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Research Article

Derivation of Asymptotic Dynamical Systems with Partial Lie Symmetry Groups

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Lie group analysis has been applied to singular perturbation problems in both ordinary differential and difference equations and has allowed us to find the reduced dynamics describing the asymptotic behavior of the dynamical system. The present study provides an extended method that is also applicable to partial differential equations. The main characteristic of the extended method is the restriction of the manifold by some constraint equations on which we search for a Lie symmetry group. This extension makes it possible to find a partial Lie symmetry group, which leads to a reduced dynamics describing the asymptotic behavior.

1. Introduction

Appropriate reduction of a dynamical system is one of the important approximation approaches when the system is too complicated to be solved analytically. The derivation of a reduced dynamical system which describes the asymptotic behavior of the solution has been achieved by using various singular perturbation methods developed for ordinary differential equations, partial differential equations, and difference equations [1–11]. On the other hand, the Lie group analysis, a systematic method to construct solutions with Lie symmetry groups that leave the manifold formed by the dynamical system invariant [12], has been extended in order to obtain approximation solutions or reduced dynamics by considering approximate symmetries [13–19], asymptotic symmetries [20–25], and renormalization group symmetries [26–30].

While the above singular perturbation methods and the Lie symmetry group analysis had been developed independently, relations between them have been studied recently. It has been shown that the asymptotic dynamics derived with the singular perturbation methods can also be derived with the Lie group analysis when it is applied to singular perturbation problems of ordinary differential equations or difference equations [31–33]. In other words, the Lie group analysis plays also a role of a singular perturbation method. While the method works for ordinary differential equations and difference equations, it has not been clear if it works or not also for partial differential equations. The straightforward application of the method does not work because, due to the increase of independent variables in the case of partial differential equations, systems have no Lie symmetry group suitable for the construction of asymptotic dynamics.

This paper reports that, by extending the method appropriately, we can reduce partial differential equations to asymptotic dynamics. The main characteristic of the extension is to add some constraint equations to the equations originally considered in order to find partial Lie symmetry groups. The extended method is applied to one ordinary differential equation and two partial differential equations in the paper. While examples taken up here are quite simple in order to make it easy to catch the essence of the procedure, this method works for all the same types of singular perturbation problems in which secular terms are included in the naïve expansion of the solution.

2. Outline of the Extended Method

The singular perturbation method with the Lie group analysis [31–33] is briefly reviewed. For the perturbed dynamical system under consideration here, referred to as the original system in what follows, we find an approximate Lie symmetry

group which includes the transformation of the perturbation parameter that is a constant in terms of the dynamics. An approximate solution is generated from the nonperturbative solution by the transformation of the group; in other words, the approximate solution is given by the group-invariant solution. The differential equation, which the group-invariant solution satisfies, provides a reduced system describing the asymptotic behavior of the solution when we consider its long-time dynamics. It should be noted that another method presented by Gazizov and Khalique [19] also considers transformations of the perturbation parameter while the type of the transformation is different from that presented here. With their method, the regular expansion of the solution of the original system is successfully derived.

The method is extended in this study. The procedure is briefly summarized as follows. (1) Specify beforehand a nonperturbative solution on which we would like to see the effect of the perturbation. (2) Construct equations composed of the independent and dependent variables and their derivatives based on the nonperturbative solution specified in step (1). Those equations are referred to as constraint equations in what follows. (3) Find an approximate Lie symmetry group of the original equations on the manifold formed by the constraint equations constructed in the previous step. (4) For the approximate Lie symmetry group found in step (3), construct the differential equation that is satisfied by the group-invariant solution and make an appropriate approximation when necessary. As a consequence, this differential equation generates the reduced system, which describes the asymptotic behavior of the original system.

The procedure is almost the same as that presented before for ordinary differential equations and difference equations [31–33]. The main difference is in the second step, that is to say, the addition of the constraint equations for finding an approximate Lie symmetry group. In general, additional Lie symmetry groups can be found by the addition of such a constraint equation, because the groups do not need to be admitted by the entire manifold formed by the original system in the jet space but need to be admitted only by that part of the manifold restricted by the added equations. In this sense, a Lie symmetry group found by using this procedure is interpreted as a partial Lie symmetry group [24]. Because of the increase in the number of the admitted Lie symmetry groups, in some cases, this method allows us to find a Lie symmetry group when it is impossible to find such a group with the normal Lie group analysis.

3. Application to Ordinary Differential Equation

The method presented in this study is particularly meaningful when it is applied to partial differential equations. However, first, we apply the method to an ordinary differential equation, because the calculation is simpler and therefore serves to clarify the essence of the method.

Consider the following ordinary differential equation [34]:

$$2u\ddot{u} + \dot{u}^2 = \varepsilon. \tag{1}$$

Here, u denotes the dependent variable, the overdot denotes the derivative with respect to the independent variable t, and ε denotes a perturbation parameter supposed to be small sufficiently.

The two independent solutions of the nonperturbative equation, $2u\ddot{u} + \dot{u}^2 = 0$, are

$$u^{(0)} = At^{2/3}, B, \tag{2}$$

where A and B are constants.

Focusing on one of these solutions, $u^{(0)} = At^{2/3}$, we investigate the effect of the perturbation on this nonperturbative solution. For this solution,

$$\dot{u}^{(0)} = \frac{2}{3}\frac{u^{(0)}}{t} \tag{3}$$

is satisfied. From this relation, an equation is constructed,

$$\dot{u} = \frac{2}{3}\frac{u}{t}, \tag{4}$$

which is used as a constraint equation in finding a Lie symmetry group.

We find a Lie symmetry group admitted by (1). Let

$$X = \partial_\varepsilon + \eta(t,u)\,\partial_u \tag{5}$$

be the infinitesimal generator of a Lie symmetry group. Its second prolongation X^* is given by

$$\begin{aligned} X^* &= X + \eta^{\dot{u}}(t,u,\dot{u})\,\partial_{\dot{u}} + \eta^{\ddot{u}}(t,u,\dot{u},\ddot{u})\,\partial_{\ddot{u}}, \\ \eta^{\dot{u}} &:= (\partial_t + \dot{u}\partial_u)\,\eta, \\ \eta^{\ddot{u}} &:= (\partial_t + \dot{u}\partial_u + \ddot{u}\partial_{\dot{u}})\,\eta^{\dot{u}}. \end{aligned} \tag{6}$$

When we adopt the normal procedure followed by Lie group analysis [12], we find a Lie symmetry group by solving the differential equation for $\eta(t,u)$,

$$X^*\left[2u\ddot{u} + \dot{u}^2 - \varepsilon\right]\Big|_{2u\ddot{u}+\dot{u}^2=\varepsilon} = 0, \tag{7}$$

which indicates the criterion for the invariance of the entire manifold formed by the original differential equation in the jet space. In the extended method presented in this study, instead, we find a Lie symmetry group by solving

$$X^*\left[2u\ddot{u} + \dot{u}^2 - \varepsilon\right]\Big|_{\dot{u}=(2/3)(u/t)} = 0. \tag{8}$$

Note that $\dot{u} = (2/3)(u/t)$ comes from (4). Then, (8) reads

$$\begin{aligned} &\left[2u\partial_t^2 + \frac{4}{3}\frac{u^2}{t}\partial_t\partial_u + \frac{8}{9}\frac{u^3}{t^2}\partial_u^2 + \frac{4}{3}\frac{u}{t}\partial_t + \frac{4}{9}\frac{u^2}{t^2}\partial_u - \frac{4}{9}\frac{u}{t^2}\right] \\ &\quad\cdot\eta - 1 = 0. \end{aligned} \tag{9}$$

Noting the homogeneity of the equation, we find an exact solution of this differential equation,

$$\eta = -\frac{9}{20}\frac{t^2}{u}. \tag{10}$$

For this Lie symmetry group, the group-invariant solution, $u = u(\varepsilon, t)$, satisfies

$$X\left[u - u(\varepsilon, t)\right]\big|_{u=u(\varepsilon,t)} = 0, \tag{11}$$

which reads

$$\frac{\partial u}{\partial \varepsilon} = -\frac{9}{20}\frac{t^2}{u}. \tag{12}$$

This equation, which can be analytically solved, has the following solution:

$$u(\varepsilon, t) = \pm t^{2/3}\left(A - \frac{9}{10}\varepsilon t^{2/3}\right)^{1/2}. \tag{13}$$

Here, we have used $u(0, t) = At^{2/3}$, which is the nonperturbative solution, as the boundary condition in the integration. This solution corresponds exactly to the solution derived with the renormalization group method [34].

The original differential equation (1) admits no exact Lie symmetry group when we adopt the normal procedure (see the Appendix). However, the addition of constraint equation (4) has allowed us to find a Lie symmetry group.

4. Application to Partial Differential Equation

In this section the present method is applied to two partial differential equations. In general, in the case of partial differential equations, because of the increase of independent variables, it often happens that a partial differential equation admits no approximate Lie symmetry group suitable for the derivation of a system describing the asymptotic dynamics. However, it is in some cases possible to find an approximate Lie symmetry group suitable for deriving the asymptotic dynamic properties by adding some constraint equations constructed from the nonperturbative solution, thereby restricting the manifold on which we find a Lie symmetry group. The reduced systems obtained here are consistent with those derived with the singular perturbation methods presented before.

4.1. First Example: $u_t + A(\varepsilon u)u_x = 0$. Consider a system of weakly nonlinear first-order partial differential equations,

$$u_t + A(\varepsilon u)\, u_x = 0. \tag{14}$$

Here, u is a real vector that has n components and depends on two independent variables t and x. A is an $n\times n$ nondegenerate matrix that depends on u. ε is a perturbation parameter. The subscripts of u denote the derivative with respect to the independent variables.

Similar to the approach followed in the previous section, we find a Lie symmetry group such that it is not admitted by the entire manifold formed by the original equation; rather, it is admitted by a part of the manifold determined from a solution of the nonperturbative system. In the following, we focus on such a nonperturbative solution, $u^{(0)}(t, x)$, that is an arbitrary function of $x - \lambda t$ where λ is an eigenvalue of $A_0 := A(0)$; namely, we set

$$u^{(0)}(t, x) = U(x - \lambda t). \tag{15}$$

This solution represents a traveling wave of which the velocity is λ. As shown by the direct substitution into the nonperturbative system, $u_x^{(0)}$ is an eigenvector of A_0:

$$A_0 u_x^{(0)} = \lambda u_x^{(0)}. \tag{16}$$

In addition,

$$u_t^{(0)} + \lambda u_x^{(0)} = 0 \tag{17}$$

is also satisfied. Based on these relations, we find a Lie symmetry group with constraints of

$$\begin{aligned} A_0 u_x &= \lambda u_x, \\ u_t + \lambda u_x &= 0, \end{aligned} \tag{18}$$

in the following.

Let

$$X = \partial_\varepsilon + \eta(t, x, u; \varepsilon)\,\partial_u \tag{19}$$

be the infinitesimal generator of a Lie symmetry group. Its first prolongation X^* is given by

$$\begin{aligned} X^* &= X + \eta^{u_t}\partial_{u_t} + \eta^{u_x}\partial_{u_x}, \\ \eta^{u_t} &= (\partial_t + u_t\partial_u)\,\eta, \\ \eta^{u_x} &= (\partial_x + u_x\partial_u)\,\eta. \end{aligned} \tag{20}$$

The criterion for the invariance of the original differential equation (14) on the manifold restricted by the constraint equations (18) is given by

$$X^*\left[u_t + A(\varepsilon u)\, u_x\right]\big|_{A_0 u_x = \lambda u_x,\, u_t + \lambda u_x = 0} = 0. \tag{21}$$

This equation reads

$$\begin{aligned} &\left[\eta^{u_t} + A(\varepsilon u)\,\eta^{u_x} + (u\cdot\nabla)\,A(\varepsilon u)\,u_x \right. \\ &\quad \left. + \varepsilon(\eta\cdot\nabla)\,A(\varepsilon u)\,u_x\right]\big|_{A_0 u_x = \lambda u_x,\, u_t + \lambda u_x = 0} = 0. \end{aligned} \tag{22}$$

Here, $(u\cdot\nabla)A(\varepsilon u)$ and $(\eta\cdot\nabla)A(\varepsilon u)$ are defined by matrices of which the ij components are given by, respectively,

$$\begin{aligned} &\left[\sum_k u_k \partial_{r_k} a_{ij}(r)\right]\Bigg|_{r=\varepsilon u}, \\ &\left[\sum_k \eta_k \partial_{r_k} a_{ij}(r)\right]\Bigg|_{r=\varepsilon u}, \end{aligned} \tag{23}$$

where a_{ij} denotes the ij component of the matrix, A.

Suppose $\eta(t, x, u; \varepsilon)$ and $A(\varepsilon u)$ can be expanded as a power series of ε. Their expanded forms,

$$\begin{aligned} \eta &=: \sum_{k=0}^{\infty} \varepsilon^k \eta_k(t, x, u), \\ A(\varepsilon u) &=: \sum_{k=0}^{\infty} \varepsilon^k A_k(u), \end{aligned} \tag{24}$$

are substituted into (22), which is then rewritten as

$$\left[\eta_0^{u_t} + A_0\eta_0^{u_x} + (u\cdot\nabla)A_0u_x\right]\Big|_{A_0u_x=\lambda u_x,\, u_t+\lambda u_x=0} = 0, \tag{25}$$

$$\left[\eta_i^{u_t} + A_0\eta_i^{u_x} + (u\cdot\nabla)A_iu_x + \sum_{j=0}^{i-1}\left(\eta_{i-j-1}\cdot\nabla\right)A_j\Big|_{u=0}u_x\right]\Bigg|_{A_0u_x=\lambda u_x,\, u_t+\lambda u_x=0} = 0, \quad (i=1,2,\ldots). \tag{26}$$

We consider only the lowest order of ε, namely, (25), which is rewritten as

$$\left(\partial_t + u_t\partial_u\right)\eta_0 + A_0\left(\partial_x + u_x\partial_u\right)\eta_0 - \left[(u\cdot\nabla)A_0\right]u_x\big|_{A_0u_x=\lambda u_x,\, u_t+\lambda u_x=0} = 0, \tag{27}$$

and reads

$$\left(\partial_t + A_0\partial_x\right)\eta_0 = \left[-(u\cdot\nabla)\lambda\right]u_x. \tag{28}$$

The vector on the right-hand side of this equation is separated into the following two components:

$$\left(\partial_t + A_0\partial_x\right)\eta_0 = -\left(e_\parallel\cdot\left[(u\cdot\nabla)\lambda\right]u_x\right)e_\parallel - \left(e_\perp\cdot\left[(u\cdot\nabla)\lambda\right]u_x\right)e_\perp, \tag{29}$$

where $e_\parallel$ denotes the unit vector with the same direction as $u_0(t,x)$ in $\mathbb{R}^n$ and $e_\perp$ is the unit vector parallel to the projection of $[-(u\cdot\nabla)\lambda]u_x$ on a $n-1$ dimensional subspace in $\mathbb{R}^n$ spanned by all the eigenvectors of A_0 except $e_\parallel$. The solution corresponding to the second term is given by a linear combination of the $n-1$ eigenvectors of A_0 whose linear coefficients are a function of $x-\lambda t$. On the other hand, a term proportional to t appears in the solution corresponding to the first term, because every vector proportional to $e_\parallel$ whose proportional coefficient is a function of $x-\lambda t$ is a zero eigenvector of the linear operator $\partial_t + A_0\partial_x$. For this reason, we can write a solution of (29) as follows:

$$\eta_0 = -t\left(e_\parallel\cdot\left[(u\cdot\nabla)\lambda\right]u_x\right)e_\parallel + \left(\text{terms not proportional to } t\right). \tag{30}$$

The group-invariant solution, $u = u(t,x)$, for this approximate Lie symmetry group satisfies

$$X\left[u - u(t,x)\right]\big|_{u=u(t,x)} = 0, \tag{31}$$

which reads

$$\frac{\partial u(t,x)}{\partial\varepsilon} = -t\left(e_\parallel\cdot\left[(u\cdot\nabla)A_0\right]u_x\right)e_\parallel + \left(\text{terms not proportional to } t\right). \tag{32}$$

The long-time behavior of the solution is obtained by considering the case in which $t \gg 1$ and by neglecting the terms other than the first one. Then, (32) is reduced to

$$\frac{d}{d\tau}u(t,x) = -\left(e_\parallel\cdot\left[(u\cdot\nabla)A_0\right]u_x\right)e_\parallel, \tag{33}$$

where τ is a long time-scale defined by $\tau := \varepsilon t$. As a result, we have obtained a dynamical system which describes asymptotic behavior of the original system. This reduced dynamics corresponds to that presented before with the reductive perturbation method [3] or the renormalization group method [7].

4.2. Second Example: $u_{tt} - u_{xx} + u - \varepsilon u^3 = 0$. We consider a perturbed Klein-Gordon equation represented by

$$u_{tt} - u_{xx} + u - \varepsilon u^3 = 0. \tag{34}$$

Here, u is a scalar function which depends on two independent variables, t and x, and ε is a perturbation parameter. The subscripts of u denote the derivative with respect to the independent variables. By introducing new dependent variables as

$$u_1 := u_t, \qquad u_2 := u_x, \tag{35}$$

we rewrite (34) in terms of four first-order partial differential equations for three dependent variables u, u_1, and u_2 as follows:

$$\begin{aligned} u_{1t} - u_{2x} + u - \varepsilon u^3 &= 0,\\ u_t - u_1 &= 0,\\ u_x - u_2 &= 0,\\ u_{1x} - u_{2t} &= 0. \end{aligned} \tag{36}$$

The last equation implies the integrable condition.

As in the previous examples, a specific nonperturbative solution $u^{(0)}(t,x)$ is prespecified, on which we investigate the effect of the perturbation. Here we focus on a nonperturbative solution describing a traveling wave with a constant velocity, v. The solution is represented by

$$u^{(0)}(t,x) = U(x - vt), \tag{37}$$

where $U(\xi) := \alpha e^{i\kappa\xi} + \overline{\alpha}e^{-i\kappa\xi}$ for $\kappa := (v^2-1)^{-1/2}$ and an arbitrary constant, α. For this solution,

$$\begin{aligned} u_t^{(0)} &= -vu_x^{(0)},\\ u_{tt}^{(0)} &= -vu_{tx}^{(0)} = -vu_{xt}^{(0)} = v^2u_{xx}^{(0)}; \end{aligned} \tag{38}$$

therefore,

$$u_{tt}^{(0)} = \frac{u_t^{(0)}}{u_x^{(0)}}u_{xt}^{(0)} = \frac{u_t^{(0)}}{u_x^{(0)}}u_{tx}^{(0)} = \left(\frac{u_t^{(0)}}{u_x^{(0)}}\right)^2 u_{xx}^{(0)} \tag{39}$$

are satisfied. By combining these relations with the nonperturbative system, $u_{tt}^{(0)} - u_{xx}^{(0)} + u^{(0)} = 0$, we obtain

$$\begin{aligned} u_{tt}^{(0)} &= \frac{\left[u_t^{(0)}\right]^2 u^{(0)}}{\left[u_x^{(0)}\right]^2 - \left[u_t^{(0)}\right]^2}, \\ u_{xt}^{(0)} &= u_{tx}^{(0)} = \frac{u_t^{(0)} u_x^{(0)} u^{(0)}}{\left[u_x^{(0)}\right]^2 - \left[u_t^{(0)}\right]^2}, \\ u_{xx}^{(0)} &= \frac{\left[u_x^{(0)}\right]^2 u^{(0)}}{\left[u_x^{(0)}\right]^2 - \left[u_t^{(0)}\right]^2}. \end{aligned} \tag{40}$$

From these relations, we construct the constraint equations,

$$\begin{aligned} u_{1t} &= \frac{u_1^2 u}{u_2^2 - u_1^2}, \\ u_{2t} = u_{1x} &= \frac{u_1 u_2 u}{u_2^2 - u_1^2}, \\ u_{2x} &= \frac{u_2^2 u}{u_2^2 - u_1^2}, \end{aligned} \tag{41}$$

which are used in the finding of Lie symmetry group.

Let

$$\begin{aligned} X(t, x, u, u_1, u_2 : \varepsilon) := {} & \partial_\varepsilon + \eta(t, x, u, u_1, u_2; \varepsilon)\partial_u \\ & + \eta^{u_1}(t, x, u, u_1, u_2; \varepsilon)\partial_{u_1} \\ & + \eta^{u_2}(t, x, u, u_1, u_2; \varepsilon)\partial_{u_2} \end{aligned} \tag{42}$$

be the infinitesimal generator of a Lie symmetry group of the four partial differential equations (36). Its first prolongation X^* is given by

$$\begin{aligned} & X^*(t, x, u, u_1, u_2 : \varepsilon) \\ & \quad := X + \eta^t(t, x, u, u_1, u_2, u_{11}, u_{12}, u_{22}; \varepsilon)\partial_{u_t} \\ & \quad + \eta^x(t, x, u, u_1, u_2, u_{11}, u_{12}, u_{22}; \varepsilon)\partial_{u_x} \\ & \quad + \eta^{u_{1t}}(t, x, u, u_1, u_2, u_{11}, u_{12}, u_{22}; \varepsilon)\partial_{u_{1t}} \\ & \quad + \eta^{u_{2x}}(t, x, u, u_1, u_2, u_{11}, u_{12}, u_{22}; \varepsilon)\partial_{u_{2x}}, \end{aligned} \tag{43}$$

where η^t, η^x, $\eta^{u_{1t}}$, and $\eta^{u_{2x}}$ are defined as follows:

$$\begin{aligned} & \eta^t(t, x, u, u_1, u_2, u_t, u_{1t}, u_{2t}; \varepsilon) \\ & \quad := D_t \eta(t, x, u, u_1, u_2; \varepsilon), \\ & \eta^x(t, x, u, u_1, u_2, u_x, u_{1x}, u_{2x}; \varepsilon) \\ & \quad := D_x \eta(t, x, u, u_1, u_2; \varepsilon), \\ & \eta^{u_{1t}}(t, x, u, u_1, u_2, u_t, u_{1t}, u_{2t}; \varepsilon) \\ & \quad := D_t \eta^{u_1}(t, x, u, u_1, u_2; \varepsilon), \\ & \eta^{u_{2x}}(t, x, u, u_1, u_2, u_x, u_{1x}, u_{2x}; \varepsilon) \\ & \quad := D_x \eta^{u_2}(t, x, u, u_1, u_2; \varepsilon), \\ & D_t := \partial_t + u_1 \partial_u + u_{1t}\partial_{u_1} + u_{2t}\partial_{u_2}, \\ & D_x := \partial_x + u_x \partial_u + u_{1x}\partial_{u_1} + u_{2x}\partial_{u_2}. \end{aligned} \tag{44}$$

The criteria for the invariance of the original system on the manifold formed by the constraint equations (41) are given by

$$\begin{aligned} X^*\left[u_{1t} - u_{2x} + u - \varepsilon u^3\right]\big|_{(41)} &= 0, \\ X^*\left[u_t - u_1\right]\big|_{(41)} &= 0, \\ X^*\left[u_x - u_2\right]\big|_{(41)} &= 0, \\ X^*\left[u_{1x} - u_{2t}\right]\big|_{(41)} &= 0. \end{aligned} \tag{45}$$

With the identities,

$$\begin{aligned} \partial_u \left\{\frac{u_1^2 u}{u_2^2 - u_1^2}\right\} &= \frac{u_1^2}{u_2^2 - u_1^2}, \\ \partial_{u_1} \left\{\frac{u_1^2 u}{u_2^2 - u_1^2}\right\} &= \frac{2u_1 u_2^2 u}{\left(u_2^2 - u_1^2\right)^2}, \\ \partial_{u_2} \left\{\frac{u_1^2 u}{u_2^2 - u_1^2}\right\} &= \frac{-2u_1^2 u_2 u}{\left(u_2^2 - u_1^2\right)^2}, \\ \partial_u \left\{\frac{u_1 u_2 u}{u_2^2 - u_1^2}\right\} &= \frac{u_1 u_2}{u_2^2 - u_1^2}, \\ \partial_{u_1} \left\{\frac{u_1 u_2 u}{u_2^2 - u_1^2}\right\} &= \frac{\left(u_1^2 + u_2^2\right) u_2 u}{\left(u_2^2 - u_1^2\right)^2}, \\ \partial_{u_2} \left\{\frac{u_1 u_2 u}{u_2^2 - u_1^2}\right\} &= \frac{-\left(u_1^2 + u_2^2\right) u_1 u}{\left(u_2^2 - u_1^2\right)^2}, \\ \partial_u \left\{\frac{u_2^2 u}{u_2^2 - u_1^2}\right\} &= \frac{u_2^2}{u_2^2 - u_1^2}, \\ \partial_{u_1} \left\{\frac{u_2^2 u}{u_2^2 - u_1^2}\right\} &= \frac{2u_1 u_2^2 u}{\left(u_2^2 - u_1^2\right)^2}, \\ \partial_{u_2} \left\{\frac{u_2^2 u}{u_2^2 - u_1^2}\right\} &= \frac{-2u_1^2 u_2 u}{\left(u_2^2 - u_1^2\right)^2}, \end{aligned} \tag{46}$$

(45) read, in the lowest order of ε,

$$\begin{aligned}
&\left(u_2^2-u_1^2\right)\left(\eta_{tt}-\eta_{xx}\right)-\left(u_2^2-u_1^2\right)^2\eta_{uu}-u_1^2u^2\eta_{u_1u_1}\\
&\quad-u_1^2u^2\eta_{u_2u_2}+2u_1u\left(u_2^2-u_1^2\right)\eta_{uu_1}\\
&\quad-2u_2u\left(u_2^2-u_1^2\right)\eta_{uu_2}-2u_1u_2u^2\eta_{u_1u_2}\\
&\quad+2u_1\left(u_2^2-u_1^2\right)\eta_{tu}+2u_1^2u\eta_{tu_1}+2u_1u_2u\eta_{tu_2}\\
&\quad-2u_2\left(u_2^2-u_1^2\right)\eta_{xu}-2u_1u_2u\eta_{xu_1}-2u_2^2u\eta_{xu_2}\\
&\quad-u_1\left(u_2^2-u_1^2\right)\eta_{u_1}-u_2\left(u_2^2-u_1^2\right)\eta_{u_2}\\
&\quad-u\left(u_2^2-u_1^2\right)\eta_u+\left(u_2^2-u_1^2\right)\eta-u^3\left(u_2^2-u_1^2\right)\\
&=0.
\end{aligned}\tag{47}$$

Noting the homogeneity of the equation, we find a solution,

$$\begin{aligned}
&\eta\left(t,x,u,u_1,u_2\right)\\
&=-\frac{3}{8}\left(u^2+u_1^2-u_2^2\right)\left(u_1^2-u_2^2\right)\left(a\frac{t}{u_1}-(1-a)\frac{x}{u_2}\right)\\
&\quad-\frac{1}{32}u\left(u^2-u_1^2-u_2^2\right),
\end{aligned}\tag{48}$$

where a is an arbitrary constant. For this infinitesimal generator of the approximate Lie symmetry group, the group-invariant solution satisfies

$$X\left[u-u\left(t,x\right)\right]\big|_{u=u(t,x)}=0,\tag{49}$$

which reads

$$\frac{\partial u\left(t,x\right)}{\partial\varepsilon}=\eta\left(t,x,u\left(t,x\right),u_1\left(t,x\right),u_2\left(t,x\right)\right).\tag{50}$$

Asymptotic dynamic behavior can be obtained by considering the cases $t\gg 1$ and $x\gg 1$ and by neglecting the terms which are not proportional to x or t. By introducing $\tau:=\varepsilon t$ and $\xi:=\varepsilon x$, we obtain

$$\begin{aligned}
\frac{\partial u}{\partial\tau}&=-\frac{3\left(u^2+u_1^2-u_2^2\right)\left(u_1^2-u_2^2\right)}{8u_1}a,\\
\frac{\partial u}{\partial\xi}&=\frac{3\left(u^2+u_1^2-u_2^2\right)\left(u_1^2-u_2^2\right)}{8u_2}\left(1-a\right).
\end{aligned}\tag{51}$$

By eliminating a, the equations are reduced to

$$u_1\frac{\partial u}{\partial\tau}-u_2\frac{\partial u}{\partial\xi}=-\frac{3}{8}\left(u^2+u_1^2-u_2^2\right)\left(u_1^2-u_2^2\right).\tag{52}$$

This reduced equation describes the asymptotic behavior of the original system. This fact can be confirmed by comparing with the result derived before. By introducing a transformation of the variables,

$$\begin{aligned}
|A|^2&:=\frac{1}{4}\left(u^2+u_1^2-u_2^2\right),\\
\omega&:=u_1\left(u_1^2-u_2^2\right)^{-1/2},\\
k&:=u_2\left(u_1^2-u_2^2\right)^{-1/2},
\end{aligned}\tag{53}$$

the reduced equation (52) is transformed into

$$\left[\frac{\partial}{\partial\tau}+\frac{k}{\omega}\frac{\partial}{\partial\xi}\right]A=-\frac{3}{2\omega}\mathrm{i}\left|A\right|^2A.\tag{54}$$

This is the nonlinear Schrödinger equation, which corresponds to the reduced equation derived with the renormalization group method [9].

5. Concluding Remarks

In summary, this paper presents an investigation of perturbation problems, which involves the application of a particular type of Lie group method to partial as well as ordinary differential equations. As a consequence, we have obtained the dynamics which describes the asymptotic behavior of the original system. The main characteristic of the present method is the addition of some constraint equations when searching for a Lie symmetry group, an approach which allows us to find a partial Lie symmetry group. The constraint equations are determined from the nonperturbative solution on which we try to investigate the effect of the perturbation. The reason why it is necessary to specify a nonperturbative solution beforehand for the partial differential equations is that, unlike for ordinary differential equations, the nonperturbative system has an infinite number of independent solutions. In terms of technique, those constraint equations are constructed in such a way that the arbitrary constants or functions included in the nonperturbative solution are eliminated as we have done in deriving (4), (18), and (41). In this study, we have investigated the effect of the perturbation of its lowest order; however, extending our work to higher-order approximations would require us to use higher-order constraint equations instead.

Examples to which the method is applied here are singular perturbation problems in which secular terms appear in the naïve expansion of the solution [3, 6, 8, 10, 11, 34]. Terms proportional to the independent variables emerge in the infinitesimal generator of the Lie symmetry group as seen in (10), (30), and (48) corresponding to the emergence of the secular terms in the naïve expansion. As inferred from this mathematical similarity, the present method is also applicable to other systems which exhibit the same type of singular perturbation problems. In the singular perturbation methods proposed thus far, it is only after the construction of the naïve expansion that it becomes clear whether the method should be used or not. However, the present method can be applied without making the expansion, enabling the appropriate reduced equations to be derived.

Future work should involve a clarification of the relations between the present method and other types of singular perturbation methods presented thus far, such as the WKB or boundary layer problem methods. In addition, there are existing methods that have enabled the successful derivation of the asymptotic behavior for various other partial differential equations [13–18, 20–30, 35–38]. Therefore, it should be clarified whether or not the present method can be appropriately extended and applied to the problems those methods have resolved.

Appendix

Searching for Lie Symmetry with the Normal Procedure

Let us search for the Lie symmetry group of (1) without constraint equations. The criterion for the invariance of the equation is given by (7); namely,

$$X^* \left[2u\ddot{u} + \dot{u} - \varepsilon\right]\big|_{2u\ddot{u}+\dot{u}-\varepsilon=0} = 0, \tag{A.1}$$

which reads

$$\begin{aligned} &2u^2\eta_{tt} - u + \dot{u}\left(4u^2\eta_{ut} + 2u\eta_t\right) \\ &\quad + \dot{u}^2\left(2u^2\eta_{uu} + u\eta - \eta\right) + \varepsilon\left(\eta + u\eta_u\right) = 0. \end{aligned} \tag{A.2}$$

In the lowest order of ε, η has to satisfy the following differential equations:

$$\begin{aligned} 2u^2\eta_{tt} - u &= 0, \\ 4u^2\eta_{ut} + 2u\eta_t &= 0, \\ 2u^2\eta_{uu} + u\eta - \eta &= 0. \end{aligned} \tag{A.3}$$

However, we can easily find out that there is no solution that satisfies all the above equations, which implies that there is no Lie symmetry group.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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