Basic Properties and Qualitative Dynamics of a Vector-borne Disease Model with Vector Stages and Vertical Transmission

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Abstract

This work systematically discusses basic properties and qualitative dynamics of vector-borne disease models, particularly those with vertical transmission in the vector population. Examples of disease include Dengue and Rift Valley fever which are endemic in Sub-Saharan Africa and understanding of the dynamics underlying their transmission is central for providing critical informative indicators useful for guiding control strategies. Of particular interest is the applicability and derivation of relevant population and epidemic thresholds and their relationships with vertical infection. This study demonstrates how the failure of R_0 derived using the next-generation method compounds itself when varying vertical transmission efficiency, and shows that the host type reproductive number gives the correct R_0. Further, novel relationships between the host type reproductive number, vertical infection and ratio of female mosquitoes to host are established and discussed.

Analytical results of the model with vector stages show that the quantities Q_0, Q_0^v and R_0^c, which represent the vector colonization threshold, the average number of female mosquitoes produced by a single infected mosquito and effective reproductive number, respectively, provide threshold conditions that determine the establishment of the vector population and invasion of the disease. Numerical simulations are also conducted to confirm and extend the analytical results. The findings imply that while vertical infection increases the size of an epidemic it in turns reduces its duration, and control efforts aiming at reducing the critical thresholds Q_0, Q_0^v and R_0^c to below unity are viable control strategies.

Keywords: Vector-borne disease, Vertical transmission, Mathematical modelling, Epidemic thresholds, Qualitative dynamics

SM: Supplementary Materials

A-1 The basic and type reproductive numbers of the basic model

In this section we provide details for the derivation of both the basic and type reproductive numbers. For instructive purpose we reconstruct the spectral matrix K by directly noting that the element (i, j) is the expected number of new infections of type i produced by an infected individual of type j [5]. For system (1) we have two types of infectious individuals: I_v and I_h. An individual of type I_v causes q_v new infections of type I_v and $\frac{p_{hv}\alpha m}{\mu_v}$ new infections of type I_h. While an individual of type I_h causes $\frac{p_{vh}\alpha}{\gamma_h+\mu_h}$ new infections of type I_v and zero new infections of type I_h. All components of the next-generation matrix have now been completed and the basic reproductive number is then the spectral radius $\rho(K)$ [16]. To obtain specific type reproductive numbers we follow the approach in [13], where the host type reproductive number T_1^h, that is the number of new individuals of type I_h resulting from one individual of type I_h is,

$$T_1^h = e_h^T K(I - (I - P_h)K)^{-1} e_h, \tag{A-1}$$

where $e_h = (0, 1)$, P_h is a 2×2 matrix with $P_{22} = 1$ and all other elements zero. T represents the transpose and I is the identity matrix. And, the vector type reproductive T_1^v, that is the number of new individuals of type I_v resulting from one individual of type I_v is then given by:

$$T_1^h = e_v^T K(I-(I-P_v)K)^{-1}e_v, \tag{A-2}$$

where $e_v = (1,0)$, P_h is a 2×2 matrix with $P_{11}=1$ and all other elements zero. T represents the transpose and I is the identity matrix.

A-2 Existence of model equilibria

Let $X^* = (L^*, S_v^*, I_v^*, S_h^*, I_h^*)$ be an arbitrary equilibrium of system (2). Then,

$$\begin{aligned}
\frac{dS_v}{dt} + \frac{dI_v}{dt} &= \frac{\theta}{\delta}(1-N_v)L_s^* + \frac{\theta}{\delta}(1-N_v)L_i^* - \mu_v N_v = 0,\\
\frac{\theta}{\delta}(1-N_v)L^* = 0,&\\
N_v &= \frac{\theta L^*}{\theta L^* + \delta\mu_v}
\end{aligned}$$

Now that we know N_v we can find L^*,

$$\begin{aligned}
\frac{dL_s}{dt} + \frac{dL_i}{dt} &= \delta r(1-L^*)N_v - (\theta+\mu_L)L^* = 0,\\
L^* &= \frac{\delta(r\theta - \mu_v(\theta+\mu_L))}{\theta(\delta r+\theta+\mu_L)} = \frac{\delta r(1-Q_0^{-1})}{\delta r+\theta+\mu_L},
\end{aligned}$$

with $Q_0 = \frac{r\theta}{\mu_v(\theta+\mu_L)}$.
Since L^* has been defined explicitly we can now define,

$$\frac{\theta}{\delta}(1-N_v) = \frac{\delta r\mu_v\theta + \mu_v\theta(\theta+\mu_L)}{\delta r(\theta+\delta\mu_v)}.$$

From $\frac{dI_v}{dt}=0$ we obtain $\lambda_{vh}I_h^*S_v^* = \mu_v I_v^* - \frac{\theta}{\delta}((1-N_v)L_i^*$ and from $\frac{dS_v}{dt}=0$ we obtain,

$$\begin{aligned}
\frac{\theta}{\delta}((1-N_v)L_s^* + \frac{\theta}{\delta}((1-N_v)L_i^* - \mu_v I_v^* - \mu_v S_v^* = 0,&\\
\frac{\theta}{\delta}((1-N_v)L^* - \mu_v I_v^* = \mu_v S_v^*,&\\
S_v^* = P^* - I_v^*, \quad \textit{with} \quad P^* = \frac{\theta(1-Q_0^{-1})}{\theta+\delta\mu_v}.&
\end{aligned}$$

From $\frac{dS_h}{dt}=0$ we obtain

$$I_v^* = \frac{\mu_h(1-S_h^*)}{\lambda_{hv}S_h^*},$$

and when replacing I_v^* on $\frac{dI_h}{dt}=0$ we obtain

$$S_h^* = 1 - \frac{\gamma_h+\mu_h}{\mu_h}I_h^*,$$

and $\frac{dL_i}{dt}=0$ yields

$$\begin{aligned}
L_i^* &= \frac{\delta r(1-L^*)q_v I_v^*}{\theta+\mu_L},\\
L_i^* &= Hq_v I_v^*, \quad \textit{with} \quad H = \frac{\delta r(\theta+\delta\mu_v)}{\theta(\delta r+\theta+\mu_L)}.
\end{aligned}$$

Note that $\frac{\theta}{\delta}((1-N_v)L_i^* = \frac{\theta}{\delta}((1-N_v)Hq_vI_v^* = q_v\mu_vI_v^*$. Thus, substituting L_i^* into $\frac{dI_v}{dt} = 0$ we obtain,

$$\lambda_{vh}I_h^*S_v^* - (1-q_v)\mu_vI_v^* = 0 \tag{A-3}$$

Now replacing S_v^* and I_v^* into equation (A-3) we obtain

$$I_h^* = \frac{(1-q_v)\mu_v(\gamma_h+\mu_h)(R_0^c-1)}{\frac{\lambda_{vh}\lambda_{hv}P^*(\gamma_h+\mu_h)}{\mu_h} + \frac{\lambda_{vh}}{(1-q_v)\mu_v}}.$$

A-3 Model parameters and references

Parameter	**Description**	**Baseline**	Units	**Reference**
$1/\mu_v$	Adult mosquito's life span	25	days	[10, 4]
$1/\mu_L$	Aquatic mosquito's life span	18	days	[9, 8]
$1/\theta$	Development time in aquatic form	5	days	[9, 8]
r	Mosquito oviposition rate	5	days^{-1}	[9, 8]
$1/\mu_h$	Livestock life span	2190	days	[6]
q_1	Probability of vertical transmission	0-1		[14]
α	Mosquito biting rate	0.71		[3]
p_{hv}	Probability of successful infection in livestock	0.21		[12, 15, 4]
β_{vh}	Probability of successful infection in mosquitoes	0.51		[12, 15, 4]
$1/\gamma_h$	Infectious duration in livestock	4	days	[1, 2, 11, 4]
m	Adult mosquitoes to livestock ratio	1.5		[7]
δ	Adult to aquatic mosquito ratio	0.4		Chosen

Table 1: Parameter description, values, units and references corresponding to RVF disease.

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