + 13u^2)\cos u - 78(-4 + u^2)\cos(2u) + 6u^2\cos(3u) - 353u\sin u + 340u\sin(2u) - 13u\sin(3u)))$

$N_{\hat{\beta}_{1,2}} = (\csc(\frac{u}{2})^4(2112 + 30u^2 - 48(65 + 7u^2)\cos u + (960 - 276u^2)\cos(2u) + 48\cos(3u) + 6u^2\cos(4u) - 1140u\sin u + 1106u\sin(2u) + 44u\sin(3u) - 13u\sin(4u)))$

$N_{\hat{\beta}_{2,2}} = -(\csc(\frac{u}{2})^4(864 + 57u^2 - 30(42 + 13u^2)\cos u + 288\cos(2u) + 108\cos(3u) - 138u^2\cos(3u) + 39u^2\cos(4u) + 80u\sin u - 104u\sin(2u) + 432u\sin(3u) - 76u\sin(4u)))$

$N_{\hat{\beta}_{3,2}} = (\csc(\frac{u}{2})^4(-1728 - 54u^2 + 2640\cos u - 12(112 + 65u^2)\cos(2u) + 432\cos(3u) + 336u^2\cos(3u) - 78u^2\cos(4u) - 1692u\sin u + 1934u\sin(2u) - 508u\sin(3u) + 125u\sin(4u)))$

$N_{\hat{\beta}_{4,2}} = (\csc(\frac{u}{2})^4(1992 + 6(-500 + 9u^2)\cos u + 6(212 + 19u^2)\cos(2u) - 264\cos(3u) - 30u^2\cos(3u) + 6u^2\cos(4u) - 337u\sin u + 212u\sin(2u) - 125u\sin(3u)))$

$N_{\hat{\gamma}_{n+4,2}} = -\sin(\frac{u}{2})(-13u\cos(\frac{u}{2}) + u\cos(\frac{3u}{2}) + 24\sin(\frac{u}{2}))$

Appendix 2: Defining the stability function ($M(q;u)$)

$$\textit{Case}\quad k=1: \quad M(q;u) = \frac{1 - \frac{((\csc(u/2)^2)(-u+\sin u)q)}{(2u)}}{1 + \frac{(\csc(u/2)^2)(u\cos u - \sin u)q}{(2u)} - \frac{(-2+uCot(u/2))(q^2)}{u^2}}$$

$$\textit{Case}\quad k=2: \quad M(q;u) \equiv \frac{\mathcal{P}(q;u)}{\mathcal{Q}(q;u)}, \quad \text{with}$$

$\mathcal{P}(q;u) = -(2q^2u + 2u^3 + 3qu^3 + 2q^2u^3 - 4u(q^2 + u^2 + qu^2)\cos u + u(2q^2 + 2u^2 + qu^2)\cos(2u) + 4q^2\sin u + 4u^2\sin u - 4q^2u^2\sin u - 2q^2\sin(2u) - 2u^2\sin(2u) + q^2u^2\sin(2u))$

$\mathcal{Q}(q;u) = (-2q^2u + 2q^3u - 2u^3 + qu^3 - 4u(-q^2 + 2q^3 - u^2 + qu^2)\cos u + u(6q^3 - 2u^2 + 3qu^2 - 2q^2(1 + u^2))\cos(2u) - 4q^2\sin u + 8q^3\sin u - 4u^2\sin u + 8qu^2\sin u - 4q^2u^2\sin u + 2q^2\sin(2u) - 4q^3\sin(2u) + 2u^2\sin(2u) - 4qu^2\sin(2u) + 3q^2u^2\sin(2u) + 2q^3u^2\sin(2u))$

$$\textit{Case}\quad k=3: \quad M(q;u) \equiv \frac{\mathcal{P}(q;u)}{\mathcal{Q}(q;u)}, \quad \text{with}$$

$\mathcal{P}(q;u) = -(2(6q^2u+12q^3u+6u^3+12qu^3+11q^2u^3+6q^3u^3-6u(5q^3+3u^2+5qu^2+3q^2(1+u^2))\cos u+3u(8q^3+6u^2+8qu^2+3q^2(2+u^2))\cos(2u)-6q^2u\cos(3u)-6q^3u\cos(3u)-6u^3\cos(3u)-6qu^3\cos(3u)-2q^2u^3\cos(3u)+45q^2\sin u+47q^3\sin u+45u^2\sin u+47qu^2\sin u-18q^3u^2\sin u-36q^2\sin(2u)-34q^3\sin(2u)-36u^2\sin(2u)-34qu^2\sin(2u)+9q^3u^2\sin(2u)+9q^2\sin(3u)+7q^3\sin(3u)+9u^2\sin(3u)+7qu^2\sin(3u)-2q^3u^2\sin(3u)))$

$\mathcal{Q}(q;u) = (-12q^2u+12q^3u-8q^4u-12u^3+12qu^3-4q^2u^3+6u(-8q^3+6q^4+6u^2-8qu^2+3q^2(2+u^2))\cos u-12u(-5q^3+6q^4+3u^2-5qu^2+3q^2(1+u^2))\cos[2u]+12q^2u\cos(3u)-24q^3u\cos(3u)+44q^4u\cos(3u)+12u^3\cos(3u)-24qu^3\cos(3u)+22q^2u^3\cos(3u)-12q^3u^3\cos(3u)-90q^2\sin u+176q^3\sin u-123q^4\sin u-90u^2\sin u+176qu^2\sin u-123q^2u^2\sin u+18q^3u^2\sin u+72q^2\sin(2u)-148q^3\sin(2u)+120q^4\sin(2u)+72u^2\sin(2u)-148qu^2\sin(2u)+120q^2u^2\sin(2u)-36q^3u^2\sin(2u)-18q^2\sin(3u)+40q^3\sin(3u)-39q^4\sin(3u)-18u^2\sin(3u)+40qu^2\sin(3u)-39q^2u^2\sin(3u)+22q^3u^2\sin(3u)+12q^4u^2\sin(3u));$

$Case\quad k=4: \quad M(q;u) \equiv \dfrac{\mathcal{P}(q;u)}{\mathcal{Q}(q;u)}, \quad \text{with}$

$\mathcal{P}(q;u) = -(72q^2u+180q^3u+210q^4u+72u^3+180qu^3+210q^2u^3+150q^3u^3+72q^4u^3-24u(26q^4+12u^2+27qu^2+3q^3(9+4u^2)+2q^2(6+13u^2))\cos u+36u(19q^4+12u^2+24qu^2+6q^3(4+u^2)+q^2(12+19u^2))\cos(2u)-288q^2u\cos(3u)-504q^3u\cos(3u)-336q^4u\cos(3u)-288u^3\cos(3u)-504qu^3\cos(3u)-336q^2u^3\cos(3u)-96q^3u^3\cos(3u)+72q^2u\cos(4u)+108q^3u\cos(4u)+66q^4u\cos(4u)+72u^3\cos(4u)+108qu^3\cos(4u)+66q^2u^3\cos(4u)+18q^3u^3\cos(4u)+1488q^2\sin u+2520q^3\sin u+1424q^4\sin u+1488u^2\sin u+2520qu^2\sin u+1424q^2u^2\sin u-288q^4u^2\sin u-1560q^2\sin(2u)-2520q^3\sin(2u)-1334q^4\sin(2u)-1560u^2\sin(2u)-2520qu^2\sin(2u)-1334q^2u^2\sin(2u)+216q^4u^2\sin(2u)+720q^2\sin(3u)+1080q^3\sin(3u)+528q^4\sin(3u)+720u^2\sin(3u)+1080qu^2\sin(3u)+528q^2u^2\sin(3u)-96q^4u^2\sin(3u)-132q^2\sin(4u)-180q^3\sin(4u)-85q^4\sin(4u)-132u^2\sin(4u)-180qu^2\sin(4u)-85q^2u^2\sin(4u)+18q^4u^2\sin(4u))$

$\mathcal{Q}(q;u) = (-72q^2u+108q^3u-66q^4u+36q^5u-72u^3+108qu^3-66q^2u^3+18q^3u^3-24u(-14q^4+8q^5-12u^2+21qu^2+q^3(21+4u^2)-2q^2(6+7u^2))\cos u+36u(-19q^4+12q^5-12u^2+24qu^2+6q^3(4+u^2)-q^2(12+19u^2))\cos(2u)+288q^2u\cos(3u)-648q^3u\cos(3u)+624q^4u\cos(3u)-576q^5u\cos(3u)+288u^3\cos(3u)-648qu^3\cos(3u)+624q^2u^3\cos(3u)-288q^3u^3\cos(3u)-72q^2u\cos(4u)+180q^3u\cos(4u)-210q^4u\cos(4u)+300q^5u\cos(4u)-72u^3\cos(4u)+180qu^3\cos(4u)-210q^2u^3\cos(4u)+150q^3u^3\cos(4u)-72q^4u^3\cos(4u)-1488q^2\sin u+3432q^3\sin u-3248q^4\sin u+1408q^5\sin u-1488u^2\sin u+3432qu^2\sin u-3248q^2u^2\sin u+1408q^3u^2\sin u-96q^4u^2\sin u+1560q^2\sin(2u)-3720q^3\sin(2u)+3734q^4\sin(2u)-1816q^5\sin(2u)+1560u^2\sin(2u)-3720qu^2\sin(2u)+3734q^2u^2\sin(2u)-1816q^3u^2\sin(2u)+216q^4u^2\sin(2u)-720q^2\sin(3u)+1800q^3\sin(3u)-1968q^4\sin(3u)+1152q^5\sin(3u)-720u^2\sin(3u)+1800qu^2\sin(3u)-1968q^2u^2\sin(3u)+1152q^3u^2\sin(3u)-288q^4u^2\sin(3u)+132q^2\sin(4u)-348q^3\sin(4u)+421q^4\sin(4u)-308q^5\sin(4u)+132u^2\sin(4u)-348qu^2\sin(4u)+421q^2u^2\sin(4u)-308q^3u^2\sin(4u)+150q^4u^2\sin(4u)+72q^5u^2\sin(4u));$

Appendix 3: Defining the predictor variables in equation (26)

$\bar{\beta}_0 = (\csc(\tfrac{u}{2}))^4(18-23u^2-18\cos u+4(100u^2-6(1+16u^2))\cos u+24\cos(2u)+(-245u^2+3(2+80u^2))\cos(2u)-6\cos(3u)+32u\sin u-4u\sin(2u)))/(96u(u+2u\cos u-3\sin u))$

$\bar{\beta}_1 = -\csc(\tfrac{u}{2})^4(3(155u^2-9(2+16u^2))\cos u+(-245u^2+3(2+80u^2))\cos(3u)+2(24-32u^2-24\cos u+27\cos(2u)-3\cos(4u)+44u\sin u-u\sin(2u)-2u\sin(3u)))/(96u(u+2u\cos u-3\sin u))$

$\bar{\beta}_2 = \csc(\tfrac{u}{2})^4(30-53u^2-30\cos u+3(155u^2-9(2+16u^2))\cos(2u)+54\cos(3u)-4(100u^2-6(1+16u^2))\cos(3u)-24\cos(4u)+34u\sin u+52u\sin(2u)-22u\sin(3u))/(96u(u+2u\cos u-3\sin u))$

$\bar{\beta}_3 = -\csc(\tfrac{u}{2})^4((-101u^2+3(10+16u^2))\cos u-30\cos(2u)-4(-64u^2+3(4+16u^2))\cos(2u)+66\cos(3u)-23u^2\cos(3u)-18\cos(4u)+68u\sin u-22u\sin(2u))/(96u(u+2u\cos u-3\sin u))$

$\bar{\gamma}_{n+3} = (\csc(\tfrac{u}{2}))^4(-12+48\cos u-72\cos(2u)+48\cos(3u)-12\cos(4u)-91u\sin u+80u\sin(2u)-23u\sin(3u)))/(96u(u+2u\cos u-3\sin u))$