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Research Article

Split Common Fixed Point Problem for a Class of Total Asymptotic Pseudocontractions

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We study the split common fixed point problem (SCFP) for a class of total asymptotically pseudocontractive mappings. We obtain some important properties of our class of mappings including the demiclosedness property and the closedness and convexity of the fixed point set. We then propose an algorithm and prove weak and strong convergence theorems for the approximation of solutions of the SCFP for certain class of these mappings.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces, K and Q nonempty closed convex subsets of H_1 and H_2, respectively, and $A : H_1 \rightarrow H_2$ a bounded linear operator. The split feasibility problem (SFP) (see, e.g., [1–9]) is

$$\begin{aligned} \text{find} \quad & x^* \in H_1 \\ \text{such that} \quad & x^* \in K, \\ & Ax^* \in Q. \end{aligned} \tag{1}$$

If $Q = \{b\}$, a singleton, we have the *convexly constrained linear inverse problem* (CCLIP):

$$\begin{aligned} \text{find} \quad & x^* \in H_1, \\ \text{such that} \quad & x^* \in K, \\ & Ax^* = b. \end{aligned} \tag{2}$$

The split feasibility problem (SFP) has various important applications in several disciplines (see, e.g., [2–9]).

Let $S : H_1 \rightarrow H_1$ and $T : H_2 \rightarrow H_2$ be mappings such that $K := F(S) = \{x \in H_1 : Sx = x\} \neq \emptyset$ and $Q := F(T) = \{x \in H_2 : Tx = x\} \neq \emptyset$. The *split common fixed point problem* (SCFP) for S and T is to find a point $x^* \in H_1$ such that $x^* \in F(S)$ and $Ax^* \in F(T)$. In sequel we use Γ to denote the set of solutions of (SCFP); that is,

$$\Gamma = \{x \in K = F(S),\ Ax \in Q = F(T)\}. \tag{3}$$

Definition 1. Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H.

A mapping $T : C \rightarrow C$ is said to be $(\{\mu_n\}_{n=1}^{\infty}, \{\xi_n\}_{n=1}^{\infty}, \phi)$-*total asymptotically k-strictly pseudocontractive* (see, e.g., [3]) if there exist a constant $k \in [0, 1)$, a continuous and strictly increasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$, and sequences $\{\mu_n\} \subset [0, \infty)$ and $\{\xi_n\} \subset [0, \infty)$ with $\mu_n \rightarrow 0$ and $\xi_n \rightarrow 0$ such that, for all $x, y \in C$,

$$\begin{aligned} \left\|T^n x - T^n y\right\|^2 \leq{} & \left\|x - y\right\|^2 \\ & + k\left\|\left(I - T^n\right)x - \left(I - T^n\right)y\right\|^2 \\ & + \mu_n \phi\left(\left\|x - y\right\|\right) + \xi_n. \end{aligned} \tag{4}$$

T is said to be $(\{\mu_n\}_{n=1}^{\infty}, \{\xi_n\}_{n=1}^{\infty}, \phi)$-*total asymptotically pseudocontractive* if $k = 1$ in (4).

Observe that if $\phi(t) = t^2$ and $\xi_n = 0\ \forall n \geq 1$ in (4), we obtain

$$\begin{aligned} \left\|T^n x - T^n y\right\|^2 \leq{} & k_n \left\|x - y\right\|^2 \\ & + k\left\|x - T^n x - \left(y - T^n y\right)\right\|^2, \end{aligned} \tag{5}$$

where $k_n = 1+\mu_n \subseteq [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$. Mappings satisfying (5) are the well-known class of *k-strictly asymptotically pseudocontractive mappings*, while mappings satisfying (5) for $k = 1$ are called asymptotically pseudocontractive mappings. These classes of mappings are generalizations of the well-known important class of *asymptotically nonexpansive mappings* introduced by Goebel and Kirk [10] (i.e., mappings $T : C \to C$ which satisfy $\|T^n x - T^n y\| \le k_n \|x - y\|$, $\forall x, y \in C$ and for some sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$).

We consider the following examples.

Example 2 (see [10]). In the real Hilbert space ℓ^2, let $C = \{x \in \ell^2 : \|x\| \le 1\}$ denote the closed unit ball, and define $S : C \to C$ by $S(x_1, x_2, x_3, \ldots) = (0, x_1^2, a_2 x_2, a_3 x_3, \ldots)$, where $\{a_j\}_{j=1}^{\infty}$ is a real sequence in $(0, 1)$ such that $\prod_{j=1}^{\infty} a_j = 1/2$. Then S is asymptotically nonexpansive and hence asymptotically strictly pseudocontractive. It follows that S is total asymptotically strictly pseudocontractive and hence total asymptotically pseudocontractive.

Example 3 (see [6, 7]). Let D be an orthogonal subspace of the Euclidean space $\mathfrak{R}^n$, and for each $x = (x_1, x_2, x_3, \ldots, x_n) \in D$, define $T : D \to D$ by

$$Tx = \begin{cases} (x_1, x_2, x_3, \ldots, x_n), & \text{if } \prod_{j=1}^{n} x_j < 0, \\ (-x_1, -x_2, -x_3, \ldots, -x_n), & \text{if } \prod_{j=1}^{n} x_j \ge 0. \end{cases} \tag{6}$$

Then T is asymptotically nonexpansive (see, e.g., [7]) and hence total asymptotically strictly pseudocontractive (see, e.g., [6]) and hence it is total asymptotically pseudocontractive.

Example 4 (see [11]). Let $\mathfrak{R}$ denote the reals with the usual norm, $C = [-6, 2]$, and define $T : C \to C$ by

$$Tx = \begin{cases} x, & \text{if } x \in [-6, 1), \\ 2x - x^2, & \text{if } x \in [1, 2]. \end{cases} \tag{7}$$

It is shown in [11] that $|Tx-Ty|^2 \le |x-y|^2+|x-Tx-(y-Ty)|^2$ and $|Tx - Ty| \le 6|x - y|$, $\forall x, y \in [-6, 2]$. It is easy to observe that, for all integers $n > 1$, we have

$$T^n x = \begin{cases} x, & \text{if } x \in [-6, 1), \\ 2x - x^2, & \text{if } x \in [1, 2]. \end{cases} \tag{8}$$

Thus we easily obtain

$$\left|T^n x - T^n y\right|^2 \le \left|x - y\right|^2 + \left|x - T^n x - \left(y - T^n y\right)\right|^2, \quad \forall x, y \in [-6, 2], \ \forall n \ge 1. \tag{9}$$

Hence T is asymptotically pseudocontractive and hence total asymptotically pseudocontractive. Furthermore, $|T^n x - T^n y| \le 6|x - y|$, $\forall x, y \in [-6, 1]$ so that T is uniformly L-Lipschitzian.

The following is an example of a total asymptotically pseudocontractive map which is not total asymptotically strictly pseudocontractive.

Example 5. Let $\mathfrak{R}$ denote the reals with the usual norm, $C = [0, 1]$, and define $T : C \to C$ by

$$Tx = \left(1 - x^{2/3}\right)^{3/2}. \tag{10}$$

Then for all integers $n \ge 1$ and for all $x \in [0, 1]$ we have

$$T^n x = \begin{cases} \left(1 - x^{2/3}\right)^{3/2}, & \text{if } n \text{ is odd}, \\ x, & \text{if } n \text{ is even}. \end{cases} \tag{11}$$

Thus

$$\left|T^n x - T^n y\right|^2 \le \left|x - y\right|^2 + \left|x - T^n x - \left(y - T^n y\right)\right|^2, \quad \forall x, y \in [0, 1], \ n \ge 1, \tag{12}$$

and hence T is total asymptotically pseudocontractive. T is not total asymptotically strictly pseudocontractive since, in every real Hilbert space H, every total asymptotically strictly pseudocontractive mapping $T : C \subseteq H \to C$ satisfies

$$\limsup_{n\to\infty} \left\|T^n x - T^n y\right\| \le \frac{\left(1 + \sqrt{k}\right)}{1 - \sqrt{k}} \left\|x - y\right\|, \quad \forall x, y \in C. \tag{13}$$

In [3] the authors studied the split common fixed point problem (SCFP) for a class of total asymptotically strictly pseudocontractive mappings in real Hilbert spaces. They proposed an algorithm and proved weak and strong convergence theorems for finding solutions of SCFP for the class of mappings studied.

It is our purpose in this work to study the split common fixed point problem (SCFP) for a class of total asymptotically pseudocontractive mappings which is much more general than the class of mappings studied in [3]. We obtain some important properties of our class of mappings including the *demiclosedness property* and then propose an algorithm and prove weak and strong convergence theorems for the approximation of solutions of the SCFP.

2. Preliminaries

In what follows, we will need the following.

Let E be a real Banach space and C a nonempty closed convex subset of E. A mapping $T : C \to C$ is said to be *semicompact* if, for any bounded sequence $\{x_n\} \subset C$ with $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $x^* \in C$.

$T : C \to C$ is said to be uniformly L-Lipschitzian if there exists a constant $L \ge 0$, such that, for all $x, y \in C$,

$$\left\|T^n x - T^n y\right\| \le L \left\|x - y\right\|. \tag{14}$$

E is said to have the Opial property if, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in E \text{ with } y \neq x. \tag{15}$$

It is well known that every Hilbert space satisfies the Opial condition.

Lemma 6 (see [10]). *Let H be a real Hilbert space. If $\{x_n\}$ is a sequence in H weakly convergent to z, then*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2, \quad \forall y \in H. \tag{16}$$

Lemma 7 (see [11]). *Let $\{a_n\}$, $\{b_n\}$, and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n) a_n + b_n, \quad \forall n \geq 1. \tag{17}$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If in addition $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges to zero, then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

We start with the following important properties of $(\{\mu_n\}_{n=1}^{\infty}, \{\xi_n\}_{n=1}^{\infty}, \phi)$-total asymptotically pseudocontractive mappings.

Proposition 8. *Let H be a real Hilbert space, C a nonempty closed convex subset of H, and $T : C \to C$ a uniformly L-Lipschitzian $(\{\mu_n\}_{n=1}^{\infty}, \{\xi_n\}_{n=1}^{\infty}, \phi)$-total asymptotically pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\beta \in (0, 1/(1 + \sqrt{1 + L^2}))$; and set $G_n = T^n[(1 - \beta)I + \beta T^n]$. Then for each $q \in F(T)$ and each $x \in C$, the following equivalent inequalities hold:*

$$\langle x - G_n x, x - q\rangle \geq \frac{\beta}{2} \|x - G_n x\|^2 - (1 + \beta) \frac{\mu_n}{2} \phi(\|x - q\|) - (1 + \beta) \frac{\xi_n}{2}, \tag{18}$$

$$\langle x - G_n x, q - G_n x\rangle \leq \frac{2 - \beta}{2} \|x - G_n x\|^2 + (1 + \beta) \frac{\mu_n}{2} \phi(\|x - q\|) + (1 + \beta) \frac{\xi_n}{2}. \tag{19}$$

Proof. For arbitrary $x \in C$ and $q \in F(T)$ we have

$$\begin{aligned}
\|G_n x - q\|^2 &= \|T^n[(1 - \beta) x + \beta T^n x] - T^n q\|^2 \\
&\leq \|(1 - \beta) x + \beta T^n x - q\|^2 \\
&\quad + \|(1 - \beta) x + \beta T^n x - G_n x\|^2 + \mu_n \phi(\|x - q\|) \\
&\quad + \xi_n \\
&= \|(1 - \beta)(x - q) + \beta(T^n x - q)\|^2 \\
&\quad + \|(1 - \beta)(x - G_n x) + \beta(T^n x - G_n x)\|^2 \\
&\quad + \mu_n \phi(\|x - q\|) + \xi_n \\
&= (1 - \beta)\|x - q\|^2 + \beta\|T^n x - q\|^2 \\
&\quad - \beta(1 - \beta)\|x - T^n x\|^2 + (1 - \beta)\|x - G_n x\|^2 \\
&\quad + \beta\|T^n x - G_n x\|^2 - \beta(1 - \beta)\|x - T^n x\|^2 \\
&\quad + \mu_n \phi(\|x - q\|) + \xi_n \\
&\leq \|x - q\|^2 + \beta\left[\|x - T^n x\|^2 + \mu_n \phi(\|x - q\|) + \xi_n\right] \\
&\quad - \beta(1 - \beta)\|x - T^n x\|^2 + (1 - \beta)\|x - G_n x\|^2 \\
&\quad + L^2 \beta^3 \|x - T^n x\|^2 - \beta(1 - \beta)\|x - T^n x\|^2 \\
&\quad + \mu_n \phi(\|x - q\|) + \xi_n \\
&= \|x - q\|^2 + (1 - \beta)\|G_n x - x\|^2 \\
&\quad + (1 + \beta)\mu_n \phi(\|x - q\|) + (1 + \beta)\xi_n \\
&\quad - \beta\left(1 - 2\beta - L^2\beta^2\right)\|x - T^n x\|^2 \\
&\leq \|x - q\|^2 + (1 - \beta)\|G_n x - x\|^2 \\
&\quad + (1 + \beta)\mu_n \phi(\|x - q\|) + (1 + \beta)\xi_n.
\end{aligned} \tag{20}$$

It follows from (20) that

$$\begin{aligned}
\langle x - G_n x, x - q\rangle &\geq \frac{\beta}{2}\|x - G_n x\|^2 - (1 + \beta)\frac{\mu_n}{2}\phi(\|x - q\|) - (1 + \beta)\frac{\xi_n}{2}, \\
\langle x - G_n x, q - G_n x\rangle &\leq \frac{2 - \beta}{2}\|x - G_n x\|^2 + (1 + \beta)\frac{\mu_n}{2}\phi(\|x - q\|) + (1 + \beta)\frac{\xi_n}{2}.
\end{aligned} \tag{21}$$

□

Proposition 9. *Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a uniformly L-Lipschitzian $(\{\mu_n\}_{n=1}^{\infty}, \{\xi_n\}_{n=1}^{\infty}, \phi)$-total asymptotically pseudocontractive mapping with $\lim_{n\to\infty}\mu_n = 0$, $\lim_{n\to\infty}\xi_n = 0$. Then*

(i) *$(I - T)$ is demiclosed at 0;*

(ii) *$F(T) = \{x \in C : Tx = x\}$ is closed and convex.*

Proof. (i) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C which converges weakly to p and $\{x_n - Tx_n\}_{n=1}^{\infty}$ converges strongly to 0. We prove that $p \in F(T)$. Since $\{x_n\}_{n=1}^{\infty}$ converges weakly, it is bounded. For each $x \in H$, define $f : H \to [0, \infty)$ by

$$f(x) := \limsup_{n\to\infty} \|x_n - x\|^2. \tag{22}$$

Observe that, for arbitrary but fixed integer $m \geq 1$, we have

$$\begin{aligned}\|x_n - T^m x_n\| &\leq \|x_n - Tx_n\| + \|Tx_n - T^2 x_n\| + \cdots \\ &\quad + \|T^{m-1}x_n - T^m x_n\| \\ &\leq mL\|x_n - Tx_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{23}$$

Set $G_m x := T^m((1-\beta)x + \beta T^m x)$, where $\beta \in (0, 1/(1 + \sqrt{L^2+1}))$. Then we obtain

$$\begin{aligned}&\|(1-\beta)x_n + \beta T^m x_n - T^m x_n\| \\ &\quad = (1-\beta)\|x_n - T^m x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \\ &\|T^m x_n - G_m x_n\| \leq L\beta\|x_n - Tx_n\| \longrightarrow 0 \\ &\qquad \text{as } n \longrightarrow \infty.\end{aligned} \tag{24}$$

Hence

$$\begin{aligned}&\|(1-\beta)x_n + \beta T^m x_n - G_m x_n\| \\ &\quad \leq \|(1-\beta)x_n + \beta T^m x_n - T^m x_n\| \\ &\quad\quad + \|T^m x_n - G_m x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{25}$$

Furthermore,

$$\begin{aligned}\|x_n - G_m x_n\| &\leq \|x_n - T^m x_n\| + \|T^m x_n - G_m x_n\| \longrightarrow 0 \\ &\qquad \text{as } n \longrightarrow \infty.\end{aligned} \tag{26}$$

Since $\{\|x_n - p\|\}$ is bounded we also obtain

$$\begin{aligned}&\|(1-\beta)x_n + \beta T^m x_n - ((1-\beta)p + \beta T^m p)\| \\ &\quad \leq (1+L)\|x_n - p\| \leq D \quad \forall n \geq 1,\end{aligned} \tag{27}$$

for some $D > 0$.

From Lemma 6, we obtain $f(x) = \limsup_{n\to\infty}\|x_n - p\|^2 + \|p - x\|^2$, $\forall x \in H$. Thus $f(x) = f(p) + \|p - x\|^2$, $\forall x \in H$, and hence

$$f(G_m p) = f(p) + \|p - G_m p\|^2. \tag{28}$$

Observe that

$$\begin{aligned}f(G_m p) &= \limsup_{n\to\infty}\|x_n - G_m p\|^2 = \limsup_{n\to\infty}\|x_n - G_m x_n \\ &\quad + G_m x_n - G_m p\|^2 = \limsup_{n\to\infty}\|G_m x_n - G_m p\|^2 \\ &= \limsup_{n\to\infty}\|T^m((1-\beta)x_n + \beta T^m x_n) - T^m((1-\beta)p \\ &\quad + \beta T^m p)\|^2 \leq \limsup_{n\to\infty}\big[\|(1-\beta)x_n + \beta T^m x_n \\ &\quad - ((1-\beta)p + \beta T^m p)\|^2 + \|(1-\beta)x_n + \beta T^m x_n \\ &\quad - G_m x_n - ((1-\beta)p + \beta T^m p - G_m p)\|^2 \\ &\quad + \mu_m\phi(\|(1-\beta)x_n + \beta T^m x_n - ((1-\beta)p + \beta T^m p)\|) \\ &\quad + \xi_m\big] \leq \limsup_{n\to\infty}\big[\|(1-\beta)(x_n - p) \\ &\quad + \beta(T^m x_n - T^m p)\|^2 + \|(1-\beta)(p - G_m p) \\ &\quad + \beta(T^m p - G_m p)\|^2 + \mu_m\phi(D) + \xi_m\big] \leq \limsup_{n\to\infty}\big[(1 \\ &\quad - \beta)\|x_n - p\|^2 + \beta\|T^m x_n - T^m p\|^2 - \beta(1-\beta)\|x_n \\ &\quad - T^m x_n - (p - T^m p)\|^2 + (1-\beta)\|p - G_m p\|^2 \\ &\quad + \beta\|T^m p - G_m p\|^2 - \beta(1-\beta)\|p - T^m p\|^2 \\ &\quad + \mu_m\phi(D) + \xi_m\big] \leq \limsup_{n\to\infty}\big[(1-\beta)\|x_n - p\|^2 \\ &\quad + \beta\big[\|x_n - p\|^2 + \|x_n - T^m x_n - (p - T^m p)\|^2 \\ &\quad + \mu_m\phi(\|x_n - p\|) + \xi_m\big] - \beta(1-\beta)\|x_n - T^m x_n \\ &\quad - (p - T^m p)\|^2 + (1-\beta)\|p - G_m p\|^2 + L^2\beta^3\|p \\ &\quad - T^m p\|^2 - \beta(1-\beta)\|p - T^m p\|^2 + \mu_m\phi(D) + \xi_m\big] \\ &\leq \limsup_{n\to\infty}\big[\|x_n - p\|^2 - \beta\big[1 - 2\beta - L^2\beta^2\big]\|p - T^m p\|^2 \\ &\quad + (1-\beta)\|p - G_m p\|^2 + (1+\beta)\mu_m\phi(D) + (1+\beta) \\ &\quad \cdot \xi_m\big].\end{aligned} \tag{29}$$

It follows that

$$\begin{aligned}f(p) + \|p - G_m p\|^2 &\leq f(p) + (1-\beta)\|p - G_m p\|^2 \\ &\quad + (1+\beta)\mu_m\phi(D) \\ &\quad + (1+\beta)\xi_m,\end{aligned} \tag{30}$$

and thus

$$\|p - G_m p\|^2 \leq \frac{1}{\beta}\left[(1+\beta)\mu_m \phi(D) + (1+\beta)\xi_m\right] \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \tag{31}$$

It follows that

$$\begin{aligned}\|p - T^m p\| &\leq \|p - G_m p\| + \|G_m p - T^m p\| \\ &\leq \|p - G_m p\| + L\beta \|p - T^m p\|,\end{aligned} \tag{32}$$

and hence

$$\|p - T^m p\| \leq \frac{1}{(1-\beta L)}\|p - G_m p\| \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \tag{33}$$

It now follows that $T^m p \to p$ as $m \to \infty$. Since T is continuous, we have that $T^{m+1}p \to Tp$ as $m \to \infty$, and hence $Tp = p$.

(ii) Let $\{p_n\}_{n=1}^{\infty} \subseteq F(T)$ be such that $p_n \to p$. We prove that $p \in F(T)$. Consider

$$\begin{aligned}\|p - Tp\| &\leq \|p - p_n\| + \|p_n - Tp\| \\ &= \|p - p_n\| + \|Tp_n - Tp\| \\ &\leq (1+L)\|p_n - p\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{34}$$

Hence $p \in F(T)$, and $F(T)$ is closed.

Let $p_1, p_2 \in F(T)$ and let $\lambda \in [0,1]$ be arbitrary. Set $p := \lambda p_1 + (1-\lambda)p_2$. We prove that $p \in F(T)$. Observe that $\|p - p_1\| = (1-\lambda)\|p_1 - p_2\|$ and $\|p - p_2\| = \lambda\|p_1 - p_2\|$. Set

$$G_n x := T^n\left((1-\beta)x + \beta T^n x\right), \tag{35}$$

where $\beta \in (0, 2/(1+\sqrt{1+4L^2}))$. Then $G_n p_1 = p_1$, and $G_n p_2 = p_2$. Observe that

$$\begin{aligned}\|p - G_n p\|^2 &= \|\lambda(p_1 - G_n p) + (1-\lambda)(p_2 - G_n p)\|^2 \\ &= \lambda\|p_1 - G_n p\|^2 + (1-\lambda)\|p_2 - G_n p\|^2 \\ &\quad - \lambda(1-\lambda)\|p_1 - p_2\|^2.\end{aligned} \tag{36}$$

Observe that

$$\begin{aligned}\|G_n p - p_1\|^2 &= \|T^n((1-\beta)p + \beta T^n p) - p_1\|^2 \\ &= \|T^n((1-\beta)p + \beta T^n p) \\ &\quad - T^n((1-\beta)p_1 + \beta T^n p_1)\|^2 \leq \|(1-\beta)(p - p_1) \\ &\quad + \beta(T^n p - T^n p_1)\|^2 + \|(1-\beta)p + \beta T^n p - G_n p \\ &\quad - ((1-\beta)p_1 + \beta T^n p_1 - G_n p_1)\|^2 \\ &\quad + \mu_n \phi(\|(1-\beta)(p - p_1) + \beta(T^n p - T^n p_1)\|) + \xi_n \\ &= (1-\beta)\|p - p_1\|^2 + \beta\|T^n p - T^n p_1\|^2 - \beta(1 \\ &\quad - \beta)\|p - T^n p\|^2 + \|(1-\beta)(p - G_n p) \\ &\quad + \beta(T^n p - G_n p)\|^2 + \mu_n \phi([1-\beta+\beta L]\|p - p_1\|) \\ &\quad + \xi_n \leq (1-\beta)\|p - p_1\|^2 + \beta\|p - p_1\|^2 + \beta\|p \\ &\quad - T^n p\|^2 + \beta\mu_n\phi(\|p - p_1\|) + \beta\xi_n - \beta(1-\beta)\|p \\ &\quad - T^n p\|^2 + (1-\beta)\|p - G_n p\|^2 + \beta\|T^n p - G_n p\|^2 \\ &\quad - \beta(1-\beta)\|p - T^n p\|^2 \\ &\quad + \mu_n\phi([1-\beta+\beta L]\|p - p_1\|) + \xi_n \leq \|p - p_1\|^2 \\ &\quad + (1-\beta)\|p - G_n p\|^2 - \beta\left[1 - 2\beta - \beta^2 L^2\right]\|p \\ &\quad - T^n p\|^2 + \mu_n[\phi([1-\beta+\beta L]\|p - p_1\|) \\ &\quad + \beta\phi(\|p - p_1\|)] + (1+\beta)\xi_n = \|p - p_1\|^2 + (1 \\ &\quad - \beta)\|p - G_n p\|^2 - \beta\left[1 - 2\beta - \beta^2 L^2\right]\|p - T^n p\|^2 \\ &\quad + \sigma_n,\end{aligned} \tag{37}$$

where $\sigma_n = \mu_n[\phi([1-\beta+\beta L]\|p-p_1\|) + \beta\phi(\|p-p_1\|)] + (1+\beta)\xi_n$.

Similarly,

$$\begin{aligned}\|G_n p - p_2\|^2 &\leq \|p - p_2\|^2 + (1-\beta)\|p - G_n p\|^2 \\ &\quad - \beta\left[1 - 2\beta - \beta^2 L^2\right]\|p - T^n p\|^2 \\ &\quad + \sigma_n.\end{aligned} \tag{38}$$

Thus

$$\begin{aligned}&\|p - G_n p\|^2 \\ &\leq \lambda\|p - p_1\|^2 + \lambda(1-\beta)\|p - G_n p\|^2 \\ &\quad - \lambda\beta\left[1 - 2\beta - \beta^2 L^2\right]\|p - T^n p\|^2 + \lambda\sigma_n \\ &\quad + (1-\lambda)\|p - p_2\|^2 \\ &\quad + (1-\lambda)(1-\beta)\|p - G_n p\|^2\end{aligned}$$

$$- (1-\lambda)\beta\left[1-2\beta-\beta^2L^2\right]\left\|p-T^np\right\|^2 + (1-\lambda)\sigma_n - \lambda(1-\lambda)\left\|p_1-p_2\right\|^2, \tag{39}$$

and it follows that

$$\beta\left\|p-G_np\right\|^2 \le \sigma_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{40}$$

Hence $G_np \to p$ as $n \to \infty$. Observe that

$$\begin{aligned}\left\|p-T^np\right\|^2 &\le \left\|p-G_np\right\| + \left\|G_np-T^np\right\| \\ &\le \left\|p-G_np\right\| + L\beta\left\|p-T^np\right\|.\end{aligned} \tag{41}$$

Thus

$$(1-L\beta)\left\|p-T^np\right\| \le \left\|p-G_np\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{42}$$

and hence $T^np \to p$ as $n \to \infty$. Since T is continuous we have

$$T^{n+1}p \longrightarrow Tp \quad \text{as } n \longrightarrow \infty. \tag{43}$$

Thus $Tp = p$. □

We now introduce our algorithm and prove weak and strong convergence theorems for solving the split common fixed point problem for a class of total asymptotically pseudocontractive mappings in real Hilbert spaces.

Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ a bounded linear operator, $S : H_1 \to H_1$ a uniformly L_1-Lipschitzian $(\{\mu_n^1\}, \{\xi_n^1\}, \phi_1)$-total asymptotically pseudocontractive mapping, with $K := F(S) \neq \emptyset$, and $T : H_2 \to H_2$ a uniformly L_2-Lipschitzian $(\{\mu_n^2\}, \{\xi_n^2\}, \phi_2)$-total asymptotically pseudocontractive mapping, with $Q := F(T) \neq \emptyset$. We now introduce the following iterative algorithm for approximating solutions of split common fixed point problem: $x \in H_1$ such that $x \in F(S)$ and $Ax \in F(T)$ (i.e., $x \in \Gamma = \{x \in H_1 : x \in F(S) \text{ and } Ax \in F(T)\}$.

For arbitrary $x_1 \in H_1$, the sequence $\{x_n\}_{n=1}^{\infty}$ is given by

$$\begin{aligned} u_n &= x_n + \gamma A^*\left[T^n\left((1-\beta)I+\beta T^n\right)-I\right]Ax_n, \quad n \ge 1, \\ y_n &= (1-\beta_n)u_n + \beta_nS^nu_n, \quad n \ge 1, \\ x_{n+1} &= (1-\alpha_n)u_n + \alpha_nS^ny_n, \quad n \ge 1, \end{aligned} \tag{44}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are suitable sequences in $(0,1)$ and β is a suitable parameter in $(0,1)$. We prove the following.

Theorem 10. *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ a bounded linear operator, $S : H_1 \to H_1$ a uniformly L_1-Lipschitzian $(\{\mu_n^{(1)}\}, \{\xi_n^{(1)}\}, \phi_1)$-total asymptotically pseudocontractive mapping, with $K := F(S) \neq \emptyset$, and $T : H_2 \to H_2$ a uniformly L_2-Lipschitzian $(\{\mu_n^{(2)}\}, \{\xi_n^{(2)}\}, \phi_2)$-total asymptotically pseudocontractive mapping, with $Q := F(T) \neq \emptyset$; let $\mu_n = \max\{\mu_n^{(1)}, \mu_n^{(2)}\}$, $\xi_n = \max\{\xi_n^{(1)}, \xi_n^{(2)}\}$, $L = \max\{L_1, L_2\}$, and $\phi(t) = \max\{\phi_1(t), \phi_2(t)\}$ such that there exist two positive real constants M and M^* such that $\phi(t) \le M^*t^2$, $\forall t \ge M$. Let $\sum_{n=1}^{\infty}\mu_n < \infty$ and $\sum_{n=1}^{\infty}\xi_n < \infty$; let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers satisfying the condition:*

$$0 < \delta \le \alpha_n \le \beta_n \le \beta < \frac{2}{(2+\mu M^*) + \sqrt{4L^2 + (2+\mu M^*)^2}}, \tag{45}$$

where $\mu = \sup_{n\ge1}\mu_n$ and $\gamma \in (0, \beta/\|A\|^2)$. Let $\Gamma = \{x \in H_1 : x \in F(S) \text{ and } Ax \in F(T)\} \neq \emptyset$. Then for arbitrary $x_1 \in H_1$, the sequence $\{x_n\}$ generated from x_1 by (44) converges weakly to a point in Γ.

If in addition S is semicompact, then $\{x_n\}$ and $\{u_n\}$ converge strongly to a point in Γ.

Proof. We will divide the proof into four steps.

Step 1. We prove that, for each $p \in \Gamma$, the following limits exist and

$$\lim_{n\to\infty}\left\|x_n-p\right\| = \lim_{n\to\infty}\left\|u_n-p\right\|. \tag{46}$$

Since ϕ is a continuous and an increasing function, it follows that $\phi(\lambda) \le \phi(M)$ $\forall\lambda \le M$, and by hypothesis $\phi(\lambda) \le M^*\lambda^2$ $\forall\lambda \ge M$. In either case, we can obtain that

$$\phi(\lambda) \le \phi(M) + M^*\lambda^2, \quad \forall\lambda \ge 0. \tag{47}$$

Let $G_nu_n = S^n((1-\beta_n)u_n + \beta_nS^nu_n)$, $n \ge 1$. Then, for arbitrary $p \in F(S)$, we obtain

$$\begin{aligned}\left\|G_nu_n-p\right\|^2 &= \left\|S^n\left((1-\beta_n)u_n+\beta_nS^nu_n\right)-S^np\right\|^2 \\ &\le \left\|(1-\beta_n)u_n+\beta_nS^nu_n-p\right\|^2 + \left\|(1-\beta_n)u_n \right. \\ &\left. + \beta_nS^nu_n - G_nu_n\right\|^2 + \mu_n\phi\left(\left\|(1-\beta_n)u_n+\beta_nS^nu_n \right.\right. \\ &\left.\left. - p\right\|\right) + \xi_n \le \left\|(1-\beta_n)(u_n-p)+\beta_n(S^nu_n-p)\right\|^2 \\ &+ \left\|(1-\beta_n)(u_n-G_nu_n)+\beta_n(S^nu_n-G_nu_n)\right\|^2 \\ &+ \mu_n\left[\phi(M) + M^*\left\|(1-\beta_n)(u_n-p)\right.\right. \\ &\left.\left. + \beta_n(S^nu_n-p)\right\|^2\right] + \xi_n = (1-\beta_n)\left\|u_n-p\right\|^2 \\ &+ \beta_n\left\|S^nu_n-p\right\|^2 - \beta_n(1-\beta_n)\left\|u_n-S^nu_n\right\|^2 + (1 \\ &- \beta_n)\left\|u_n-G_nu_n\right\|^2 + \beta_n\left\|S^nu_n-G_nu_n\right\|^2 - \beta_n(1 \\ &- \beta_n)\left\|u_n-S^nu_n\right\|^2 + \mu_n\phi(M) + \mu_nM^*\left[(1-\beta_n)\right.\end{aligned}$$

$$
\begin{aligned}
&\cdot \|u_n - p\|^2 + \beta_n \|S^n u_n - p\|^2 - \beta_n (1-\beta_n) \|u_n \\
&- S^n u_n\|^2\big] + \xi_n \le (1-\beta_n) \|u_n - p\|^2 + \beta_n \big[\|u_n \\
&- p\|^2 + \|u_n - S^n u_n\|^2 + \mu_n \phi(M) + \mu_n M^* \|u_n \\
&- p\|^2 + \xi_n\big] - \beta_n (1-\beta_n) \|u_n - S^n u_n\|^2 + (1-\beta_n) \\
&\cdot \|u_n - G_n u_n\|^2 + \beta_n^3 L^2 \|u_n - S^n u_n\|^2 - \beta_n (1-\beta_n) \\
&\cdot \|u_n - S^n u_n\|^2 + \mu_n \phi(M) + \mu_n M^* \big[(1-\beta_n) \|u_n \\
&- p\|^2 + \beta_n \big(\|u_n - p\|^2 + \|u_n - S^n u_n\|^2 + \mu_n \phi(M) \\
&+ \mu_n M^* \|u_n - p\|^2 + \xi_n\big) - \beta_n (1-\beta_n) \|u_n \\
&- S^n u_n\|^2\big] + \xi_n = [1 + \mu_n M^* (1 \\
&+ \beta_n (1 + \mu_n M^*))] \|u_n - p\|^2 + (1-\beta_n) \|u_n \\
&- G_n u_n\|^2 - \beta_n \big[1 - \beta_n (2 + \mu M^*) - \beta_n^2 L^2\big] \|u_n \\
&- S^n u_n\|^2 + (1 + \beta_n (1 + \mu_n M^*)) \xi_n + (1 + \beta_n (1 \\
&+ \mu_n M^*)) \mu_n \phi(M) = [1 + \delta_n] \|u_n - p\|^2 + (1 \\
&- \beta_n) \|u_n - G_n u_n\|^2 - \beta_n \big[1 - \beta_n (2 + \mu M^*) \\
&- \beta_n^2 L^2\big] \|u_n - S^n u_n\|^2 + \sigma_n,
\end{aligned}
\tag{48}
$$

where $\delta_n = \mu_n M^*(1 + \beta_n(1 + \mu_n M^*))$ and

$$
\begin{aligned}
\sigma_n &= (1 + \beta_n (1 + \mu_n M^*)) \xi_n \\
&\quad + (1 + \beta_n (1 + \mu_n M^*)) \mu_n \phi(M) \\
&= [1 + \beta_n (1 + \mu_n M^*)] [\xi_n + \mu_n \phi(M)].
\end{aligned}
\tag{49}
$$

Observe that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n) u_n + \alpha_n S^n y_n - p\|^2 \\
&= \|(1-\alpha_n) u_n + \alpha_n G_n u_n - p\|^2 \\
&= (1-\alpha_n) \|u_n - p\|^2 + \alpha_n \|G_n u_n - p\|^2 \\
&\quad - \alpha_n (1-\alpha_n) \|u_n - G_n u_n\|^2.
\end{aligned}
\tag{50}
$$

Using (48) in (50) we obtain

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \le (1-\alpha_n) \|u_n - p\|^2 \\
&\quad + \alpha_n \big[(1+\delta_n) \|u_n - p\|^2 + (1-\beta_n) \|u_n - G_n u_n\|^2 \\
&\quad - \beta_n \big(1 - \beta_n (2 + \mu_n M^*) - \beta_n^2 L^2\big) \|u_n - S^n u_n\|^2 \\
&\quad + \sigma_n\big] - \alpha_n (1-\alpha_n) \|u_n - G_n u_n\|^2 \le [1 + \alpha_n \delta_n] \|u_n \\
&\quad - p\|^2 - \alpha_n (\beta_n - \alpha_n) \|u_n - G_n u_n\|^2 - \alpha_n \beta_n \big[1 \\
&\quad - \beta_n (2 + \mu M^*) - \beta_n^2 L^2\big] \|u_n - S^n u_n\|^2 + \alpha_n \sigma_n.
\end{aligned}
\tag{51}
$$

Observe also that if we let $F_n = T^n((1-\beta)I + \beta T^n)$, then,

$$
\begin{aligned}
&\|u_n - p\|^2 \\
&\quad = \|x_n + \gamma A^* [T^n ((1-\beta) I + \beta T^n) - I] A x_n - p\|^2 \\
&\quad = \|x_n - p + \gamma A^* (F_n - I) A x_n\|^2 \\
&\quad = \|x_n - p\|^2 + \gamma^2 \|A^* (F_n - I) A x_n\|^2 \\
&\quad\quad + 2\gamma \langle x_n - p, A^* (F_n - I) A x_n \rangle.
\end{aligned}
\tag{52}
$$

But

$$
\begin{aligned}
&\gamma^2 \|A^* (F_n - I) A x_n\|^2 \\
&\quad = \gamma^2 \langle A^* (F_n - I) A x_n, A^* (F_n - I) A x_n \rangle \\
&\quad = \gamma^2 \langle A A^* (F_n - I) A x_n, (F_n - I) A x_n \rangle \\
&\quad \le \gamma^2 \|A\|^2 \|(F_n - I) A x_n\|^2.
\end{aligned}
\tag{53}
$$

Furthermore,

$$
\begin{aligned}
&2\gamma \langle x_n - p, A^* (F_n - I) A x_n \rangle = 2\gamma \langle A x_n \\
&\quad - Ap, (F_n - I) A x_n \rangle = 2\gamma \langle (A x_n - Ap) \\
&\quad + (F_n - I) A x_n - (F_n - I) A x_n, (F_n - I) A x_n \rangle \\
&\quad = 2\gamma \big\{\langle F_n A x_n - Ap, F_n A x_n - A x_n \rangle \\
&\quad - \|(F_n - I) A x_n\|^2\big\}.
\end{aligned}
\tag{54}
$$

Since $Ap \in F(T)$, we set $x = Ax_n$ and $q = Ap$ in (19) to obtain

$$
\begin{aligned}
&\langle F_n A x_n - Ap, F_n A x_n - A x_n \rangle - \|(F_n - I) A x_n\|^2 \\
&\quad \le \left(\frac{2-\beta}{2}\right) \|(F_n - I) A x_n\|^2 \\
&\quad\quad + \frac{1+\beta}{2} \mu_n \phi(\|A x_n - Ap\|) + \frac{1+\beta}{2} \xi_n \\
&\quad\quad - \|(F_n - I) A x_n\|^2 \\
&\quad \le -\left(\frac{\beta}{2}\right) \|F_n A x_n - A x_n\|^2 \\
&\quad\quad + \left(\frac{1+\beta}{2}\right) \mu_n \phi(\|A x_n - Ap\|) + \left(\frac{1+\beta}{2}\right) \xi_n
\end{aligned}
$$

$$\leq -\left(\frac{\beta}{2}\right)\left\|F_nAx_n - Ax_n\right\|^2 + \left(\frac{1+\beta}{2}\right)\mu_n\left(M^*\left\|A\right\|^2\left\|x_n - p\right\|^2 + \phi(M)\right) + \left(\frac{1+\beta}{2}\right)\xi_n. \tag{55}$$

Substituting (55) into (54) yields

$$\begin{aligned} &2\gamma\left\langle x_n - p, A^*\left(F_n - I\right)Ax_n\right\rangle \\ &\quad\leq 2\gamma\left[\frac{-\beta}{2}\left\|F_nAx_n - Ax_n\right\|^2 + \frac{(1+\beta)}{2}\mu_nM^*\left\|A\right\|^2\left\|x_n - p\right\|^2 + \frac{(1+\beta)}{2}\mu_n\phi(M) + \frac{(1+\beta)}{2}\xi_n\right]; \end{aligned} \tag{56}$$

and it follows from (52) that

$$\begin{aligned} \left\|u_n - p\right\|^2 &\leq \left\|x_n - p\right\|^2 + \gamma^2\left\|A\right\|^2\left\|F_nAx_n - Ax_n\right\|^2 - \gamma\beta\left\|F_nAx_n - Ax_n\right\|^2 \\ &\quad + (1+\beta)\gamma\mu_n\left(M^*\left\|A\right\|^2\left\|x_n - p\right\|^2 + \phi(M)\right) + (1+\beta)\gamma\xi_n \\ &= \left[1 + (1+\beta)\gamma\mu_nM^*\left\|A\right\|^2\right]\left\|x_n - p\right\|^2 - \gamma\left(\beta - \gamma\left\|A\right\|^2\right)\left\|F_nAx_n - Ax_n\right\|^2 \\ &\quad + (1+\beta)\gamma\mu_n\phi(M) + (1+\beta)\gamma\xi_n \\ &= \left[1 + \omega_n\right]\left\|x_n - p\right\|^2 - \gamma\left(\beta - \gamma\left\|A\right\|^2\right)\left\|F_nAx_n - Ax_n\right\|^2 + \nu_n, \end{aligned} \tag{57}$$

where $\omega_n = (1+\beta)\gamma\mu_nM^*\|A\|^2$ and $\nu_n = (1+\beta)\gamma\mu_n\phi(M) + (1+\beta)\gamma\xi_n$.

Substituting (57) in (51) we obtain

$$\begin{aligned} \left\|x_{n+1} - p\right\|^2 &\leq \left(1 + \alpha_n\delta_n\right)\left[\left(1+\omega_n\right)\left\|x_n - p\right\|^2 - \gamma\left(\beta - \gamma\left\|A\right\|^2\right)\left\|F_nAx_n - Ax_n\right\|^2 + \nu_n\right] - \alpha_n\left(\beta_n - \alpha_n\right)\left\|u_n - G_nu_n\right\|^2 \\ &\quad - \alpha_n\beta_n\left[1 - \beta_n\left(2 + \mu M^*\right) - \beta_n^2L^2\right]\left\|u_n - S^nu_n\right\|^2 + \alpha_n\sigma_n \leq \left[1 + \omega_n + \left(1+\omega_n\right)\alpha_n\delta_n\right]\left\|x_n - p\right\|^2 - \gamma\left(\beta - \gamma\left\|A\right\|^2\right) \\ &\quad \cdot\left\|F_nAx_n - Ax_n\right\|^2 + \left(1 + \alpha_n\delta_n\right)\alpha_n\nu_n - \alpha_n\left(\beta_n - \alpha_n\right)\left\|u_n - G_nu_n\right\|^2 - \alpha_n\beta_n\left[1 - \beta_n\left(2 + \mu M^*\right) - \beta_n^2L^2\right]\left\|u_n - S^nu_n\right\|^2 + \alpha_n\sigma_n \\ &= \left[1 + \theta_n\right]\left\|x_n - p\right\|^2 - \gamma\left(\beta - \gamma\left\|A\right\|^2\right)\left\|F_nAx_n - Ax_n\right\|^2 - \alpha_n\left(\beta_n - \alpha_n\right) \\ &\quad \cdot\left\|u_n - G_nu_n\right\|^2 - \alpha_n\beta_n\left[1 - \beta_n\left(2 + \mu M^*\right) - \beta_n^2L^2\right] \cdot\left\|u_n - S^nu_n\right\|^2 + \eta_n, \end{aligned} \tag{58}$$

where $\theta_n = \omega_n + (1+\omega_n)\alpha_n\delta_n$ and $\eta_n = \nu_n + \alpha_n(\delta_n\nu_n + \sigma_n)$.

It follows from (58) and condition (45) that

$$\left\|x_{n+1} - p\right\|^2 \leq \left[1 + \theta_n\right]\left\|x_n - p\right\|^2 + \eta_n. \tag{59}$$

The conditions $\sum_{n=1}^{\infty}\mu_n < \infty$ and $\sum_{n=1}^{\infty}\xi_n < \infty$ imply that $\sum_{n=1}^{\infty}\theta_n < \infty$ and $\sum_{n=1}^{\infty}\eta_n < \infty$. It now follows from Lemma 7 that $\lim_{n\to\infty}\|x_n - p\|$ exists. It now follows from (58) that

$$\lim_{n\to\infty}\left\|u_n - S^nu_n\right\| = 0; \tag{60}$$

$$\lim_{n\to\infty}\left\|F_nAx_n - Ax_n\right\| = 0. \tag{61}$$

Consequently,

$$\begin{aligned} \left\|T^nAx_n - Ax_n\right\| &= \left\|\left(T^n - I\right)Ax_n\right\| \\ &= \left\|T^nAx_n - F_nAx_n + F_nAx_n - Ax_n\right\| \\ &\leq \left\|T^nAx_n - F_nAx_n\right\| + \left\|F_nAx_n - Ax_n\right\| \\ &\leq L\beta\left\|T^nAx_n - Ax_n\right\| + \left\|F_nAx_n - Ax_n\right\|. \end{aligned} \tag{62}$$

Hence

$$\left\|T^nAx_n - Ax_n\right\| \leq \frac{1}{1 - L\beta}\left\|F_nAx_n - Ax_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{63}$$

This together with (61) implies that

$$\lim_{n\to\infty}\left\|T^nAx_n - Ax_n\right\| = 0. \tag{64}$$

Since $\lim_{n\to\infty}\|x_n - p\|$ exists, it follows from (52) and (64) that $\lim_{n\to\infty}\|u_n - p\|$ exists and

$$\lim_{n\to\infty}\left\|u_n - p\right\| = \lim_{n\to\infty}\left\|x_n - p\right\|. \tag{65}$$

Step 2. We prove that

$$\begin{aligned} &\lim_{n\to\infty}\left\|x_{n+1} - x_n\right\| = 0, \\ &\lim_{n\to\infty}\left\|u_{n+1} - u_n\right\| = 0. \end{aligned} \tag{66}$$

From (44) we obtain

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|(1-\alpha_n)u_n+\alpha_n S^n y_n - x_n\| \\
&= \|(1-\alpha_n)u_n+\alpha_n G_n u_n - x_n\| = \|(1-\alpha_n) \\
&\cdot (x_n+\gamma A^*[T^n((1-\beta)I+\beta T^n)-I]Ax_n) \\
&+\alpha_n G_n u_n - x_n\| = \|(1-\alpha_n) \\
&\cdot (x_n+\gamma A^*(F_n-I)Ax_n)+\alpha_n G_n u_n - x_n\| \\
&= \|(1-\alpha_n)(\gamma A^*(F_n-I)Ax_n)+\alpha_n(G_n u_n-u_n) \\
&+\alpha_n(u_n-x_n)\| = \|(1-\alpha_n)(\gamma A^*(F_n-I)Ax_n) \\
&+\alpha_n(G_n u_n-u_n) \\
&+\alpha_n(x_n+\gamma A^*(F_n-I)Ax_n-x_n)\| \\
&= \|\gamma A^*(F_n-I)Ax_n+\alpha_n(G_n u_n-u_n)\| \\
&\le \|\gamma A^*(F_n-I)Ax_n\|+\alpha_n\|G_n u_n-u_n\|.
\end{aligned} \tag{67}$$

Observe that

$$\begin{aligned}
\|u_n-G_n u_n\| &\le \|u_n-S^n u_n\|+\|S^n u_n-G_n u_n\| \\
&\le \|u_n-S^n u_n\|+L\beta_n\|u_n-S^n u_n\| \\
&= (1+L\beta_n)\|u_n-S^n u_n\|.
\end{aligned} \tag{68}$$

Using (68) in (67) we obtain

$$\begin{aligned}
\|x_{n+1}-x_n\| \le{}& \|\gamma A^*(F_n-I)Ax_n\| \\
&+\alpha_n(1+L\beta_n)\|u_n-S^n u_n\|,
\end{aligned} \tag{69}$$

and it follows from (60) and (61) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{70}$$

Similarly, it follows from (44), (61), and (70) that

$$\begin{aligned}
\|u_{n+1}-u_n\| &= \|x_{n+1} \\
&+\gamma A^*\left[T^{n+1}\left((1-\beta)I+\beta T^{n+1}\right)-I\right]Ax_{n+1}-x_n \\
&-\gamma A^*[T^n((1-\beta)I+\beta T^n)-I]Ax_n\| = \|x_{n+1} \\
&+\gamma A^*(F_{n+1}-I)Ax_{n+1} \\
&-(x_n+\gamma A^*(F_n-I)Ax_n)\| \le \|x_{n+1}-x_n\| \\
&+\|\gamma A^*(F_{n+1}-I)Ax_{n+1}\| \\
&+\|\gamma A^*(F_n-I)Ax_n\| \longrightarrow 0 \quad \text{as } n\longrightarrow\infty.
\end{aligned} \tag{71}$$

Step 3. We prove that

$$\begin{aligned}
&\|u_n-Su_n\| \longrightarrow 0, \\
&\|Ax_n-TAx_n\| \longrightarrow 0 \\
&\qquad (\text{as } n\longrightarrow\infty).
\end{aligned} \tag{72}$$

In fact, from (60), we have

$$q_n := \|u_n-S^n u_n\| \longrightarrow 0, \quad (\text{as } n\longrightarrow\infty). \tag{73}$$

Since S is uniformly L_1-Lipschitzian, it follows from (71) and (73) that

$$\begin{aligned}
\|u_n-Su_n\| &\le \|u_n-S^n u_n\|+\|S^n u_n-Su_n\| \\
&\le q_n+L\left\|S^{n-1}u_n-u_n\right\| \\
&\le q_n+L\left\{\left\|S^{n-1}u_n-S^{n-1}u_{n-1}\right\|+\left\|S^{n-1}u_{n-1}-u_n\right\|\right\} \\
&\le q_n+L^2\|u_n-u_{n-1}\| \\
&\quad +L\left\|S^{n-1}u_{n-1}-u_{n-1}+u_{n-1}-u_n\right\| \\
&\le q_n+L(1+L)\|u_n-u_{n-1}\|+Lq_{n-1} \longrightarrow 0 \\
&\qquad (\text{as } n\longrightarrow\infty).
\end{aligned} \tag{74}$$

Similarly, from (64), we obtain

$$w_n := \|Ax_n-T^n Ax_n\| \longrightarrow 0, \quad (\text{as } n\longrightarrow\infty). \tag{75}$$

Since T is uniformly L_2-Lipschitzian, it follows from (70) and (75) that

$$\begin{aligned}
\|Ax_n-TAx_n\| &\le \|Ax_n-T^n Ax_n\|+\|T^n Ax_n-TAx_n\| \\
&\le w_n+L\left\|T^{n-1}Ax_n-Ax_n\right\| \le w_n \\
&+L\left\{\left\|T^{n-1}Ax_n-T^{n-1}Ax_{n-1}\right\|\right. \\
&\left.+\left\|T^{n-1}Ax_{n-1}-Ax_n\right\|\right\} \le w_n+L^2\|Ax_n-Ax_{n-1}\| \\
&+L\left\|T^{n-1}Ax_{n-1}-Ax_{n-1}+Ax_{n-1}-Ax_n\right\| \le w_n \\
&+L(1+L)\|Ax_n-Ax_{n-1}\|+Lw_{n-1} \longrightarrow 0 \\
&\qquad (\text{as } n\longrightarrow\infty).
\end{aligned} \tag{76}$$

This implies that

$$\|Ax_n-TAx_n\| \longrightarrow 0, \quad (\text{as } n\longrightarrow\infty). \tag{77}$$

Step 4. We prove that the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to $x^*\in\Gamma$.

Observe that since $\{u_n\}$ is bounded, then there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ which converges weakly to a point $x^*\in H_1$. Since $\lim_{n\to\infty}\|u_n-Su_n\|=0$, we obtain

$$\lim_{i\to\infty}\left\|u_{n_i}-Su_{n_i}\right\| = 0. \tag{78}$$

It follows from Proposition 9 that $x^*\in F(S)$.

Furthermore, from (44) and (61), we obtain

$$x_{n_i} = u_{n_i}-\gamma A^*\left(F_{n_i}-I\right)Ax_{n_i} \rightharpoonup x^*. \tag{79}$$

Since A is linear and bounded, we obtain $Ax_{n_i}\rightharpoonup Ax^*$, and it follows from (72) that

$$\lim_{i\to\infty}\left\|Ax_{n_i}-TAx_{n_i}\right\| = 0. \tag{80}$$

The demiclosedness of T at zero now yields that $Ax^* \in F(T)$, and thus $x^* \in \Gamma$. Since every Hilbert space is an Opial space and $\{u_n\}$ has a subsequence $\{u_{n_i}\}$ which converges weakly to a point $x^* \in \Gamma$, it follows from a standard argument that $\{u_n\}$ converges weakly to x^*.

If S is semicompact, then since $\{u_n\}$ is bounded and $\lim_{n\to\infty}\|u_n - Su_n\| = 0$, we have that there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ which converges strongly to a point $u^* \in H_1$. Since $\{u_n\}$ converges weakly to x^*, we have $u^* = x^*$. Thus $\lim_{j\to\infty}\|u_{n_j} - x^*\| = 0$ and it follows from Lemma 7 that $\{u_n\}$ (and hence $\{x_n\}$) converges strongly to $x^* \in \Gamma$. □

Corollary 11. *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ a bounded linear operator, $S : H_1 \to H_1$ a uniformly L_1-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n^{(1)}\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty}(k_n^{(1)} - 1) < \infty$ and $K := F(S) \neq \emptyset$, and $T : H_2 \to H_2$ a uniformly L_2-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n^{(2)}\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty}(k_n^{(2)} - 1) < \infty$ and $Q := F(T) \neq \emptyset$; let $k_n = \max\{k_n^{(1)}, k_n^{(2)}\}$ and $L = \max\{L_1, L_2\}$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers satisfying the condition:*

$$0 < \delta \le \alpha_n \le \beta_n \le \beta < \frac{2}{(2+k) + \sqrt{4L^2 + (2+k)^2}}, \quad (81)$$

where $k = \sup_{n\ge 1}(k_n - 1)$ and $\gamma \in (0, \beta/\|A\|^2)$. Let $\Gamma = \{x \in H_1 : x \in F(S) \text{ and } Ax \in F(T)\} \neq \emptyset$. Then for arbitrary $x_1 \in H_1$ the sequence $\{x_n\}$ generated from x_1 by (44) converges weakly to a point in Γ.

If in addition S is semicompact, then $\{x_n\}$ and $\{u_n\}$ converge strongly to a point in Γ.

Example 12. Let C and S be as in Example 2, and let D and T be as in Example 3. Then $F(T) = \{(0,0,0,\dots,0)\} \cup \{(x_1, x_2, x_3, \dots, x_n) : \prod_{j=1}^{n} x_j < 0\}$, and $F(S) = \{(0,0,0,\dots)\}$. Furthermore, C and D are nonempty closed convex subsets of ℓ^2 and $\mathfrak{R}^n$, respectively. Define $A : C \to D$ by $Ax = (x_1, x_2, x_3, \dots, x_n)$ for each $x = (x_1, x_2, x_3, \dots) \in C$. Then A is a bounded linear operator with adjoint operator $A^* y = (x_1, x_2, x_3, \dots, x_n, 0, 0, 0, \dots)$ for $y = (x_1, x_2, x_3, \dots, x_n) \in D$. Furthermore, $\|A\| = \|A^*\| = 1$. Thus using algorithm (44) with $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ satisfying the condition $0 < \delta < \alpha_n \le \beta_n < \beta < 1/(1+\sqrt{5})$ and $\gamma \in (0, \beta)$, it follows from Theorem 10 that $x_n \rightharpoonup (0,0,0,\dots) \in F(S)$ and $A(0,0,0,\dots) = (0,0,0,\dots,0) \in F(T)$.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final paper.

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