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Research Article

A Truncation Method for Solving the Time-Fractional Benjamin-Ono Equation

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We deem the time-fractional Benjamin-Ono (BO) equation out of the Riemann–Liouville (RL) derivative by applying the Lie symmetry analysis (LSA). By first using prolongation theorem to investigate its similarity vectors and then using these generators to transform the time-fractional BO equation to a nonlinear ordinary differential equation (NLODE) of fractional order, we complete the solutions by utilizing the power series method (PSM).

1. Introduction

Lie symmetry method provides an effective tool for deriving the analytic solutions of the nonlinear partial differential equations (NLPDEs) [1–4]. In recent years, many authors have studied the nonlinear fractional differential equations (NLFDEs) because these equations express many nonlinear physical phenomena and dynamic forms in physics, electrochemistry, and viscoelasticity [5–9].

Time-fractional NLDEs arise from classical NLPDEs by replacing its time derivative with the fractional derivative. The methods applied to derive the analytic solutions of NLFPDEs are the exp-function, the G'/G expansion, fractional su-equation, Lie symmetry method, and many more [10–19].

The one-dimensional Benjamin-Ono equation is considered here as follows (see [20]):

$$u_t + hu_{xx} + uu_x = 0 \tag{1}$$

In fact, the BO equation describes one-dimensional internal waves in deep water. We consider LSA for the analytic solutions by using PS expansion for the time-fractional BO equation:

$$u_t^\alpha + hu_{xx} + uu_x = 0, \quad 0 < \alpha < 1 \tag{2}$$

In division 2 of this paper, some basic properties of the Riemann–Liouville fractional derivative are shown firstly and then the Lie group method for FPDEs is presented. In division 3, the Lie group to the time-fractional BO equation (FBO) and the symmetry reductions are determined. In division 4, we derive anew arrangement of the FBO equation (2) via the PSM. In division 5, we study the convergence for the series solution. We conclude our work in division 6.

2. Notations and Delineations

2.1. Description of Lie Symmetry Reduction Method for NLFPDEs. We present the principal notations and definitions that detecting the symmetries of the NLFPDEs.

Here, the time-fractional NLFPDEs are

$$\partial_t^\alpha u = F\left(t, x, u, u_x, u_{xx}, \ldots\ldots\right) \tag{3}$$

Suppose that the infinitesimal vector X has the form

$$X = \xi^1\left(x,t,u\right)\frac{\partial}{\partial x} + \xi^2\left(x,t,u\right)\frac{\partial}{\partial t} + \eta\left(x,t,u\right)\frac{\partial}{\partial u} \tag{4}$$

The Lie group parameter of infinitesimal transformations [8, 21, 22] has the formula

$$\begin{aligned}
\overline{x} &= x + \varepsilon\xi^1(t,x,u) + O\left(\varepsilon^2\right),\\
\overline{t} &= t + \varepsilon\xi^2(t,x,u) + O\left(\varepsilon^2\right),\\
\overline{u} &= u + \varepsilon\eta(t,x,u) + O\left(\varepsilon^2\right),\\
\frac{\partial^\alpha \overline{u}}{\partial \overline{t}^\alpha} &= \frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon\eta_\alpha^0(t,x,u) + O\left(\varepsilon^2\right),\\
\frac{\partial \overline{u}}{\partial \overline{x}} &= \frac{\partial u}{\partial x} + \varepsilon\eta^x(t,x,u) + O\left(\varepsilon^2\right),\\
\frac{\partial^2 \overline{u}}{\partial \overline{x}^2} &= \frac{\partial^2 u}{\partial x^2} + \varepsilon\eta^{xx}(t,x,u) + O\left(\varepsilon^2\right),
\end{aligned} \tag{5}$$

where ξ^1, ξ^2, and η are considered as the infinitesimals of the transformation's variables (x,t,u), respectively, and εń1 is considered as the group parameter; we will take it to be equal to one. The explicit expressions of η^x and η^{xx}, which we consider as the prolongation of the infinitesimals, are given by

$$\eta^x = D_x(\eta) - u_x D_x\left(\xi^1\right) - u_t D_t\left(\xi^2\right), \tag{6}$$

and

$$\eta^{xx} = D_x\left(\eta^x\right) - u_{xt} D_x\left(\xi^1\right) - u_{xx} D_t\left(\xi^2\right), \tag{7}$$

where D_x is in [8] assigned as

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial x} + u_{xx}\frac{\partial}{\partial u_x} + \dots \tag{8}$$

Theorem 1. *Equation (2) coincides with a one-parameter group of transformations (5) with the infinitesimal generator X if and only if the accompanying infinitesimal conditions holds true:*

$$Pr^{(\alpha,2)}\ X(\Delta)|_{\Delta=0} = 0 \tag{9}$$

where $\Delta = D_t^\alpha u - F(t,x,u,u_x,u_{xx},\dots\dots)$ and Pr is the second prolongation of the infinitesimal generator X.

Definition 2. The prolonged vector is demonstrated by

$$\begin{aligned}
Pr^{(n)}X = X &+ \sum_{i=1}^{p}\sum_{\alpha=1}^{q}\xi_i^\alpha\frac{\partial}{\partial u_i^\alpha} + \dots\\
&+ \sum_{j_1=1}^{p}\dots\sum_{j_n=1}^{p}\sum_{\alpha=1}^{q}\xi_{j_1\dots j_n}^\alpha\frac{\partial}{\partial u_{j_1\dots j_n}^\alpha}
\end{aligned} \tag{10}$$

where q is the number of dependent variables, p is the number of independent variables, $\partial/\partial u_{j_1}^\alpha = \partial/\partial u_x^\alpha$, and the PDE involves derivatives of up to the order n. The condition [21–23] is given by

$$\xi^2\ (t,x,u)|_{t=0} = 0 \tag{11}$$

Lemma 3. *The function $u = \theta(x,t)$ is an invariant solution of (3) if and only if*

$$\text{(i)}\ \xi^2(x,t,\theta)\,\theta_t + \xi^1(x,t,\theta)\,\theta_x = \eta(x,t,\theta).$$

Lemma 4. *The $\boldsymbol{\alpha^{th}}$ extended infinitesimal [24, 25] for the fractional derivative part utilizing the RL definition with (11) is given by*

$$\begin{aligned}
\eta_\alpha^0 = &\frac{\partial^\alpha \eta}{\partial t^\alpha} + \left(\eta_u - \alpha D_t\left(\xi^2\right)\right)\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \mu\\
&- \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n\left(\xi^1\right)D_t^{\alpha-n}\left(u_x\right) + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^\alpha \eta_u}{\partial t^\alpha}\right.\\
&\left.- \left[\binom{\alpha}{n+1}D_t^{n+1}\left(\xi^2\right)\right]D_t^{\alpha-n}(u)\right],
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
\mu = &\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=2}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\\
&\cdot\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^r\frac{\partial^m}{\partial t^m}\left[u^{k-r}\right]\frac{\partial^{n-m+k}}{\partial t^{n-m}\partial u^k}
\end{aligned} \tag{13}$$

Remember that

$$\binom{\alpha}{n} = \frac{(-1)^{\boldsymbol{n-1}}\,\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\,\Gamma(n+1)} \tag{14}$$

3. Reduction of Time-Fractional Benjamin-Ono Equation

We use the LSA to find the similarity solution for 1D time-factional BO equation (1). Suppose that (2) is an invariant under (5), so that we have

$$\overline{u}_t^\alpha + h\overline{u}_{xx} + \overline{u}\,\overline{u}_x = 0 \tag{15}$$

Thus, $u(x,t)$ satisfies (2). Applying the second prolongation to (2), symmetry invariant equation is

$$\eta_\alpha^0 + u\eta^x + h\eta^{xx} + u_x\eta = 0 \tag{16}$$

Substituting the values from (6), (7), and (12) into (16) and isolating coefficients in partial derivatives regarding x and power of u, we have

$$\binom{\alpha}{n}\partial_t^n\eta_u - \binom{\alpha}{n+1}D_t^{n+1}\left(\xi^2\right) = 0, \quad n = 1,2,3\dots,$$

$$\xi_u{}^2 = \xi_x{}^2 = \xi_u{}^1 = \xi_t{}^1 = \eta_{uu} = 0,$$

$$\alpha\xi_t{}^2 - 2\xi_x{}^1 = 0,$$
$$h\eta_{xx} - u\partial_t^\alpha \eta_u + \partial_t^\alpha \eta + u\eta_x = 0,$$
$$\xi_{xx}{}^1 - 2\eta_{xu} = 0. \tag{17}$$

Solving the obtained determining equation, we get

$$\xi^1 = c_2 + \alpha x c_1,$$
$$\xi^2 = 2tc_1, \tag{18}$$
$$\eta = -\alpha u c_1,$$

where c_1 and c_2 are constants, for simplicity. We take their values equal to one. So, (2) has two vector fields that can generate its infinitesimal symmetry. These Lie vectors are considered as follows:

$$X_1 = \frac{\partial}{\partial x}, \tag{19}$$

$$X_2 = \alpha x \frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - \alpha u \frac{\partial}{\partial u}. \tag{20}$$

Case 1. For (19), we have

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0} \tag{21}$$

Solving this equation, $u = f(t)$. Putting $u = f(t)$ into (1), we get

$$D_t^\alpha f(t) = 0 \tag{22}$$

where $u = a_1 t^{\alpha-1}$.

Case 2. For X_2 in (20), we have

$$\frac{dx}{\alpha x} = \frac{dt}{2t} = -\frac{du}{\alpha u} \tag{23}$$

This is the characteristic equation. By solving it, the resulting similarity variable in the form

$$z_1 = xt^{-\alpha/2},$$
$$z_2 = ut^{-\alpha/2}. \tag{24}$$

The variables transformation is as follows:

$$u = t^{-\alpha/2} f(\xi), \quad \xi = xt^{-\alpha/2}, \tag{25}$$

where $f(\xi)$ is a function in one variable ξ. We use (25) to transform (2) into a fractional ODE.

Theorem 5. *Transformation (25) reduces (2) to the nonlinear FODE as follows:*

$$\left(P_{3/\alpha}^{1-3\alpha/2,\alpha} f\right)(\xi) + hf_{\xi\xi} + ff_\xi = 0 \tag{26}$$

utilizing the Erdelyi-Kober (EK) fractional derivative operator [20]:

$$\left(P_\beta^{\xi^2,\alpha} f\right)(\xi) = \prod_{j=0}^{n-1}\left(\xi^2 + j - \frac{1}{\beta}\frac{d}{d\xi}\right)\left(K_\beta^{\xi^2+\alpha,n-\alpha} f\right)(\xi), \tag{27}$$

where

$$\left(K_\beta^{\xi^2,\alpha} f\right)(\xi) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^\infty (u-1)^{\alpha-1} u^{-(\xi^2+\alpha)} f\left(\xi u^{1/\beta}\right) du, & \alpha > 0 \\ f(\xi), & \alpha = 0 \end{cases} \tag{28}$$

and

$$n = \begin{cases} [\alpha] + 1, & \alpha \neq N \\ \alpha, & \alpha \in N \end{cases} \tag{29}$$

Proof. Utilizing the definition of the RL fractional derivative in (25), we get

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)} \cdot \int_1^t (t-s)^{n-\alpha-1} s^{-\alpha/2} f\left(xs^{-(\alpha/2)}\right) ds\right], \quad n-1 < \alpha < 1,\ n = 1, 2, \ldots \tag{30}$$

Assume that $v = t/s$, $ds = -(t/v^2)dv$. Thus, (30) becomes

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[t^{n-3\alpha/2}\frac{1}{\Gamma(n-\alpha)} \cdot \int_0^\infty (v-1)^{n-\alpha-1} v^{-(n+1-3\alpha/2)} f\left(\xi v^{(\alpha/2)}\right) dv\right] \tag{31}$$

Applying the EK fractional integral operator (28) in (31), we have

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^{n}}{\partial t^{n}}\left[t^{n-3\alpha/2}\left(K_{2/\alpha}^{1-\alpha/2,n-\alpha} f\right)(\xi)\right] \tag{32}$$

For simplicity, we consider $= x\, t^{-\alpha/2}$, $\phi \in (0, \infty)$. We thus find that

$$t\frac{\partial}{\partial t}\phi(\xi) = tx\left(-\frac{\alpha}{2}\right)t^{-\alpha/2-1}\phi(\xi) = -\frac{\alpha}{2}\xi\frac{\partial}{\partial \xi}\phi(\xi) \tag{33}$$

Hence, we have

$$\begin{aligned}\frac{\partial^{n}}{\partial t^{n}}\left[t^{n-3\alpha/2}\left(K_{2/\alpha}^{1-\alpha/2,n-\alpha} f\right)(\xi)\right] &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\right.\\ &\left.\cdot\, t^{n-3\alpha/2}\left(K_{2/\alpha}^{1-\alpha/2,n-\alpha} f\right)(\xi)\right]\\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-3\alpha/2-1}\left(n-\frac{3\alpha}{2}-\frac{\alpha}{2}\xi\frac{\partial}{\partial \xi}\right)\right.\\ &\left.\cdot\left(K_{2/\alpha}^{1-\alpha/2,n-\alpha} f\right)(\xi)\right].\end{aligned} \tag{34}$$

Repeating $n-1$ times, we have

$$\begin{aligned}\frac{\partial^{n}}{\partial t^{n}}\left[t^{n-3\alpha/2}\left(K_{2/\alpha}^{1-\alpha/2,n-\alpha} f\right)(\xi)\right] &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\right.\\ &\left.\cdot\, t^{n-3\alpha/2}\left(K_{2/\alpha}^{1-\alpha/2,n-\alpha} f\right)(\xi)\right] = \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-3\alpha/2-1}\left(n\right.\right.\\ &\left.\left.-\frac{3\alpha}{2}-\frac{\alpha}{2}\xi\frac{\partial}{\partial \xi}\right)\left(K_{2/\alpha}^{1-\alpha/2,n-\alpha} f\right)(\xi)\right]\\ &= t^{-\alpha/2}\prod_{j=0}^{n-1}\left[\left(1-\frac{3\alpha}{2}+j-\frac{\alpha}{2}\xi\frac{\partial}{\partial \xi}\right)\left(K_{2/\alpha}^{1-\alpha/2,n-\alpha} f\right)\right.\\ &\cdot(\xi)\end{aligned} \tag{35}$$

Applying the EK fractional differential operator (27) in (35), we get

$$\begin{aligned}&\frac{\partial^{n}}{\partial t^{n}}\left[\left(t^{n-\alpha/3}\left(K_{2/\alpha}^{1-\alpha/2,n-\alpha} f\right)(\xi)\right)\right]\\ &\quad = t^{-\alpha/2}\left(P_{2/\alpha}^{1-3\alpha/2,\alpha} f\right)(\xi)\end{aligned} \tag{36}$$

Substituting (36) into (32), we get

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = t^{-\alpha/2}\left(P_{2/\alpha}^{1-3\alpha/2,\alpha} f\right)(\xi), \tag{37}$$

Thus, (2) is reduced to a fractional-order ODE as follows:

$$\left(P_{2/\alpha}^{1-3\alpha/2,\alpha} f\right)(\xi) + ff_{\xi} + hf_{\xi\xi} = 0. \tag{38}$$

□

4. The Explicit Solution for the Time-Fractional Benjamin-Ono Equation by Using PSM

The analytic solutions via PSM [26] are demonstrated. We assume that

$$f(\xi) = \sum_{n=0}^{\infty} a_n \phi(\xi)^n, \tag{39}$$

Differentiating (39) twice regarding ξ, we get

$$f'(\xi) = \sum_{n=0}^{\infty} n a_n \phi(\xi)^{n-1}, \tag{40}$$

and

$$f''(\xi) = \sum_{n=0}^{\infty} n(n-1) a_n \phi(\xi)^{n-2}, \tag{41}$$

Substituting (39), (40), and (41) into (38), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{\Gamma(2-3\alpha/2-n\alpha/2)}{\Gamma(2-\alpha/2-n\alpha/2)}a_n\phi(\xi)^n\\ &\quad+\sum_{n=0}^{\infty}a_n\phi(\xi)^n\sum_{n=0}^{\infty}(n+1)a_{n+1}\phi(\xi)^n\\ &\quad+h\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\phi(\xi)^n = 0\end{aligned} \tag{42}$$

Comparing coefficients in (42) when $n = 0$, we obtain

$$a_2 = \frac{-1}{2h}\left(\frac{\Gamma(2-3\alpha/2)}{\Gamma(2-\alpha/2)}a_0 + a_0 a_1\right), \tag{43}$$

When $n \geq 1$, the recurrence relations between the series coefficients are

$$a_{n+2} = \frac{-1}{2h(n+2)(n+1)}\left(\frac{\Gamma(2-3\alpha/2-n\alpha/2)}{\Gamma(2-\alpha/2-n\alpha/2)}a_n + (n+1)a_n a_{n+1}\right) \tag{44}$$

Using (44), the series solution for (39) can be represented by substituting (43) and (44) into (39):

$$\begin{aligned} f(\xi) &= a_0 + a_1\xi + a_2\xi^2 + \sum_{n=1}^{\infty} a_{n+2}\xi^{n+2} = a_0 + a_1\xi \\ &- \frac{1}{2h}\left(\frac{\Gamma(2-3\alpha/2)}{\Gamma(2-\alpha/2)}a_0 + a_0 a_1\right)\xi^2 \\ &- \sum_{n=1}^{\infty}\frac{1}{2h(n+2)(n+1)}\left(\frac{\Gamma(2-3\alpha/2-n\alpha/2)}{\Gamma(2-\alpha/2-n\alpha/2)}a_n \right. \\ &\left. + (n+1)a_n a_{n+1}\right)\xi^{n+2}. \end{aligned} \tag{45}$$

Upon substitution using similarity variables in (25), the following explicit solutions for (2) are

$$\begin{aligned} u(x,t) &= a_0 t^{-\alpha/2} + a_1 x t^{-\alpha} - \frac{1}{2h}\left(\frac{\Gamma(2-3\alpha/2)}{\Gamma(2-\alpha/2)}a_0 \right. \\ &\left. + a_0 a_1\right) t^{-\alpha/2}\left(x t^{-\alpha/2}\right)^2 \\ &- \sum_{n=1}^{\infty}\left\{\frac{1}{2h(n+2)(n+1)}\frac{\Gamma(2-3\alpha/2-n\alpha/2)}{\Gamma(2-\alpha/2-n\alpha/2)}a_n \right. \\ &\left. + \sum_{k=0}^{n}\sum_{j=0}^{k}\left(a_j + a_{k-j}\right)a_{n-k}(n+1)\right\}(x)^{n+2} \\ &\cdot t^{-\alpha(2n+\alpha\beta)/2\beta}. \end{aligned} \tag{46}$$

5. Convergence Analysis

To satisfy the convergence test, there are many kinds of tests as the ratio, the comparison, and the quotient tests. The convergence of the solution equation (46) will be presented as follows. We revamp (46) as follows:

$$\begin{aligned} |a_{n+2}| &\leq \left(\frac{|\Gamma(2-3\alpha/2-n\alpha/2)|}{|\Gamma(2-\alpha/2-n\alpha/2)|}|a_n| \right. \\ &\left. - \sum_{k=0}^{n}\sum_{j=0}^{k}|a_j||a_{k-j}||a_{n-k}| - |a_n|\right) \end{aligned} \tag{47}$$

Equation (47), utilizing the Gamma function, shows that $|\Gamma(2-3\alpha/2-n\alpha/2)|/|\Gamma(2-\alpha/2-n\alpha/2)| < 1$ for arbitrary n that

$$|a_{n+2}| \leq M\left(|a_n| - \sum_{k=0}^{n}\sum_{j=0}^{k}|a_j||a_{k-j}||a_{n-k}| - |a_n|\right) \tag{48}$$

where $M = \max\{(|c_1|, |c_2|)$. We now assume another form of the PSM:

$$B(\xi) = \sum_{n=0}^{\infty} c_n \xi^n \tag{49}$$

By comparing the two series, we can observe that $|c_n| \leq a_n$, $n = 0, 1, \ldots$. Hence, the series $B(\xi) = \sum_{n=0}^{\infty} c_n\xi^n$ is the majorant series of (47). So, we find that

$$\begin{aligned} B(\xi) &= c_0 + c_1\xi \\ &+ M\left(\sum_{n=0}^{\infty}c_n\xi^2 B(\xi) + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}c_j c_{k-j}c_{n-k} + \sum_{n=0}^{\infty}c_n\right) \\ &\cdot \xi^{n+2} \end{aligned} \tag{50}$$

Consider an implicit functional system regarding ξ as follows:

$$\begin{aligned} &\beta(\xi, B) \\ &= B - c_0 - c_1\xi - c_2\xi^2 \\ &- M\left(\xi^2 B(\xi) + 2B(\xi)^2 + \left(\xi^2 - c_1\xi - 3c_0\right)B\right) \end{aligned} \tag{51}$$

since β is analytic in a neighborhood of $(0, c_0)$, where $\beta(0, c_0) = 0$, and $(\partial/\partial B)\beta(0, c_0) = 0$. Then, the series $B(\xi) = \sum_{n=0}^{\infty}\xi^n$ is analytic around $(0, c_0)$ and this is verified utilizing [27] and the radius of convergence of this series belongs to a positive domain. This shows that (46) converges around $(0, c_0)$.

6. Physical Performance of the Power Series Technique for Eqs. (46)

To have expressed and convenient conception of the physical characteristic of the power series solution, the 3D plots for the explicit solution equations (46) is plotted in Figures 1–4 at $h = 1$ by utilizing appropriate parameter forms. The spectacle vision of the real portion of (46) can be visible in the 3D plots proof in Figures 1, 2, 3, and 4, respectively.

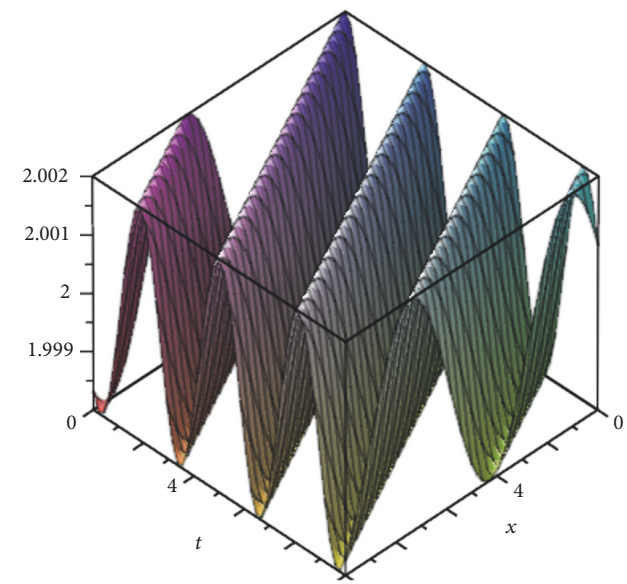

Figure 1: 3D plot of (46) with $a_0 = 1.9$, $a_1 = 1.7$, $a_2 = 0.77$, $\gamma = 1$, $\alpha = 0.8$, $\beta = 2$.

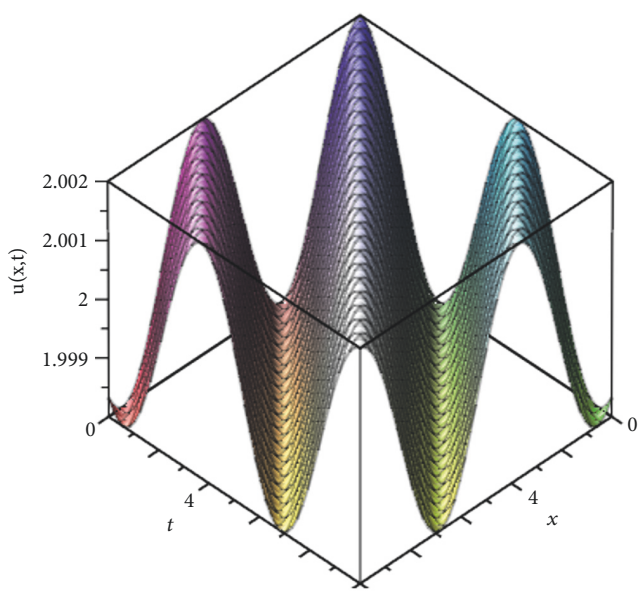

Figure 2: 3D plot of (46) with $a_0 = 3.9$, $a_1 = 1.7$, $a_2 = 0.87$, $\gamma = 1$, $\alpha = 0.6$, $\beta = 1$.

7. Conclusions

Lie point symmetry properties of (1 + 1)-dimensional time-fractional Benjamin-Ono equation have been considered with the Riemann–Liouville fractional derivative. These symmetries are used here to transform the FPDEs into NLFODEs. Closed-form solutions are determined by using PSM in the last division. The accuracy exhibits the assembly of the solution. Considerable frames for the acquired explicit solutions were approached.

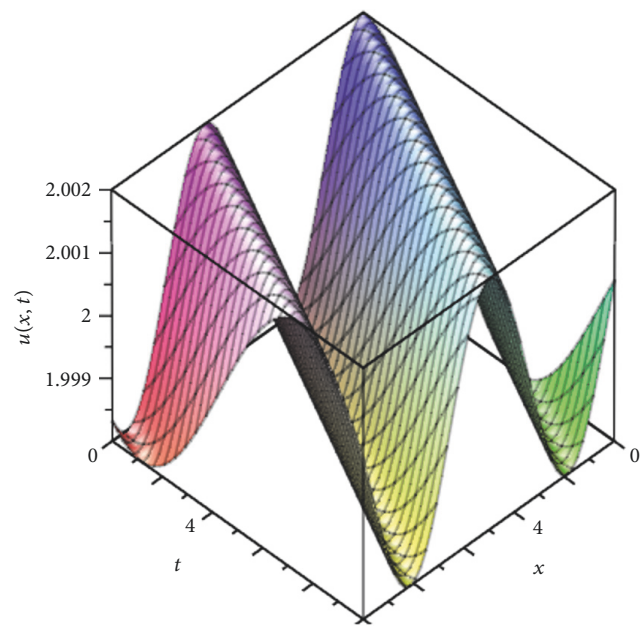

Figure 3: 3D plot of (46) with $a_0 = 1$, $a_1 = 1.1$, $a_2 = 0.57$, $\gamma = 1$, $\alpha = 0.5$, $\beta = 0.7$.

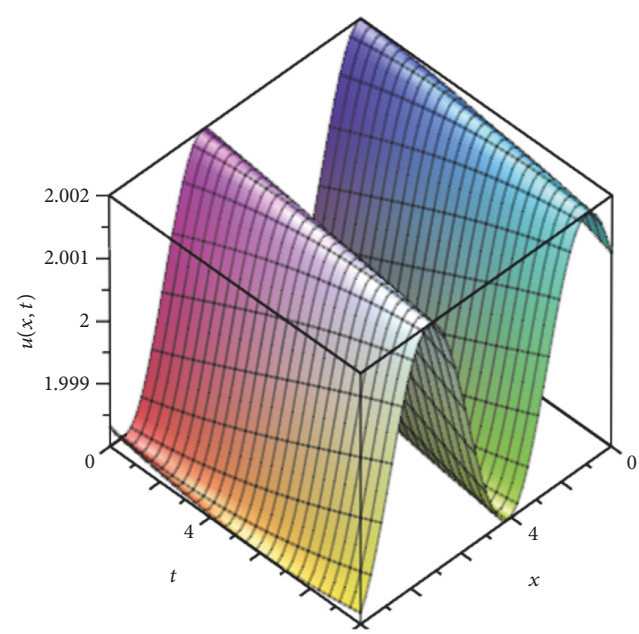

Figure 4: 3D plot of (46) with $a_0 = 1$, $a_1 = 1$, $a_2 = 0.4$, $\gamma = 1$, $\alpha = 0.3$, $\beta = 0.4$.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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