

Research Article

Per-Spectral Characterizations of Bicyclic Networks

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Spectral techniques are used for the study of several network properties: community detection, bipartition, clustering, design of highly synchronizable networks, and so forth. In this paper, we investigate which kinds of bicyclic networks are determined by their per-spectra. We find that the permanent spectra cannot determine sandglass graphs in general. When we restrict our consideration to connected graphs or quadrangle-free graphs, sandglass graphs are determined by their permanent spectra. Furthermore, we construct countless pairs of per-cospectra bicyclic networks.

1. Introduction

It was recognized in about last decade that graph spectra have several important applications in computer science. Graph spectra appear in internet technologies, pattern recognition, computer vision, data mining, multiprocessor systems, statistical databases, and many other areas. For example, spectral filtering is applied in the study of Internet structure [1]. This method uses the eigenvectors of the adjacency and other graph matrices and some clusters in data sets represented by graphs. For more information about the applications of graph spectra in computer science see [2–6], among others.

The *permanent* of $n \times n$ matrix $Y = (y_{ij})$ ($i, j = 1, 2, \dots, n$) is defined as

$$\text{per}(Y) = \sum_{\sigma} \prod_{i=1}^n y_{i\sigma(i)}, \quad (1)$$

where the sum is taken over all permutations σ of $\{1, 2, \dots, n\}$. Valiant [7] showed that computing the permanent is #P-complete even when restricted to $(0, 1)$ -matrices.

Let G be a graph with n vertices, and let $A(G)$ be $(0,1)$ -adjacency matrix of G . The *permanent polynomial* of G , denoted by $\pi(G, x)$, is defined as $\pi(G, x) = \text{per}(xI - A(G))$. The *permanent spectrum* (per-spectrum for short) of graph G , denoted by $ps(G)$, is the set of all roots (together with their multiplicities) of $\pi(G, x)$.

Two graphs are *per-cospectral* if they share the same per-spectrum. A graph G is said to be *determined by its per-spectrum* (DPS for short) if there is no other nonisomorphic graph with the same per-spectrum.

Which graphs are determined by their adjacency spectra is an old problem in graph spectra theory. van Dam and Haemers [8, 9] gave an excellent survey of answers to the question of which graphs are determined by the spectra of some graph polynomials. Merris et al. [10] first considered the problem which graph is DPS. And they showed that the five pairs adjacency cospectral graphs (see [11]) are DPS. Based on the result, they formulated that the per-spectrum seems a little better than the adjacency spectrum when it comes to distinguishing graphs which are not trees. In fact, characterizing what kinds of graphs are determined by the per-spectra is generally a very hard problem. Up to now, only a few types of graphs are proved to be DPS; see [12–17].

A *bicyclic network* is a simple connected graph in which the number of edges equals the number of vertices plus one [18]. The *sandglass graph* is a bicyclic network, denoted by $S(C_3, P_h, C_3)$, obtained by appending a triangle to each pendant vertex of the path P_h . Lu et al. [19] proved that sandglass graphs are determined by their adjacency spectra. Motivated by the statement of Merris et al., a natural problem is whether sandglass graphs are determined by their per-spectra. In this paper, we give a solution of this question.

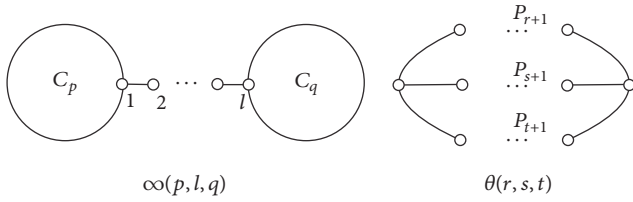


FIGURE 1: ∞ -graph and θ -graph.

In what follows, we begin with some definitions and notions. Let $G \cup H$ be the union of two graphs G and H which have no common vertices. For any positive integer l , let lG denote the union of l disjoint copies of graph G . The path and cycle on n vertices are denoted by P_n and C_n , respectively. Let $c_i(G)$ denote the number of i -cycles in G .

Let C_p and C_q be two vertex-disjoint cycles. Suppose that v_1 is a vertex of C_p and v_l is a vertex of C_q . Joining v_1 and v_l by a path $v_1 v_2 \dots v_l$ of length $l - 1$, where $r \geq 1$ and $l = 1$ means identifying v_1 with v_l , the resulting graph (see Figure 1), denoted by $\infty(p, l, q)$, is called ∞ -graph. Let P_{r+1} , P_{s+1} , and P_{t+1} be three vertex-disjoint paths, where $r, s, t \geq 1$ and at most one of them is 1. Identifying the three initial vertices and terminal vertices of them, respectively, the resulting graph (see Figure 1), denoted by $\theta(r, s, t)$, is called θ -graph. Then bicyclic networks can be partitioned into two classes: the class of graphs which contain ∞ -graph as its induced subgraph and the class of graphs which contain θ -graph as its induced subgraph.

A subgraph H of G is a *Sachs subgraph* if each component of H is a single edge or a cycle. Merris et al. [10] gave a modified Sachs formula to compute the coefficients of the permanental polynomials of graphs.

Lemma 1 (see [10]). *Let G be a graph with $\pi(G, x) = \sum_{k=0}^n b_k(G)x^{n-k}$. Then*

$$b_k(G) = (-1)^k \sum_H 2^{c(H)}, \quad 1 \leq k \leq n, \tag{2}$$

where the sum is taken over all Sachs subgraphs H of G on k vertices, and $c(H)$ is the number of cycles in H .

Lemma 2 (see [13]). *Let G be a graph with n vertices and m edges, and let (d_1, d_2, \dots, d_n) be the degree sequence of G . Then*

$$\begin{aligned} b_0(G) &= 1, \\ b_1(G) &= 0, \\ b_2(G) &= m, \\ b_3(G) &= -2c_3(G), \\ b_4(G) &= \binom{m}{2} - \sum_{i=1}^n \binom{d_i}{2} + 2c_4(G). \end{aligned} \tag{3}$$

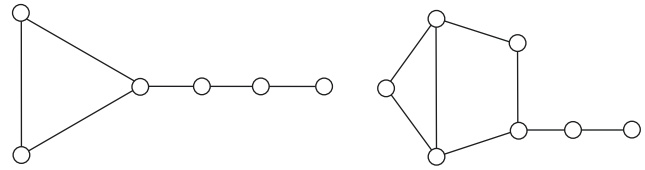


FIGURE 2: Graph G' .

Lemma 3 (see [17]). *Let G be a graph with m edges, and let $t_i(G)$ denote the degree sum of the three vertices on i th triangle in G . Then*

$$b_5(G) = -2 \left(\sum_{i=1}^{c_3(G)} (m + 3 - t_i(G)) + c_5(G) \right). \tag{4}$$

Lemma 4 (see [13]). *The following can be deduced from the permanental polynomial of a graph G :*

- (i) *The number of vertices*
- (ii) *The number of edges*
- (iii) *The number of triangles*
- (iv) *The length of the shortest odd cycle*
- (v) *The number of the shortest odd cycles*
- (vi) *Whether G is bipartite*

2. Sandglass Graphs Are DPS

In this section, we will give the solutions of the problem which sandglass graphs are DPS?

Checking graph G' depicted in Figure 2, direct computation yields $\pi(G', x) = x^{13} + 14x^{11} - 4x^{10} + 74x^9 - 40x^8 + 186x^7 - 136x^6 + 230x^5 - 180x^4 + 130x^3 - 76x^2 + 25x - 4 = \pi(S(C_3, P_9, C_3), x)$. This implies that the permanental spectra cannot determine sandglass graphs in general. Examining graph G' again, we know that G' is not connected and contains a quadrangle. It is natural to consider the problem which sandglass graphs are DPS when we restrict our consideration to connected graphs or quadrangle-free graphs, where the quadrangle-free graph is one which contains no quadrangles (i.e., cycles of length 4). We will answer these questions one by one in the following.

First, we give some lemmas which will play an important role in the proof of main theorems.

Lemma 5. *Let $S(C_3, P_h, C_3)$ be a graph with n vertices. Then $b_5(S(C_3, P_h, C_3)) = -4n + 12$.*

Proof. By Lemma 3, the proof is trivial. □

Lemma 6. *Let $S(C_3, P_h, C_3)$ be a graph with n vertices. Then n -th coefficient of $\pi(S(C_3, P_h, C_3), x)$ is*

$$b_n(S(C_3, P_h, C_3)) = \begin{cases} -4, & \text{if } n \text{ is odd;} \\ 5, & \text{if } n \text{ is even.} \end{cases} \tag{5}$$

Proof. Suppose that n is odd. Then the Sachs subgraph of order n in $S(C_3, P_h, C_3)$ is only $C_3 \cup ((n-3)/2)P_2$. By Lemma 1, we have $b_n(G) = -4$. Otherwise, if n is even, then the Sachs subgraph of order n in $S(C_3, P_h, C_3)$ is $C_3 \cup C_3 \cup ((n-6)/2)P_2$ and $(n/2)P_2$. By Lemma 1, we have $b_n(G) = 5$. \square

Lemma 7. *Let G be a quadrangle-free graph with n vertices. If $\pi(G, x) = \pi(S(C_3, P_h, C_3), x)$, then the degree sequence of G is $(2^{n-2}, 3^2)$, where a^k means $\overbrace{a, a, \dots, a}^k$.*

Proof. Suppose that the degree sequence G is $(3+t_1, 3+t_2, 2+t_3, \dots, 2+t_n)$, where $t_1, t_2 > -3$, and $t_i > -2$ ($i = 3, 4, \dots, n$) are integers. Since G and $S(C_3, P_h, C_3)$ have the same number of edges, then

$$\sum_{i=1}^n t_i = 0. \tag{6}$$

By Lemma 2, we have

$$b_4(G) = \binom{n+1}{2} - \binom{3+t_1}{2} - \binom{3+t_2}{2} - \sum_{i=3}^n \binom{2+t_i}{2}. \tag{7}$$

Since $\pi(G, x) = \pi(S(C_3, P_h, C_3), x)$, we have $b_4(G) = b_4(S(C_3, P_h, C_3))$. By a simple calculation, we obtain

$$\sum_{i=1}^n t_i^2 = -2(t_1 + t_2). \tag{8}$$

Checking (8), it is easy to see that if $|t_1| \geq 3$ or $|t_2| \geq 3$, then $\sum_{i=3}^n t_i^2 < 0$, a contradiction. Hence,

$$-2 \leq t_1, t_2 \leq 2. \tag{9}$$

Furthermore, if $t_1 = -t_2$ except $t_1 = t_2 = 0$, then $\sum_{i=3}^n t_i^2 < 0$, a contradiction. Thus

$$t_1 + t_2 \neq 0. \tag{10}$$

Solving simultaneous equations (6)–(10), we have

$$\begin{aligned} t_1 &= t_2 = -2, \\ t_3 &= \dots = t_n = 0, \\ t_1 &= -2, \\ t_2 &= -1, \\ t_3 &= \pm 1, \\ t_4 &= \dots = t_n = 0, \\ t_1 &= -2, \\ t_2 &= 0, \\ t_3 &= \dots = t_n = 0, \\ t_1 &= t_2 = -1, \\ t_3 &= t_4 = \pm 1, \end{aligned}$$

$$\begin{aligned} t_5 &= \dots = t_n = 0, \\ t_1 &= -1, \\ t_2 &= 0, \\ t_3 &= \pm 1, \\ t_4 &= \dots = t_n = 0, \\ t_1 &= 0, \\ t_2 &= -2, \\ t_3 &= \dots = t_n = 0, \\ t_1 &= 0, \\ t_2 &= -1, \\ t_3 &= \pm 1, \\ t_4 &= \dots = t_n = 0, \\ t_1 &= t_2 = 0, \\ t_3 &= \dots = t_n = 0. \end{aligned} \tag{11}$$

Thus the degree sequence of G is possible $(1^2, 2^{n-2})$, $(1, 2^{n-2}, 3)$, or $(2^{n-2}, 3^2)$. It is not difficult to check that only $(2^{n-2}, 3^2)$ satisfies the well-known hand-shaking theorem. So, the degree sequence of G is $(2^{n-2}, 3^2)$. \square

Theorem 8. *Restricting consideration on quadrangle-free graphs, sandglass graphs are determined by their per-spectra.*

Proof. Let G be a quadrangle-free graph with n vertices per-cospectral with $S(C_3, P_h, C_3)$. By Lemma 7, we know that the degree sequence of G is $(2^{n-2}, 3^2)$. Then G is isomorphic to $\infty(p, l, q) \cup (\bigcup_{i=1}^k C_{k_i})$ or $\theta(r, s, t) \cup (\bigcup_{i=1}^k C_{k_i})$, where $\bigcup_{i=1}^k C_{k_i}$ denotes the union of $k \geq 0$ disjoint cycles C_{k_i} of length k_i . By the above, it implies that $|V(G)| \geq 5$. It is not difficult to see that if $n = 5$, then G is isomorphic to $S(C_3, P_1, C_3)$. So, assume $n > 5$ and consider the following two cases.

Case 1. Assume that G is isomorphic to $\infty(p, l, q) \cup (\bigcup_{i=1}^k C_{k_i})$. We further discuss the following three subcases.

Subcase 1.1. Exact one of the two triangles belongs to the bicyclic component. Then $G = \infty(p, l, q) \cup C_3 \cup \bigcup_{i=1}^{k-1} C_{k_i}$ for $p = 3$, $q \geq 4$, and $k \geq 1$. By Lemma 3, we obtain that $b_5(G) = -4n + 10 - 2c_5(G)$. By Lemma 5, we have $b_5(S(C_3, P_1, C_3)) - b_5(G) = 2 + 2c_5(G) > 0$. This contradicts the assumption that G and $S(C_3, P_1, C_3)$ are per-cospectral.

Subcase 1.2. Both of the two triangles belong to the bicyclic component. Then $G = \infty(3, l, 3) \cup \bigcup_{i=1}^k C_{k_i}$. If $k = 0$, then G is isomorphic to $S(C_3, P_1, C_3)$. Next we suppose that $k > 0$. By the structure of G and Lemma 1, we can obtain that $b_n(G) > 5$ when n is even, and $b_n(G) < -4$ when n is odd. By Lemma 6, this is a contradiction.

Subcase 1.3. Neither of the two triangles belongs to the bicyclic component. Then $G = \infty(p, l, q) \cup C_3 \cup C_3 \cup \bigcup_{i=1}^{k-2} C_{k_i}$ for $p, q \geq 4$. By Lemma 3, we have $b_5(G) = -4n + 8 - 2c_5(G)$. So, $b_5(S(C_3, P_l, C_3)) - b_5(G) = 4 + 2c_5(G) > 0$. This contradicts the assumption that G and $S(C_3, P_l, C_3)$ are per-cospectral.

Case 2. Assume that G is isomorphic to $\theta(r, s, t) \cup (\bigcup_{i=1}^k C_{k_i})$. We further consider the following three subcases.

Subcase 2.1. Exact one of the two triangles belongs to the bicyclic component. Then $G = \theta(1, 2, t) \cup C_3 \cup \bigcup_{i=1}^{k-1} C_{k_i}$ for $t > 2$ and $k \geq 1$. By the structure of G and Lemma 1, we can obtain that if n is even, then $b_n(G) > 5$; and if n is odd, then $b_n(G) < -4$. By Lemma 6, this is a contradiction.

Subcase 2.2. Both of the two triangles belong to the bicyclic component. Then $G = B_2(1, 2, 2) \cup (\bigcup_{i=1}^k C_{k_i})$, which is a contradiction to G being a quadrual-free graph.

Subcase 2.3. Neither of the two triangles belongs to the bicyclic component. Then $G = \theta(r, s, t) \cup C_3 \cup C_3 \cup (\bigcup_{i=1}^{k-2} C_{k_i})$ for $r > 1, s, t \geq 2$ or $r \geq 1, s, t > 2$. By Lemma 3, we have $b_5(G) = -4n + 8 - 2c_5(G)$. By Lemma 6, we have a contradiction, and the theorem follows. \square

Theorem 9. *Restricting consideration on connected graphs, sandglass graphs are determined by their per-spectra.*

Proof. Let G be a connected graph, where $|G| = n$ and $n \geq 5$, and let G be per-cospectral with $S(C_3, P_h, C_3)$. By Lemma 4, G is a bicyclic graph with two triangles and must be isomorphic to a graph containing a sandglass graph $S(C_3, P_l, C_3)$ as its induced subgraph or $K_4 - e$ (isomorphic to θ -graph when $r = 1$ and $s = t = 2$) as its induced subgraph.

Suppose that G is isomorphic to a graph which contains a sandglass graph $S(C_3, P_l, C_3)$ as its induced subgraph. Then G contains no quadrangles. By Lemma 7, G must be isomorphic to the sandglass graph $S(C_3, P_h, C_3)$.

In the following, we will prove that G is isomorphic to a graph containing $K_4 - e$ as its induced subgraph.

Suppose that n is even. By Lemma 6, we know that $b_n(S(C_3, P_h, C_3)) = 5$. This implies, by Lemma 1, that G must have odd perfect matchings. Examining the structure of G , we see that G has at most two perfect matchings. So, G only has uniquely one perfect matching. This implies that all triangles or 4-cycle in G are not a component of some Sachs subgraph of order n . Thus, the perfect matching of G is a unique Sachs subgraph of order n . By Lemma 1, $b_n(G) = 1$, which contradicts the fact that $b_n(S(C_3, P_h, C_3)) = 5$.

Assume that n is odd. By Lemma 1 and examining the structure of G , we know that the Sachs subgraphs of order n in G is only the union of a triangle and a perfect matching of G deleting all edges on the triangle. Then $b_n(G) = 2$. This contradicts $b_n(S(C_3, P_h, C_3)) = 5$.

This completes the proof. \square

For any bicyclic network, it is difficult to discuss which is determined by its per-spectrum. We can construct countless pairs per-cospectral bicyclic networks. Let H be an arbitrary

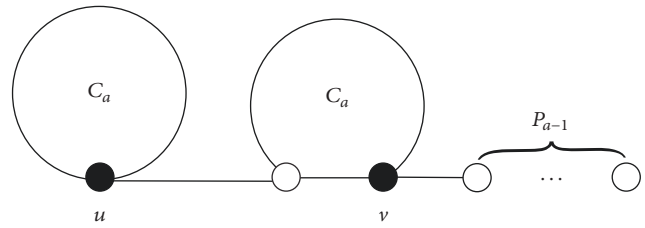


FIGURE 3: The bicyclic network B .

graph with a fixed vertex w and let $G_u \cdot H$ denote the coalescence of G and H with respect to u and w , which is the graph obtained from $G \cup H$ by identifying u and w . Similarly, we define $G'_v \cdot H$. Borowiecki [20] showed that if both $G - u$ and $G' - v$ are per-cospectral, then both $G_u \cdot H$ and $G'_v \cdot H$ are also per-cospectral. As an example, let $B = B'$ be the bicyclic network depicted in Figure 3. As $B - u$ and $B - v$ are isomorphic, they are per-cospectral. By the above-mentioned result of Borowiecki [20], for any graph H , both $B_u \cdot H$ and $B_v \cdot H$ are per-cospectral.

3. Summary

Per-spectra is an important part of graph spectra. In this paper, we discuss properties of permanent spectra of bicyclic networks. We show that without some limitations bicyclic networks are not DPS. Particularly, we find a pair of per-cospectral graphs. Combining the result of Lu et al. [19], our results (Theorems 8 and 9) are beyond Merris et al.'s imagination. Finally, we pose the following conjecture.

Conjecture 10. *Sandglass graphs with a perfect matching are DPS.*

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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