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Research Article

Generated Surfaces via Inextensible Flows of Curves in $\mathbb{R}^3$

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We study the inextensible flows of curves in 3-dimensional Euclidean space $\mathbb{R}^3$. The main purpose of this paper is constructing and plotting the surfaces that are generated from the motion of inextensible curves in $\mathbb{R}^3$. Also, we study some geometric properties of those surfaces. We give some examples about the inextensible flows of curves in $\mathbb{R}^3$ and we determine the curves from their intrinsic equations (curvature and torsion). Finally, we determine and plot the surfaces that are generated by the motion of those curves by using Mathematica 7.

1. Introduction

The evolution of curves and surfaces has important applications in many fields such ascomputer vision [1, 2], computer animation [3], and image processing [4]. The motion of curves and surfaces in $\mathbb{R}^3$ leads to nonlinear evolution equations, which are often integrable. The connection between integrable systems and the differential geometry of curves has been studied extensively. Some integrable systems arise from invariant curve flows in certain geometries such as affine and centroaffine geometries [5–7] and similarity and projective geometries [8, 9]. Motion of curves in Minkowski space R_1^3 is studied in [10–12]. Hashimoto [13] showed that the integrable nonlinear Schrodinger equation (NLS)

$$i\phi_t + \phi_{ss} + |\phi|^2 \phi = 0 \tag{1}$$

is equivalent to the system for the curvatures k and τ of curves in $\mathbb{R}^3$:

$$\begin{aligned} k_t &= -\tau_s k - 2\tau k_s, \\ \tau_t &= k k_s + \frac{\partial}{\partial s}\left(\frac{-\tau^2 k + k_{ss}}{k}\right), \end{aligned} \tag{2}$$

via the so-called Hashimoto transformation $\phi = k \exp^{i\int \tau(s,t)ds}$. System (2) is equivalent to the vortex filament equation

$$\gamma_t = \gamma_s \times \gamma_{ss} = kb, \tag{3}$$

where b is the binormal vector of γ.

In [14], Schief and Rogers studied the binormal motion of curves of constant curvature or torsion.

This line of research has been extended to motions of curves in three-dimensional space forms.

Recently, Abdel-All et al. [15–18] constructed new geometrical models of motion of plane curves. Also, they constructed a Hashimoto surface from its fundamental coefficients via numerical integration of Gauss-Weingarten equations and fundamental theorem of surfaces. Also, they studied kinematics of moving generalized curves in n-dimensional Euclidean space in terms of intrinsic geometries. Mohamed [19] studied the motions of inextensible curves in spherical space $\mathbb{S}^3$.

In this paper, we will present the flows of curves in $\mathbb{R}^3$. The outline of this paper is as follows.

In Section 2, we study the geometry of curves in $\mathbb{R}^3$. In Sections 3 and 4, we study the motion of curves in $\mathbb{R}^3$ and we get the time evolution of Serret-Frenet frame and the evolution of curvatures. In Section 5, we study the geometric properties of the surfaces that are generated by the motion of the family of curves C_t. In Section 6, we give some examples of motions of inextensible curves in $\mathbb{R}^3$, and we plot the surfaces that are generated by the motion of those curves. For these surfaces, we study the Gaussian and Mean curvatures. Finally, Section 7 is devoted to conclusion.

2. Geometric Preliminaries

Consider a smooth curve in a 3-dimensional space. Assume that u is the parameter along the curve in $\mathbb{R}^3$. Let $r(u)$ denote the position vector of a point on the curve. The metric on the curve is

$$g(u) = \left\langle \frac{\partial r}{\partial u}, \frac{\partial r}{\partial u} \right\rangle, \tag{4}$$

where u is the parameter of the curve. The arc length along the curve is given by

$$\begin{gathered} s(u) = \int_0^u \sqrt{g(\sigma)}d\sigma, \\ \frac{\partial}{\partial s} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial u}; \end{gathered} \tag{5}$$

we use $\{u, t\}$ as coordinates of a point on the curve.

Consider the orthonormal frame $\mathfrak{T} = \{T, N, B\}$, such that T is the tangent vector and N, B denote the normal vectors at any point on the curve.

Lemma 1. *The Frenet frame for the curve in $\mathbb{R}^3$ satisfies the following:*

$$F_s = Q \cdot F, \tag{6}$$

where

$$\begin{gathered} F = \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \\ Q = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}. \end{gathered} \tag{7}$$

Lemma 2. *Consider the curve $\gamma(s(u))$ with an arbitrary parameter $u \in I$. Then the Serret-Frenet frame satisfies*

$$F_u = \sqrt{g}Q \cdot F, \tag{8}$$

where Q and F are given as in (6).

3. Curve Evolution

An evolving curve can be considered as a family of curves parametrized by time. This means that each curve in the family is a mapping $\gamma : I \times (0, 1] \to \mathbb{R}^2$ that assigns for each space parameter $u \in I$ and each time parameter $t \in (0, 1]$; there is a point $\gamma(u, t) \in \mathbb{R}^2$. An evolution equation which is a differential equation that describes the evolution of $\gamma(u, t)$ in time can be specified by the form

$$\dot{\gamma} = \frac{d\gamma}{dt} = WT + UN + VB, \tag{9}$$

where W, U, V are the velocities along the frame T, N, B. Consider a local motion; that is, the velocities W, U, V depend only on the local values of the curvatures $\{k, \tau\}$.

4. Main Results

From [18], we considered the curve $\gamma(u, t)$. For the curve flow

$$\frac{d\gamma}{dt} = \sum_{j=1}^{n} v_j e_j, \tag{10}$$

where v_j are the velocities in the direction of e_j, we had the following.

Lemma 3. *The evolution equation for the metric g is given by*

$$\dot{g} = 2g\lambda, \tag{11}$$

where $\lambda = (\partial v_1/\partial s - kv_2)$.

Theorem 4. *The evolution for the frame $F = (e_1, e_2, \ldots, e_n)^t$ can be given in a matrix form*

$$F_t = M \cdot F, \tag{12}$$

where M is the evolution matrix and it takes the form

$$M = \begin{pmatrix} 0 & M_{12} & M_{13} & \cdots & M_{1n} \\ -M_{12} & 0 & M_{23} & \cdots & M_{2n} \\ -M_{13} & -M_{23} & 0 & \cdots & M_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -M_{1(n-1)} & -M_{2(n-1)} & -M_{3(n-1)} & \cdots & M_{(n-1)n} \\ -M_{1n} & -M_{2n} & -M_{3n} & \cdots & 0 \end{pmatrix}, \tag{13}$$

where the elements of the matrix M are given explicitly by

$$\begin{aligned} M_{1j} &= A_j = v_{j,s} + k_{j-1}v_{j-1} - k_j v_{j+1}, \quad j = 2, 3, \ldots, n, \\ M_{\alpha\mu} &= \frac{1}{k_{\mu-\alpha}} \Big(M_{(\alpha-1)\mu,s} + k_{\mu-1} M_{(\alpha-1)(\mu-1)} \\ &\quad + k_\mu M_{(\alpha-1)(\mu+1)} + k_{\alpha-2} M_{(\alpha-2)\mu} \Big), \\ &\qquad \alpha = 2, \ldots, n-1, \ \alpha < \mu \le n, \ k_0 = k_n = 0. \end{aligned} \tag{14}$$

Theorem 5. *The time evolution equations for the curvatures take the form*

$$\begin{aligned} k_{1,t} &= M_{12,s} - k_1\lambda - k_2 M_{13}, \\ k_{\alpha,t} &= M_{\alpha\mu,s} - k_\alpha\lambda + k_{\alpha-1}M_{(\alpha-1)\mu} - k_{\alpha+1}M_{\alpha(\mu+1)}, \\ &\qquad k_0 = k_n = 0. \end{aligned} \tag{15}$$

For $n = 3$, we can study the motion of curves in $\mathbb{R}^3$; we choose $k_1 = k$, $k_2 = \tau$, $v_1 = W$, $v_2 = U$, and $v_3 = V$; then we have the following.

Theorem 6. *The time evolution of the Serret-Frenet frame can be written in matrix form as follows:*

$$F_t = M \cdot F, \tag{16}$$

where

$$F = \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & M_{12} & M_{13} \\ -M_{12} & 0 & M_{23} \\ -M_{13} & -M_{23} & 0 \end{pmatrix}, \tag{17}$$

$$\begin{aligned} M_{12} &= kW + U_s - \tau V, \\ M_{13} &= V_s + \tau U, \\ M_{23} &= \frac{1}{k}\left(M_{13,s} + \tau M_{12}\right). \end{aligned} \tag{18}$$

Theorem 7. *The time evolution of the curvature and torsion of the curve C_t can be given by*

$$\begin{pmatrix} k \\ \tau \end{pmatrix}_t = \begin{pmatrix} -\dfrac{g_t}{2g} & -M_{13} \\ M_{13} & -\dfrac{g_t}{2g} \end{pmatrix} \begin{pmatrix} k \\ \tau \end{pmatrix} + \begin{pmatrix} M_{12,s} \\ M_{23,s} \end{pmatrix}. \tag{19}$$

Definition 8. An inextensible curve is a curve whose length is preserved; that is, it does not evolve in time:

$$\frac{\partial s}{\partial t} = 0, \qquad \text{i.e., } g_t = \dot{g} = 0. \tag{20}$$

The necessary and sufficient conditions for inextensible flows are then given by the following theorem.

Theorem 9. *The flow of the curve is inextensible if and only if $\partial W / \partial s = kU$.*

Lemma 10. *If the curve $\gamma(s,t)$ is inextensible ($g_t = \dot{g} = 0$), then the evolution equations for the curvature and torsion (19) are*

$$\begin{pmatrix} k \\ \tau \end{pmatrix}_t = \begin{pmatrix} 0 & -M_{13} \\ M_{13} & 0 \end{pmatrix} \begin{pmatrix} k \\ \tau \end{pmatrix} + \begin{pmatrix} M_{12,s} \\ M_{23,s} \end{pmatrix}. \tag{21}$$

Then the PDE system (21) can be written explicitly in the following form:

$$\begin{aligned} k_t &= \left(k^2 - \tau^2\right) U + U_{ss} + k_s W - \tau_s V - 2\tau V_s, \\ \tau_t &= k\left(V_s + \tau U\right) \\ &\quad + \left(\frac{1}{k}\left(-\tau^2 V + \tau\left(kW + 2U_s\right) + V_{ss} + \tau_s U\right)\right)_{,s}. \end{aligned} \tag{22}$$

5. Examples of Inextensible Flows of Curves in $\mathbb{R}^3$

Example 11. If

$$\begin{aligned} W &= \text{constant} = a \neq 0, \\ U &= 0, \\ V &= \frac{k(s,t)}{a}, \end{aligned} \tag{23}$$

the PDE system (22) takes the form

$$\begin{aligned} k_t &= \frac{1}{a}\left(-k\tau_s + \left(a^2 - 2\tau\right) k_s\right), \\ \tau_t &= \frac{k}{a} k_s + \left(\frac{\tau}{a}\left(-\tau^2 + a^2\right) + \frac{k_{ss}}{k}\right)_{,s}. \end{aligned} \tag{24}$$

One solution of this system is

$$\begin{aligned} k(s,t) &= -2c_1 \operatorname{sech}\left(c_1 s + c_2 t + c_3\right), \\ \tau(s,t) &= \frac{a\left(ac_1 - c_2\right)}{2c_1}, \end{aligned} \tag{25}$$

where c_1, c_2, and c_3 are constants.

The curvature of the family of curves C_t as a function of the coordinates s and t is plotted in Figure 1.

Substitute (23) and (25) into systems (6) and (16) and solve them numerically. Then we can get the family of curves $C_t = \gamma(s,t)$, so we can determine the surface that is generated by this family of curves (Figure 2).

Example 12. If

$$\begin{aligned} W &= \text{constant} = a, \\ U &= 0, \end{aligned} \tag{26}$$

and assuming that $\tau(s,t) = k(s,t)$, then PDE system (22) takes the form

$$\begin{aligned} \tau_t &= ak_s - \tau_s V - 2\tau V_s, \\ \tau_t &= kV_s + \left(\frac{1}{k}\left(-\tau^2 V + a\tau k + V_{ss}\right)\right)_{,s}. \end{aligned} \tag{27}$$

One solution of this system is

$$\begin{aligned} \tau(s,t) &= k(s,t) = c_4 \tanh\left(c_1 s + c_2 t + c_3\right), \\ V(s,t) &= \frac{ac_1 - c_2}{c_1}, \end{aligned} \tag{28}$$

where c_1, c_2, c_3, and c_4 are constants.

The curvature of the family of curves C_t as a function of the coordinates s and t is plotted in Figure 3.

Substitute (26) and (28) into systems (6) and (16) and solve them numerically. Then we can get the family of curves $C_t = \gamma(s,t)$, so we can determine the surface that is generated by this family of curves (Figure 4).

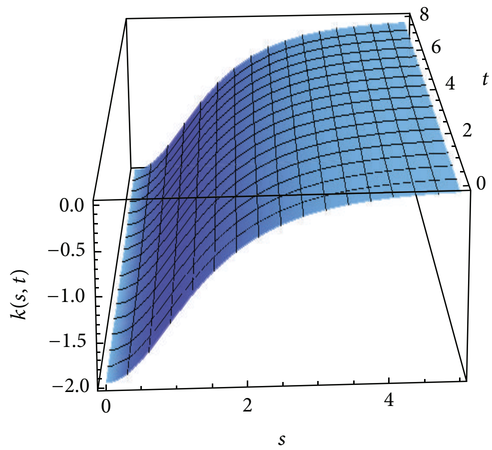

Figure 1: The curvature of the family of curves C_t for $s \in [0,5]$, $t \in [0,8]$, $a = 0.5$, $c_1 = 1$, $c_2 = -0.001$, and $c_3 = 0$.

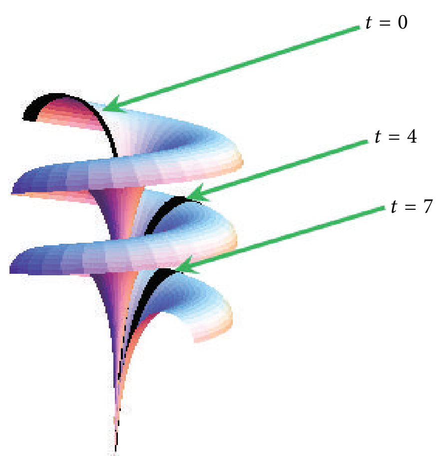

Figure 2: The surface that is generated by the motion of the family of curves C_t for $s \in [0,5]$, $t \in [0,8]$, $a = 0.5$, $c_1 = 1$, $c_2 = -0.001$, and $c_3 = 0$. The bold black curves in this surface represent the family of curves C_t for $t = 0, 4, 7$.

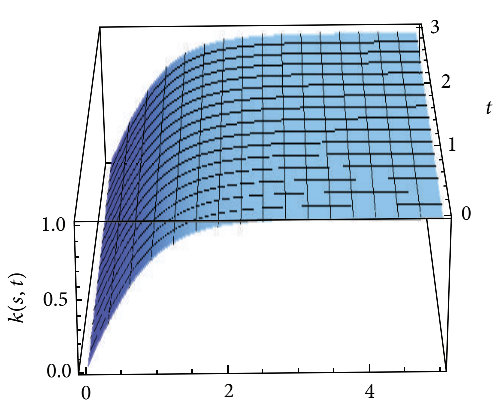

Figure 3: The curvature of the family of curves C_t for $s \in [0,5]$, $t \in [0,3]$, $a = 0.6$, $c_1 = 1$, $c_2 = 0.01$, $c_3 = 0$, and $c_4 = 1$.

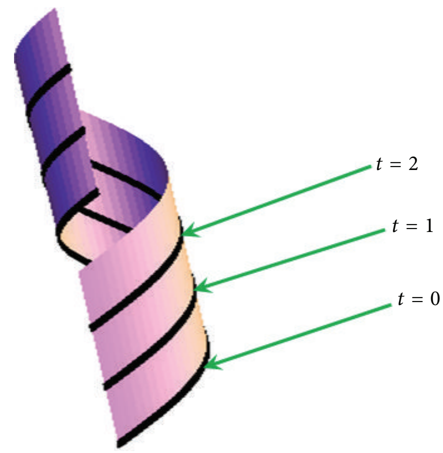

Figure 4: The surface that is generated by the motion of the family of curves C_t for $s \in [0,5]$, $t \in [0,3]$, $a = 0.6$, $c_1 = 1$, $c_2 = 0.01$, $c_3 = 0$, and $c_4 = 1$. The bold black curves in this surface represent the family of curves C_t for $t = 0, 1, 2$.

Example 13. If

$$\begin{aligned} W &= k(s,t), \\ U &= \frac{k_s}{k}, \\ V &= b = \text{constant} \neq 0, \end{aligned} \tag{29}$$

then the PDE system (22) takes the form

$$\begin{aligned} k_t &= \left(k^2 - \tau^2\right)\left(\frac{k_s}{k}\right) + \left(\frac{k_s}{k}\right)_{ss} + kk_s - b\tau_s, \\ \tau_t &= k\tau\frac{k_s}{k} \\ &\quad + \left(\frac{1}{k}\left(-\tau^2 b + \tau\left(k^2 + 2\left(\frac{k_s}{k}\right)_s\right) + \frac{\tau_s k_s}{k}\right)\right)_{,s}. \end{aligned} \tag{30}$$

One solution of this system is

$$k(s,t) = \tau(s,t) = -\sqrt{2}\frac{c_1}{b}\operatorname{sech}\left(\frac{c_1(-s+bt) + bc_2}{b}\right), \tag{31}$$

where c_1 and c_2 are constants.

The curvature of the family of curves C_t as a function of the coordinates s and t is plotted in Figure 5.

Substitute (29) and (31) into systems (6) and (16) and solve them numerically. Then we can get the family of curves $C_t = \gamma(s,t)$, so we can determine the surface that is generated by this family of curves (Figure 6).

5.1. Examples of Binormal Motion of Inextensible Curves. Consider the binormal motion of inextensible curves in $\mathbb{R}^3$, so $W = U = 0$. Then the evolution equation (9) takes the form

$$\dot{r} = \frac{dr}{dt} = VB. \tag{32}$$

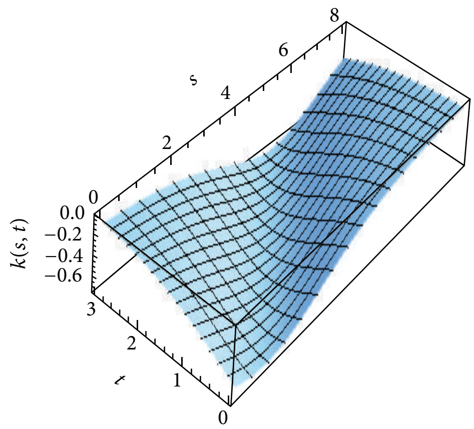

FIGURE 5: The curvature of the family of curves C_t for $s \in [0, 8]$, $t \in [0, 3]$, $b = 1.5$, $c_1 = 0.8$, and $c_2 = 0.5$.

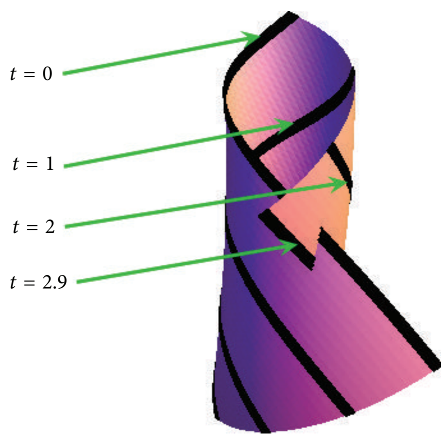

FIGURE 6: The surface that is generated by the motion of the family of curves C_t for $s \in [0, 8]$, $t \in [0, 3]$, $b = 1.5$, $c_1 = 0.8$, and $c_2 = 0.5$. The bold black curves in this surface represent the family of curves C_t for $t = 0, 1, 2, 2.9$.

Lemma 14. *Consider the binormal motion of inextensible curves in $\mathbb{R}^3$; then the time evolution of the Serret-Frenet frame can be written in matrix form as follows:*

$$F_t = M \cdot F, \tag{33}$$

where

$$F = \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$
$$M = \begin{pmatrix} 0 & -\tau V & V_s \\ \tau V & 0 & \frac{1}{k}\left(-\tau^2 V + V_{ss}\right) \\ -V_s & -\frac{1}{k}\left(-\tau^2 V + V_{ss}\right) & 0 \end{pmatrix}. \tag{34}$$

The PDE system (22) takes the form

$$\begin{aligned} k_t &= -\tau_s V - 2\tau V_s, \\ \tau_t &= kV_s + \left(\frac{1}{k}\left(-\tau^2 V + V_{ss}\right)\right)_{,s}. \end{aligned} \tag{35}$$

Example 15. The famous example of binormal motion is the motion of the vortex filament in $\mathbb{R}^3$, where the binormal velocity equals the curvature of the curve $V = k$, and the evolution equation is

$$\dot{r} = \frac{dr}{dt} = kB. \tag{36}$$

If $V = k$, then the PDE system (35) takes the form

$$\begin{aligned} k_t &= -k\tau_s - 2\tau k_s, \\ \tau_t &= kk_s + \left(\frac{1}{k}\left(-\tau^2 k + k_{ss}\right)\right)_{,s}. \end{aligned} \tag{37}$$

The solution of the PDE (37) is

$$\begin{aligned} k(s,t) &= -2c_1 \operatorname{sech}\left(c_1 s + c_2 t + c_3\right), \\ \tau(s,t) &= \tau_0 = \frac{a}{2c_1}\left(ac_1 - c_2\right), \end{aligned} \tag{38}$$

where c_1, c_2, and c_3 are constants.

The curvature of the family of curves C_t as a function of the coordinates s and t is plotted in Figures 7 and 8.

Substitute (38) into (6), (33) for $V = k$ and solve them numerically. Then we can get the family of curves $\gamma(s,t)$, so we can determine the surface that is generated by this family of curves. For $\tau = \tau_0 = -1/2$ see (Figure 9), and for $\tau = \tau_0 = 0$ see (Figure 10).

Example 16. If $V = \text{constant} = a$, then (35) takes the form

$$\begin{aligned} k_t &= -a\tau_s, \\ \tau_t &= \left(\frac{-a\tau^2}{k}\right)_{,s}. \end{aligned} \tag{39}$$

The solution of the PDE system (39) is

$$\begin{aligned} k(s,t) &= \frac{ac_1}{c_2}\left(c_4 + c_5 \tanh\left(c_1 s + c_2 t + c_3\right)\right), \\ \tau(s,t) &= c_4 + c_5 \tanh\left(c_1 s + c_2 t + c_3\right), \end{aligned} \tag{40}$$

where c_1, c_2, c_3, c_4, and c_5 are constants.

The curvature of the family of curves C_t as a function of the coordinates s and t is plotted in Figure 11.

Substitute (40) into (6), (33) and solve them numerically. Then we can get the family of curves $\gamma(s,t)$, so we can determine the surface that is generated by this family of curves (Figure 12).

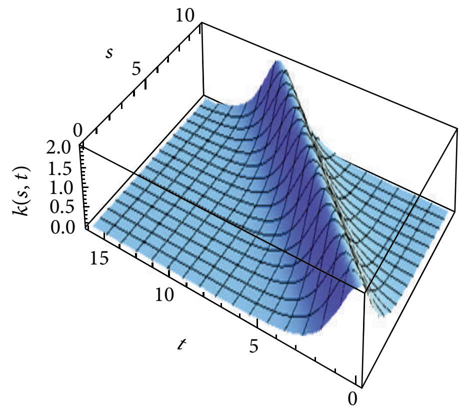

FIGURE 7: The curvature of the family of curves C_t for $s \in [0, 10]$, $t \in [0, 15]$, $c_1 = 1$, $c_2 = -1$, and $c_3 = 0$.

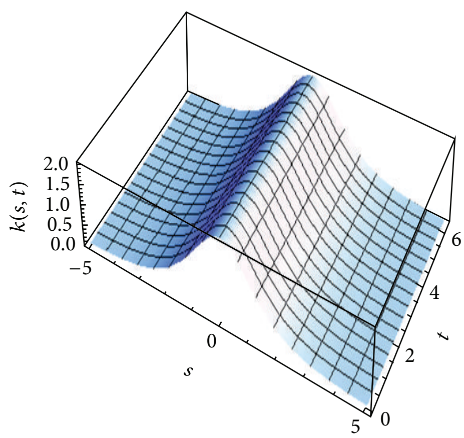

FIGURE 8: The curvature of the family of curves C_t for $s \in [0, 5]$, $t \in [0, 6.3]$, $c_1 = 1$, $c_2 = 0$, and $c_3 = 0$.

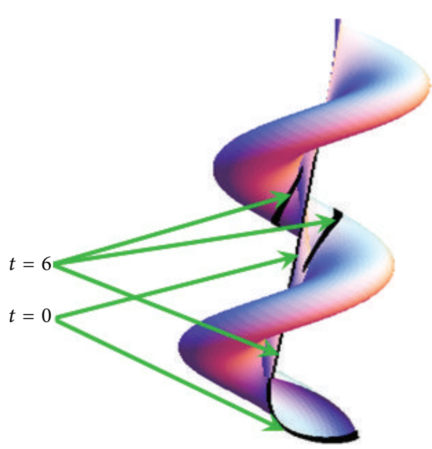

FIGURE 9: The surface that is generated by the motion of the family of curves C_t for $\tau_0 = -1/2$, $s \in [0, 10]$, $t \in [0, 15]$, $c_1 = 1$, $c_2 = -1$, and $c_3 = 0$. The bold black curves in this surface represent the family of curves C_t for $t = 0, 6$.

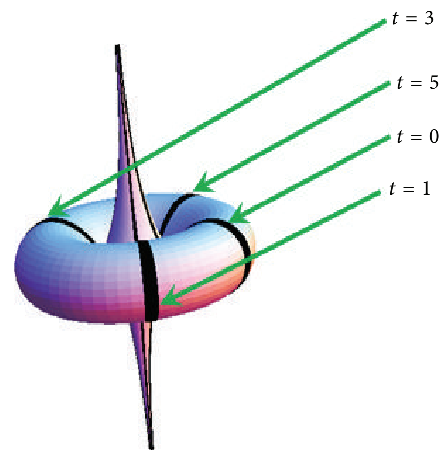

FIGURE 10: The surface that is generated by the motion of the family of curves C_t for $\tau_0 = 0$, $s \in [0, 5]$, $t \in [0, 6.3]$, $c_1 = 1$, $c_2 = 0$, and $c_3 = 0$. The bold black curves in this surface represent the family of curves C_t for $t = 0, 1, 3, 5$.

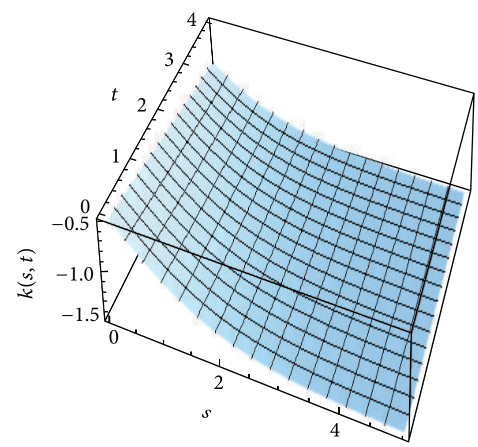

FIGURE 11: The curvature of the family of curves C_t for $s \in [0, 8]$, $t \in [0, 3]$, $b = 1.5$, $c_1 = 0.8$, and $c_2 = 0.5$.

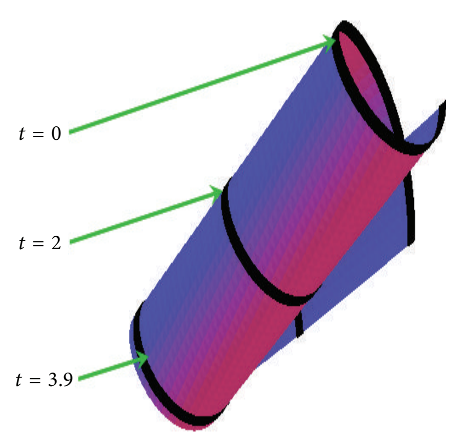

FIGURE 12: The surface that is generated by the motion of the family of curves C_t for $s \in [0, 8]$, $t \in [0, 3]$, $b = 1.5$, $c_1 = 0.8$, and $c_2 = 0.5$. The bold black curves in this surface represent the family of curves C_t for $t = 0, 1, 2, 2.9$.

6. Geometric Properties of the Generated Surfaces

Let $\Sigma = \gamma(s,t)$ be the surface that is generated by the motion of the family of curves C_t. In this section, we study some geometric properties of these surfaces.

Lemma 17. *The first fundamental form of the surface $\Sigma = \gamma(s,t)$ in $\mathbb{R}^3$ is given by*

$$I = g_{11}(ds)^2 + 2g_{12}ds\,dt + g_{22}(dt)^2, \tag{41}$$

where g_{11}, g_{12}, g_{22} are the first fundamental quantities and they are given by

$$\begin{aligned} g_{11} &= \langle \gamma_s, \gamma_s \rangle = 1, \\ g_{12} &= \langle \gamma_s, \gamma_t \rangle = W^2 + U^2 + V^2, \\ g_{22} &= \langle \gamma_t, \gamma_t \rangle = W. \end{aligned} \tag{42}$$

Lemma 18. *The unit normal vector $n(s,t)$ to the surface $\Sigma = \gamma(s,t)$ in $\mathbb{R}^3$ at a point $p = \gamma(s,t)$ on the surface is given by*

$$n(s,t) = \frac{1}{\sqrt{U^2+V^2}}(UB - VN). \tag{43}$$

Lemma 19. *The second fundamental form of the surface $\Sigma = \gamma(s,t)$ in $\mathbb{R}^3$ is given by*

$$II = L_{11}(ds)^2 + 2L_{12}ds\,dt + L_{22}(dt)^2, \tag{44}$$

where L_{11}, L_{12}, L_{22} are the second fundamental quantities and they are given by

$$\begin{aligned} L_{11} &= \langle \gamma_{ss}, n \rangle = \frac{-kV}{\sqrt{U^2+V^2}}, \\ L_{12} &= \langle \gamma_{st}, n \rangle = \frac{1}{\sqrt{U^2+V^2}}(-UM_{13} + VM_{12}), \\ L_{22} &= \langle \gamma_{tt}, n \rangle = \frac{1}{\sqrt{U^2+V^2}}\Big(UV_t - U_tV \\ &\quad + W(UM_{13} - VM_{12}) + \left(U^2+V^2\right)M_{23}\Big), \end{aligned} \tag{45}$$

where M_{12}, M_{13}, and M_{23} are given from (18).

Lemma 20. *For the surface $\Sigma = \gamma(s,t)$ in $\mathbb{R}^3$, the Gaussian and the Mean curvatures K and H, respectively, are given by*

$$\begin{aligned} K &= \frac{\det II}{\det I} = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2}, \\ H &= \frac{1}{2}\frac{L_{11}g_{22} + L_{22}g_{11}}{g_{11}g_{22} - g_{12}^2}. \end{aligned} \tag{46}$$

Lemma 21. *The Gaussian curvature K and the Mean curvature H for the surface in (Figure 2) are given by*

$$\begin{aligned} K(s,t) &= \frac{-k_{ss}}{k^2}, \\ H(s,t) &= \frac{1}{2}\frac{k^3 + k\tau^2 - k_{ss}}{k^2}. \end{aligned} \tag{47}$$

Lemma 22. *The Gaussian curvature K and the Mean curvature H for the surface in (Figure 4) are given by*

$$\begin{aligned} K(s,t) &= 0, \\ H(s,t) &= \frac{-(c_2 - ac_1)\tau}{ac_1 - c_2}. \end{aligned} \tag{48}$$

Hence the surface in Figure 4 is developable surface.

Lemma 23. *The Gaussian curvature K and the Mean curvature H for the surface in (Figure 6) are given by*

$$\begin{aligned} K(s,t) &= \frac{-k_{ss}}{k^2}, \\ H(s,t) &= \frac{1}{2}\frac{k^3 + k\tau^2 - k_{ss}}{k^2}. \end{aligned} \tag{49}$$

Lemma 24. *The Gaussian curvature K and the Mean curvature H for the surfaces in Figures 9 and 10 are given by*

$$\begin{aligned} K(s,t) &= -\frac{k_{ss}}{k}, \\ H(s,t) &= -\frac{1}{2}\frac{k^3 + k\tau_0^2 - k_{ss}}{k^2}. \end{aligned} \tag{50}$$

Lemma 25. *The Gaussian curvature K and the Mean curvature H for the surface in (Figure 12) are given by*

$$\begin{aligned} K(s,t) &= 0, \\ H(s,t) &= -\frac{1}{2}\frac{k^2 + k\tau^2}{k}. \end{aligned} \tag{51}$$

Hence the surface in (Figure 12) is developable surface.

7. Conclusion

In this paper we studied the inextensible flows of curves in $\mathbb{R}^3$. We constructed and plotted the surfaces that are generated from the motion of inextensible curves in $\mathbb{R}^3$. Also, we studied some geometric properties of those surfaces.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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