Research Article

Explicit Solutions to the (3+1)-Dimensional Kudryashov-Sinelshchikov Equations in Bubbly Flow Dynamics

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A modified tanh-coth method with Riccati equation is used to construct several explicit solutions of (3+1)-dimensional Kudryashov-Sinelshchikov equations in bubble gas liquid flow. The solutions include solitons and periodic solutions. The method applied can be used in further works to obtain entirely new solutions to many other nonlinear evolution equations.

1. Introduction

Many complex phenomena in nature are describe in various scientific and industrial fields and especially in areas of physics such as fluid mechanics [1], optical fibres [2], plasma physics [3], and so on. Nonlinear evolution equations are used to describe these nonlinear phenomena, and that has led to the development of methods to look for exact solutions of nonlinear partial differential equations [4].

Recently, Kudryashov and Sinelshchikov [5] developed a (3+1)-dimensional nonlinear evolution equation in a model of wave propagation in bubbly fluid flow. The model equation has gained a lot of attention where Bäcklund transformation and conservation laws [6], bifurcation [7], and density-fluctuation [8] analysis were carried out the important evolution equation.

Many methods have been developed to find the explicit solutions of nonlinear evolution equations; example of such methods are the first integral method [9], Jacobi elliptic function method [10], Hirota bilinear method [11], Wronskian determinant technique [12], F-expansion method [13], Darboux Transformations [14], Backlund transformation method [6], Miura transformation [15], homotopy perturbation method [16], and Adomian decomposition method [17]. Many algebraic methods were proposed so far, such as tanh method which was proposed by Malflië [18]. Fan [19] extended the tanh method and obtained new exact solution that cannot be obtained by using the conventional tanh

method. Further extension called tanh-coth was proposed by Wazwaz [4] which provides a wider applicability for solving nonlinear evolution equations. A modification was also proposed by El-Wakil [20] and Soliman [21, 22].

In this paper, we present two equations (3+1)-dimensional Kudryashov and Sinelshchikov equation written as

$$\begin{aligned} (u_t + \alpha uu_x + \gamma u_{xxx})_x + du_{yy} + eu_{zz} &= 0, \\ (u_t + \alpha uu_x - \beta u_{xx} + \gamma u_{xxx})_x + du_{yy} + eu_{zz} &= 0 \end{aligned} \quad (1)$$

where α , β , and γ represent the nonlinearity, dissipation dispersion terms, while d and e stand for transverse variation of wave in y and z directions; we assume all the coefficient to be constant parameters. We shall use modified tanh-coth to obtain many explicit exact solutions for (3+1)-dimensional Kudryashov-Sinelshchikov equations. The travelling wave solution of a special case of (1) is given in [7]. Note that when $e = 0$, (1) reduces to their two-dimensional counterparts

$$\begin{aligned} (u_t + \alpha uu_x + \gamma u_{xxx})_x + du_{yy} &= 0, \\ (u_t + \alpha uu_x - \beta u_{xx} + \gamma u_{xxx})_x + du_{yy} &= 0, \end{aligned} \quad (2)$$

which were widely studied in [16, 17, 23, 24]. Furthermore when $d = e = 0$, (1) reduces to one-dimensional Korteweg-de-Vries equations (KdV) and one-dimensional Korteweg-de-Vries-Burgers equations (KdVB). These equations have many applications in fluid dynamics [25], plasma physics

[26], cold atomic gases [27], and so on. It is well known that the KdV and KdVB equations take the form

$$\begin{aligned} u_t + \alpha uu_x + \gamma u_{xxx} &= 0, \\ u_t + \alpha uu_x - \beta u_{xx} + \gamma u_{xxx} &= 0. \end{aligned} \quad (3)$$

These two equations have been extensively investigated by many researcher in recent years [28–32].

In the following section, we briefly describe the modified tanh-coth method in three variables and the application of the modified tanh-coth to (3+1)-dimensional Kudryashov and Sinelshchikov equations and their explicit solutions will be given in Section 3. Conclusions will follow in Section 4.

2. Description of the Method

Let us consider a three-dimensional nonlinear partial differential equation in the form

$$u_t = F(u, u_x, u_y, u_z, u_{xx}, u_{yy}, u_{zz}, \dots). \quad (4)$$

Upon using the transformation $u(x, y, z, t) = u(\xi)$, where $\xi = \mu(x + y + z - ct)$, (4) can be reduced to an ordinary differential equation (ODE)

$$\begin{aligned} -c\mu u'(\xi) \\ = F(u(\xi), \mu u'(\xi), \mu^2 u''(\xi), \mu^3 u'''(\xi), \dots), \end{aligned} \quad (5)$$

where the independent variable composed of the new variable $\xi = \mu(x + y + z - ct)$, μ is the wave number, c is wave speed, x, y, z are the spatial coordinates, and t is time. The ODE (5) is then integrated as long as all terms contain derivatives, where integration constants are considered zero. The simplified ODE is then solved via the tanh-coth method [28], which admits the use of finite expansion

$$u(Y) = \sum_{i=0}^m a_i Y^i + \sum_{i=1}^m b_i Y^{-i} \quad (6)$$

with the Riccati equation

$$Y' = P + QY + RY^2. \quad (7)$$

By changing of variable

$$\begin{aligned} \frac{d}{d\xi} &= (P + QY + RY^2) \frac{d}{dY} \\ \frac{d^2}{d\xi^2} &= (P + QY + RY^2) \\ &\cdot \left((Q + 2RY) \frac{d}{dY} + (P + QY + RY^2) \frac{d^2}{dY^2} \right), \end{aligned} \quad (8)$$

where P, Q , and R shall be given, while a_i, b_i are constants to be determined later. The positive integer m can be determined by considering the homogeneous balance [33] between the highest order derivatives and the most nonlinear terms appearing in (5). If m is not an integer, then

a transformation formula should be used to overcome the difficulty. Substituting (6) into (5) and make use of (7) and (8) yield an equation in terms of Y^i . Equating the coefficients of all the power Y^i to zero, we obtain a set of algebraic equations for P, Q, R, a_i, b_i, c , and μ .

For the aforementioned method, expansion (8) reduces to the standard tanh-coth method. In this work, we shall use the following solutions of Riccati equation:

$$P = Q = 1 \text{ and } R = 0, \text{ then } Y = \exp(\xi) - 1,$$

$$P = 1/2, \quad Q = 0, \quad R = -1/2, \text{ then } Y = \coth \xi \pm \operatorname{csch} \xi.$$

3. Application of the Method

In this section, we will apply the method to the (3+1)-dimensional Kudryashov-Sinelshchikov equation for both dispersive and dissipative cases.

3.1. The (3+1)-Dimensional Kudryashov-Sinelshchikov Equation with Dispersion. The generalised equation is given as

$$(u_t + \alpha uu_x + \gamma u_{xxx})_x + du_{yy} + eu_{zz} = 0. \quad (9)$$

By using the transformation

$$\begin{aligned} u(x, y, z, t) &= u(\xi), \\ \xi &= \mu(x + y + z - ct), \end{aligned} \quad (10)$$

(9) is reduced to an ODE of the form

$$\mu(-c\mu u' + \alpha\mu u u' + \gamma\mu^3 u''')' + d\mu^2 u'' + e\mu^2 u'' = 0. \quad (11)$$

Integrating (11) twice and assuming the integration constant to be zero, we obtain

$$(-c + d + e)u + \frac{\alpha}{2}u^2 + \gamma\mu^2 u'' = 0. \quad (12)$$

Balancing terms u^2 and u'' in (12), we get $m = 2$, which enables us to make the following ansatz:

$$u(\xi) = a_0 + a_1 Y + a_2 Y^2 + \frac{b_1}{Y} + \frac{b_2}{Y^2}. \quad (13)$$

Substituting (13) into (12) and making use of (7) and (8), we obtain a system of equation in terms of a_i ($i = 0, 1, 2$), b_i ($i = 1, 2$), $\alpha, \gamma, d, e, c, P, Q, R$, and μ .

$$12\gamma R^2 \mu^2 a_2 + \alpha a_2^2 = 0,$$

$$4\gamma a_1 R^2 \mu^2 + 20\gamma Q a_2 R \mu^2 + 2\alpha a_1 a_2 = 0,$$

$$8\gamma a_2 Q^2 \mu^2 + 6\gamma R Q \mu^2 a_1 + 16\gamma P R a_2 \mu^2 + \alpha a_1^2 + 2da_2$$

$$+ 2ea_2 - 2ca_2 + 2\alpha a_0 a_2 = 0,$$

$$2\gamma a_1 Q^2 \mu^2 + 12\gamma P a_2 Q \mu^2 + 4\gamma P R a_1 \mu^2 + 2da_1 + 2ea_1$$

$$- 2ca_1 + 2\alpha a_0 a_1 + 2\alpha a_2 b_1 = 0,$$

$$\begin{aligned}
& 2da_0 + 2ea_0 - 2ca_0 + \alpha a_0^2 + 2\alpha a_1 b_1 + 2\alpha a_2 b_2 \\
& + 4\gamma P^2 \mu^2 a_2 + 4\gamma R^2 \mu^2 b_2 + 2\gamma PQ \mu^2 a_1 \\
& + 2\gamma QR \mu^2 b_1 = 0, \\
& 2\gamma b_1 Q^2 \mu^2 + 12\gamma R b_2 Q \mu^2 + 4\gamma P R b_1 \mu^2 + 2db_1 + 2eb_1 \\
& - 2cb_1 + 2\alpha a_0 b_1 + 2\alpha a_1 b_2 = 0, \\
& 8\gamma b_2 Q^2 \mu^2 + 6\gamma PQ \mu^2 b_1 + 16\gamma P R b_2 \mu^2 + \alpha b_1^2 + 2db_2 \\
& + 2eb_2 - 2cb_2 + 2\alpha a_0 b_2 = 0, \\
& 4\gamma b_1 P^2 \mu^2 + 20\gamma Q b_2 P \mu^2 + 2\alpha b_1 b_2 = 0, \\
& 12\gamma P^2 \mu^2 b_2 + \alpha b_2^2 = 0.
\end{aligned} \tag{14}$$

Case I. If $P = 1, Q = 1, R = 0$ are substituted into the system of (14) and solving the equations, we have the following solutions:

$$\begin{aligned}
a_0 &= \frac{2(c-d-e)}{\alpha}, \\
a_1 &= a_2 = 0, \\
b_1 &= b_2 = 6a_0,
\end{aligned} \tag{15a}$$

$$\begin{aligned}
\mu &= \frac{\sqrt{-\gamma(c-d-e)}}{\gamma}, \\
a_0 &= a_1 = a_2 = 0, \\
b_1 &= b_2 = -\frac{12(c-d-e)}{\alpha}, \\
\mu &= \frac{\sqrt{\gamma(c-d-e)}}{\gamma}.
\end{aligned} \tag{15b}$$

Substituting (15a) and (15b) into (13) using $Y = \exp(\xi) - 1, \gamma > 0$, with $\Omega = x + y + z - ct$, we get the solutions in simple form as

$$\begin{aligned}
u_1(x, y, z, t) &= \frac{2(c-d-e)}{\alpha} \\
&+ \frac{3(c-d-e)}{\alpha \sinh^2 \left(\left(\sqrt{-\gamma(c-d-e)}/\gamma \right) (\Omega/2) \right)}, \\
& \hspace{15em} c < d + e
\end{aligned} \tag{16}$$

$$\begin{aligned}
u_2(x, y, z, t) &= -\frac{3(c-d-e)}{\alpha \sinh^2 \left(\left(\sqrt{\gamma(c-d-e)}/\gamma \right) (\Omega/2) \right)}, \\
& \hspace{15em} c > d + e.
\end{aligned} \tag{17}$$

Case II. If we set $P = 1/2, Q = 0, R = -1/2$ into the system of (14) and solve the equations, we have the following solutions:

$$\begin{aligned}
a_0 &= -\frac{c-d-e}{\alpha}, \\
a_1 &= 0, \\
a_2 &= -3a_0,
\end{aligned} \tag{18a}$$

$$\begin{aligned}
b_1 &= b_2 = 0, \\
\mu &= \frac{\sqrt{-\gamma(c-d-e)}}{\gamma}, \\
a_0 &= \frac{3(c-d-e)}{\alpha},
\end{aligned}$$

$$\begin{aligned}
a_1 &= 0, \\
a_2 &= -3a_0,
\end{aligned} \tag{18b}$$

$$\begin{aligned}
b_1 &= b_2 = 0, \\
\mu &= \frac{\sqrt{\gamma(c-d-e)}}{\gamma},
\end{aligned}$$

$$\begin{aligned}
a_0 &= -\frac{c-d-e}{\alpha}, \\
a_1 &= a_2 = 0, \\
b_1 &= 0,
\end{aligned} \tag{18c}$$

$$\begin{aligned}
b_2 &= -3a_0, \\
\mu &= \frac{\sqrt{-\gamma(c-d-e)}}{\gamma}, \\
a_0 &= \frac{3(c-d-e)}{\alpha}, \\
a_1 &= a_2 = 0, \\
b_1 &= 0,
\end{aligned} \tag{18d}$$

$$\begin{aligned}
b_2 &= -a_0, \\
\mu &= \frac{\sqrt{\gamma(c-d-e)}}{\gamma}, \\
a_0 &= \frac{c-d-e}{2\alpha}, \\
a_1 &= b_1 = 0, \\
a_2 &= b_2 = \frac{3(c-d-e)}{4\alpha},
\end{aligned} \tag{18e}$$

$$\begin{aligned}
\mu &= \frac{\sqrt{-\gamma(c-d-e)}}{2\gamma}, \\
a_0 &= \frac{3(c-d-e)}{2\alpha}, \\
a_1 &= b_1 = 0,
\end{aligned}$$

$$\begin{aligned}
 a_2 = b_2 &= -\frac{3(c-d-e)}{4\alpha}, & &= \frac{2(c-d-e)}{\alpha} \\
 \mu &= \frac{\sqrt{\gamma(c-d-e)}}{2\gamma}. & & - \frac{3(c-d-e)}{\alpha \cosh^2\left(\left(\sqrt{-\gamma(c-d-e)}/\gamma\right)(\Omega/2)\right)},
 \end{aligned}
 \tag{18f}$$

$$c < d + e \tag{19}$$

Substituting $Y = \coth \xi \pm \operatorname{csch} \xi$, $\gamma > 0$, $\Omega = x + y + z - ct$ into (13), making use of (18a), we obtain a soliton solution

$$u_3(x, y, z, t)$$

and periodic solutions

$$u_4(x, y, z, t) = \frac{2(c-d-e)}{\alpha} - \frac{3(c-d-e)}{\alpha \cos^2\left(\left(\sqrt{\gamma(c-d-e)}/\gamma\right)(\Omega/2)\right)}, \quad c > d + e \tag{20}$$

$$u_5(x, y, z, t) = \frac{2(c-d-e)}{\alpha} - \frac{6(c-d-e)}{\alpha \left(\cos\left(\left(\sqrt{\gamma(c-d-e)}/\gamma\right)(\Omega/2)\right) + \sin\left(\left(\sqrt{\gamma(c-d-e)}/\gamma\right)(\Omega/2)\right)\right)^2}, \tag{21}$$

where $c > d + e$. Using (18b), we have a soliton solution, written in simple form as

$$u_6(x, y, z, t) = \frac{3(c-d-e)}{\alpha \cosh^2\left(\left(\sqrt{\gamma(c-d-e)}/\gamma\right)(\Omega/2)\right)}, \tag{22}$$

$$c > d + e$$

and periodic solutions

$$u_7(x, y, z, t) = \frac{3(c-d-e)}{\alpha \cos^2\left(\left(\sqrt{-\gamma(c-d-e)}/\gamma\right)(\Omega/2)\right)}, \quad c < d + e \tag{23}$$

$$u_8(x, y, z, t) = \frac{6(c-d-e)}{\alpha \left(\cos\left(\left(\sqrt{-\gamma(c-d-e)}/\gamma\right)(\Omega/2)\right) + \sin\left(\left(\sqrt{-\gamma(c-d-e)}/\gamma\right)(\Omega/2)\right)\right)^2}, \quad c < d + e. \tag{24}$$

Equation (18c) gives a soliton solution

$$u_9(x, y, z, t)$$

$$= \frac{2(c-d-e)}{\alpha}$$

$$+ \frac{3(c-d-e)}{\alpha \sinh^2\left(\left(\sqrt{-\gamma(c-d-e)}/\gamma\right)(\Omega/2)\right)},$$

$$c < d + e \tag{25}$$

and periodic solutions

$$u_{10}(x, y, z, t) = \frac{2(c-d-e)}{\alpha} - \frac{3(c-d-e)}{\alpha \sin^2\left(\left(\sqrt{\gamma(c-d-e)}/\gamma\right)(\Omega/2)\right)}, \quad c > d + e \tag{26}$$

$$u_{11}(x, y, z, t) = \frac{2(c-d-e)}{\alpha} - \frac{6(c-d-e)}{\alpha \left(\cos\left(\left(\sqrt{\gamma(c-d-e)}/\gamma\right)(\Omega/2)\right) - \sin\left(\left(\sqrt{\gamma(c-d-e)}/\gamma\right)(\Omega/2)\right)\right)^2}, \tag{27}$$

where $c > d + e$. Equation (18d) gives a soliton solution

$$= -\frac{3(c-d-e)}{\alpha \sinh^2\left(\left(\sqrt{\gamma(c-d-e)}/\gamma\right)(\Omega/2)\right)},$$

$$c > d + e \tag{28}$$

$$u_{12}(x, y, z, t)$$

and periodic solutions

$$u_{13}(x, y, z, t) = \frac{3(c-d-e)}{\alpha \sin^2 \left(\left(\sqrt{-\gamma(c-d-e)/\gamma} \right) (\Omega/2) \right)}, \quad c < d+e \quad (29)$$

$$u_{14}(x, y, z, t) = \frac{6(c-d-e)}{\alpha \left(\cos \left(\left(\sqrt{-\gamma(c-d-e)/\gamma} \right) (\Omega/2) \right) + \sin \left(\left(\sqrt{-\gamma(c-d-e)/\gamma} \right) (\Omega/2) \right) \right)^2}, \quad c < d+e. \quad (30)$$

Equation (18e) gives soliton solutions

$$\begin{aligned} u_{15}(x, y, z, t) &= \frac{2(c-d-e)}{\alpha} \\ &+ \frac{3(c-d-e)}{\alpha \sinh^2 \left(\left(\sqrt{-\gamma(c-d-e)/\gamma} \right) (\Omega/2) \right)}, \end{aligned} \quad (31)$$

$c < d+e$

$$\begin{aligned} u_{16}(x, y, z, t) &= \frac{2(c-d-e)}{\alpha} \\ &- \frac{3(c-d-e)}{\alpha \cosh^2 \left(\left(\sqrt{-\gamma(c-d-e)/\gamma} \right) (\Omega/2) \right)}, \end{aligned} \quad (32)$$

$c < d+e$

and periodic solutions

$$\begin{aligned} u_{17}(x, y, z, t) &= \frac{2(c-d-e)}{\alpha} \\ &- \frac{3(c-d-e)}{\alpha \sin^2 \left(\left(\sqrt{\gamma(c-d-e)/\gamma} \right) (\Omega/2) \right)}, \end{aligned} \quad (33)$$

$c > d+e$

$$\begin{aligned} u_{18}(x, y, z, t) &= \frac{2(c-d-e)}{\alpha} \\ &- \frac{3(c-d-e)}{\alpha \cos^2 \left(\left(\sqrt{\gamma(c-d-e)/\gamma} \right) (\Omega/2) \right)}, \end{aligned} \quad (34)$$

$c > d+e$.

Finally (18f) gives soliton solutions

$$u_{19}(x, y, z, t) = -\frac{3(c-d-e)}{\alpha \sinh^2 \left(\left(\sqrt{\gamma(c-d-e)/\gamma} \right) (\Omega/2) \right)}, \quad (35)$$

$c > d+e$

$$u_{20}(x, y, z, t) = \frac{3(c-d-e)}{\alpha \cosh^2 \left(\left(\sqrt{\gamma(c-d-e)/\gamma} \right) (\Omega/2) \right)}, \quad (36)$$

$c > d+e$

and periodic solutions

$$u_{21}(x, y, z, t) = \frac{3(c-d-e)}{\alpha \sin^2 \left(\left(\sqrt{-\gamma(c-d-e)/\gamma} \right) (\Omega/2) \right)}, \quad (37)$$

$c < d+e$

$$u_{22}(x, y, z, t) = \frac{3(c-d-e)}{\alpha \cos^2 \left(\left(\sqrt{-\gamma(c-d-e)/\gamma} \right) (\Omega/2) \right)}, \quad (38)$$

$c < d+e$.

The graphical representation of the solitary wave profile to the solution $u_3(x, y, z, t)$ of (9) at $t = 0$ is given in Figure 1, while for $t = 0.5$ is given in Figure 2.

3.2. The (3+1)-Dimensional Kudryashov-Sinelshchikov Equation with Dispersion and Dissipation. The equation is given as

$$(u_t + \alpha u u_x - \beta u_{xx} + \gamma u_{xxx})_x + du_{yy} + eu_{zz} = 0. \quad (39)$$

By using the transformation

$$\begin{aligned} u(x, y, z, t) &= u(\xi), \\ \xi &= \mu(x + y + z - ct), \end{aligned} \quad (40)$$

(39) is reduced to an ODE of the form

$$\begin{aligned} \mu \left(-c\mu u' + \alpha\mu u u' - \beta\mu^2 u'' + \gamma\mu^3 u''' \right)' + d\mu^2 u'' \\ + e\mu^2 u'' = 0. \end{aligned} \quad (41)$$

Integrating (41) twice and assuming the integration constant to be zero, we obtain

$$(-c + d + e)u + \frac{\alpha}{2}u^2 - \beta\mu u' + \gamma\mu^2 u'' = 0. \quad (42)$$

Balancing u^2 and u'' in (42), we get $m = 2$, which enables us to make the following ansatz:

$$u(\xi) = a_0 + a_1 Y + a_2 Y^2 + \frac{b_1}{Y} + \frac{b_2}{Y^2}. \quad (43)$$

FIGURE 1: Solitary wave profile of u_3 , $\alpha = 1$, $d = 0.5$, $e = 0.5$, $c = 0.6$, $x = 0$, and $t = 0$.

FIGURE 2: Solitary wave profile of u_3 , $\alpha = 1$, $d = 0.5$, $e = 0.5$, $c = 0.6$, $x = 0$, and $t = 0.5$.

Substituting (43) into (42) and making use of (7) and (8), we obtained a system of equations in a_i ($i = 0, 1, 2$), b_i ($i = 1, 2$), c , P , Q , R , α , β , γ , and μ .

$$\begin{aligned}
 12\gamma R^2 \mu^2 a_2 + \alpha a_2^2 &= 0, \\
 4\gamma a_1 R^2 \mu^2 + 20\gamma Q a_2 R \mu^2 - 4\beta a_2 R \mu + 2\alpha a_1 a_2 &= 0, \\
 8\gamma a_2 Q^2 \mu^2 + 6\gamma R Q \mu^2 a_1 - 4\beta a_2 Q \mu + 16\gamma P R a_2 \mu^2 \\
 - 2\beta R \mu a_1 + \alpha a_1^2 + 2da_2 + 2ea_2 - 2ca_2 \\
 + 2\alpha a_0 a_2 &= 0, \\
 2\gamma a_1 Q^2 \mu^2 + 12\gamma P a_2 Q \mu^2 - 2\beta a_1 Q \mu + 4\gamma A R a_1 \mu^2 \\
 - 4\beta P a_2 \mu + 2da_1 + 2ea_1 - 2ca_1 + 2\alpha a_0 a_1 \\
 + 2\alpha a_2 b_1 &= 0,
 \end{aligned}$$

$$\begin{aligned}
 2da_0 + 2ea_0 - 2wa_0 + \alpha a_0^2 + 2\alpha a_1 b_1 + 2\alpha a_2 b_2 \\
 + 4\gamma P^2 \mu^2 a_2 + 4\gamma R^2 \mu^2 b_2 - 2\beta P \mu a_1 + 2\beta R \mu b_1 \\
 + 2\gamma P Q \mu^2 a_1 + 2\gamma Q R \mu^2 b_1 &= 0, \\
 2\gamma b_1 Q^2 \mu^2 + 12\gamma R b_2 Q \mu^2 + 2\beta b_1 Q \mu + 4\gamma P R b_1 \mu^2 \\
 + 4\beta R b_2 \mu + 2db_1 + 2eb_1 - 2cb_1 + 2\alpha a_0 b_1 \\
 + 2\alpha a_1 b_2 &= 0, \\
 8\gamma b_2 Q^2 \mu^2 + 6\gamma P Q \mu^2 b_1 + 4\beta b_2 Q \mu + 16\gamma P Q b_2 \mu^2 \\
 + 2\beta P \mu b_1 + \alpha b_1^2 + 2db_2 + 2eb_2 - 2cb_2 \\
 + 2\alpha a_0 b_2 &= 0, \\
 4\gamma b_1 P^2 \mu^2 + 20\gamma Q b_2 P \mu^2 + 4\beta b_2 P \mu + 2\alpha b_1 b_2 &= 0, \\
 12\gamma P^2 \mu^2 b_2 + \alpha b_2^2 &= 0.
 \end{aligned} \tag{44}$$

Case I. If $P = 1$, $Q = 1$, $R = 0$ are used in the system (44) and solving the equations, we have obtained the following solutions:

$$\begin{aligned}
 a_0 &= a_1 = a_2 = 0, \\
 b_1 &= -\frac{24\beta^2}{25\gamma\alpha}, \\
 b_2 &= \frac{1}{2}b_1, \\
 \mu &= \frac{\beta}{5\gamma},
 \end{aligned} \tag{45a}$$

$$\begin{aligned}
 c &= \frac{6\beta^2 + 25\gamma d + 25\gamma e}{25\gamma}, \\
 a_0 &= b_2 = -\frac{12\beta^2}{25\gamma\alpha}, \\
 a_1 &= a_2 = 0, \\
 b_1 &= 2b_2, \\
 \mu &= \frac{\beta}{5\gamma}, \\
 c &= \frac{25\gamma d + 25\gamma e - 6\beta^2}{25\gamma}.
 \end{aligned} \tag{45b}$$

Substituting (45a) and (45b) into (43) and using $Y = \exp(\xi) - 1$, we get

$$\begin{aligned}
 u_1(x, y, z, t) &= \frac{12\beta^2}{25\gamma\alpha} [1 - (\exp(-\xi) - 1)^{-2}], \\
 \xi &= \frac{\beta}{5\gamma} \left(x + y + z - \frac{25\gamma d + 25\gamma e + 6\beta^2}{25\gamma} t \right),
 \end{aligned} \tag{46}$$

and

$$u_2(x, y, z, t) = -\frac{12\beta^2}{25\gamma\alpha} (\exp(-\xi) - 1)^{-2}, \quad (47)$$

$$\xi = \frac{\beta}{5\gamma} \left(x + y + z - \frac{25\gamma d + 25\gamma e - 6\beta^2}{25\gamma} t \right).$$

Case II. If we set $P = 1/2$, $Q = 0$, and $R = -1/2$ into system (44) and solve the equation, we have

$$a_0 = a_2 = -\frac{3\beta^2}{25\gamma\alpha},$$

$$a_1 = 2a_0 = 0,$$

$$b_1 = b_2 = 0, \quad (48a)$$

$$\mu = \frac{\beta}{5\gamma},$$

$$c = \frac{25\gamma d + 25\gamma e - 6\beta^2}{25\gamma},$$

$$a_0 = \frac{9\beta^2}{25\gamma\alpha},$$

$$a_1 = \frac{-2a_0}{3},$$

$$a_2 = -\frac{a_0}{3}, \quad (48b)$$

$$b_1 = b_2 = 0,$$

$$\mu = \frac{\beta}{5\gamma},$$

$$c = \frac{25\gamma d + 25\gamma e + 6\beta^2}{25\gamma},$$

$$a_0 = b_2 = -\frac{3\beta^2}{25\gamma\alpha},$$

$$a_1 = a_2 = 0,$$

$$b_1 = 2b_2, \quad (48c)$$

$$\mu = \frac{\beta}{5\gamma},$$

$$c = \frac{25\gamma d + 25\gamma e - 6\beta^2}{25\gamma},$$

$$a_0 = \frac{9\beta^2}{25\gamma\alpha},$$

$$a_1 = a_2 = 0,$$

$$b_1 = -\frac{2a_0}{3},$$

$$b_2 = -\frac{a_0}{3},$$

$$\mu = \frac{\beta}{5\gamma},$$

$$c = \frac{25\gamma d + 25\gamma e + 6\beta^2}{25\gamma}, \quad (48d)$$

$$a_0 = -\frac{9\beta^2}{50\gamma\alpha},$$

$$a_1 = b_1 = \frac{2a_0}{3},$$

$$a_2 = b_2 = \frac{a_0}{6}, \quad (48e)$$

$$\mu = \frac{q}{10\gamma},$$

$$c = \frac{25\gamma d + 25\gamma e - 6\beta^2}{25\gamma},$$

$$a_0 = \frac{3\beta^2}{10\gamma\alpha},$$

$$a_1 = b_1 = -\frac{2a_0}{5},$$

$$a_2 = b_2 = -\frac{a_0}{10}, \quad (48f)$$

It can be shown that if $Y = \coth \xi \pm \operatorname{csch} \xi$, the set of values in (48a), (48c), and (48e) give the same solution as

$$u_{3,4}(x, y, z, t)$$

$$= -\frac{3\beta^2}{25\gamma\alpha} - \frac{6\beta^2}{25\gamma\alpha} (\coth \xi \pm \operatorname{csch} \xi)$$

$$- \frac{3\beta^2}{25\gamma\alpha} (\coth \xi \pm \operatorname{csch} \xi)^2 \quad (49)$$

$$= -\frac{12\beta^2}{25\gamma\alpha} (\exp(-\xi) \mp 1)^{-2},$$

$$\xi = \frac{\beta}{5\gamma} \left(x + y + z - \frac{25\gamma d + 25\gamma e - 6\beta^2}{25\gamma} t \right).$$

The set values in (48b), (48d), and (48f) give exactly the same solution as

$$u_{5,6}(x, y, z, t)$$

$$= \frac{9\beta^2}{25\gamma\alpha} - \frac{6\beta^2}{25\gamma\alpha} (\coth \xi \pm \operatorname{csch} \xi)$$

FIGURE 3: Kink wave profile of $u_{5,6}$, $\alpha = 1$, $\gamma = 1$, $d = 0.5$, $e = 0.5$, $\beta = 4$, $x = 0$, and $t = 0$.

FIGURE 4: Kink wave profile of $u_{5,6}$, $\alpha = 1$, $\gamma = 1$, $d = 0.5$, $e = 0.5$, $\beta = 4$, $x = 0$, and $t = 0.5$.

$$\begin{aligned}
 & -\frac{3\beta^2}{25\gamma\alpha} (\coth \xi \pm \operatorname{csch} \xi)^2 \\
 & = \frac{12\beta^2}{25\gamma\alpha} \left[1 - (\exp(-\xi) \mp 1)^{-2} \right], \\
 & \xi = \frac{\beta}{5\gamma} \left(x + y + z - \frac{25\gamma d + 25\gamma e + 6\beta^2}{25\gamma} t \right).
 \end{aligned} \tag{50}$$

We use $u_{5,6}(x, y, z, t)$ to represent the solution to (39) graphically; Figure 3 gives a kink wave profile at $t = 0$, while at $t = 0.5$ it is given in Figure 4.

4. Conclusion

In this paper, the modified tanh-coth method has been applied to nonlinear evolution equations. The explicit travelling wave solutions of the generalised (3+1)-dimensional Kudryashov-Sinelshchikov equation are derived. Some of the solutions obtained are in good agreement with the known exact solutions [7] using specific parameter values. The results show that the modified tanh-coth method is an efficient approach for the seeking solutions of such type of nonlinear evolution equations.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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