


Research Article

A Framework for Nonconforming Mixed Finite Element Method for Elliptic Problems in \mathbb{R}^3

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In this paper, we suggest a new patch condition for nonconforming mixed finite elements (MFEs) on parallelepiped and provide a framework for the convergence. Also, we introduce a new family of nonconforming MFE space satisfying the new patch condition. The numerical experiments show that the new MFE shows optimal order convergence in $H(\text{div})$ and L^2 -norm for various problems with discontinuous coefficient case.

1. Introduction

The finite element method has achieved great success in many fields, and it has become a powerful tool for solving partial differential equations [1–3]. The main idea of the finite element method is using a finite dimensional space to approximate the exact solution on the given space according to a certain kind of variational principle. In the finite element discretizations, a basic distinction can be made between conforming and nonconforming methods. When the finite element space is a subspace of the solution space, the method is called conforming. In this case, the error between the true solution and the finite element method (FEM) solution is bounded by the distance between the FEM space and the given space by Céa's lemma. Meanwhile, the nonconforming finite element space is not contained in the space the exact solution lives in. Hence, extra error committed by the nonconformity has to be estimated.

There are some situations in which finite element methods for the primary variable do not yield satisfactory results, such as elliptic problem with large jumps in the diffusion coefficient. Sometimes, other quantities such as Darcy velocity along with pressure of the flow in the porous media become variables of main interest. In this case, the mixed finite element method (MFEM) is preferred. Many

mixed finite element methods have been developed since it was first suggested in the late 1970s [4–6]. The idea of the mixed methods is to introduce the velocity as a new variable and change the given equation into a system of equations. By discretizing this system, we can compute two variables, velocity and pressure, simultaneously and expect a more accurate velocity. MFEM has been used in many applications such as porous media problem [7, 8] and chemical engineering [9].

The nonconforming approaches have been widely studied for Lagrangian finite elements. So far, all the well-known mixed finite element (MFE) space are conforming in the sense that the space is contained in $H(\text{div}, \Omega)$, defined as the space of all vector functions whose divergence belongs to $L^2(\Omega)$. It is natural to ask whether there exists a nonconforming counterpart of the MFE space. Hiptmair [10] has investigated some conditions for the nonconforming MFE space. But, under the conditions suggested there, they only show suboptimal convergence. Meanwhile, a family of high-order nonconforming MFE space was introduced in [11] a few years ago, and numerical examples show the optimal order of convergence. However, there is no analysis.

In this paper, we suggest a new condition under which the nonconforming MFEM may have optimal convergence. In addition, we introduce another family of the

nonconforming MFE space which satisfies this condition. To the author's best knowledge, nonconforming element having optimal order has not been suggested by others.

The organization of this paper is as follows: in the next section, we present the model problem. In Section 3, we introduce nonconforming mixed finite element spaces on parallelepiped in \mathbb{R}^3 together with a new patch condition. A framework for the convergence is given in Section 4. We give numerical experiments in Section 5.

2. Model Problem

Given $\Omega \subset \mathbb{R}^3$, a simply connected bounded Lipschitz polyhedral domain with connected boundary $\partial\Omega$, we consider the following second-order elliptic problem:

$$\begin{cases} -\operatorname{div}(\kappa\nabla p) = f, & \text{in } \Omega, \\ p = 0, & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $\kappa \in L^\infty(\Omega)$ is assumed to be uniformly positive definite and bounded. And f is a given function in $L^2(\Omega)$. To write the given equation into a mixed system, we use Darcy's law, $\mathbf{u} = -\kappa\nabla p$. Then, we can rewrite problem (1) in the following mixed form:

$$\begin{cases} \mathbf{u} + \kappa\nabla p = 0, & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = f, & \text{in } \Omega, \\ p = 0, & \text{on } \partial\Omega. \end{cases} \quad (2)$$

Denote by $\mathbf{V} = H(\operatorname{div}, \Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 : \operatorname{div} \mathbf{v} \in L^2(\Omega)\}$ and $W = L^2(\Omega)$ the usual Sobolev spaces with obvious norms. Then, we have the following variational form for (2): find $(\mathbf{u}, p) \in \mathbf{V} \times W$ such that

$$\int_{\Omega} \kappa^{-1} \mathbf{u} \mathbf{v} \, dx - \int_{\Omega} p \operatorname{div} \mathbf{v} \, dx = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \quad (3)$$

$$\int_{\Omega} \operatorname{div} \mathbf{u} \cdot q \, dx = \int_{\Omega} f q \, dx, \quad \forall q \in W. \quad (4)$$

For the convenience of the presentation, we let

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_{\Omega} \kappa^{-1} \mathbf{u} \mathbf{v} \, dx, \\ b(\mathbf{v}, p) &= - \int_{\Omega} p \operatorname{div} \mathbf{v} \, dx. \end{aligned} \quad (5)$$

Then, saddle point problems (3) and (4) can be expressed simply as follows: find $(\mathbf{u}, p) \in \mathbf{V} \times W$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= 0, \quad \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) &= -(f, q), \quad \forall q \in W, \end{aligned} \quad (6)$$

where (\cdot, \cdot) indicates the inner product in $L^2(\Omega)$. If familiar inf-sup condition holds, then problem (6) has a unique solution [12].

3. Nonconforming Mixed Finite Element Spaces

The main idea of MFEM is solving problem (6) over suitable locally defined finite dimensional spaces. Their construction depends on triangulations \mathcal{T}_h of Ω . Let $\Omega = [0, 1]^3$ and

$\{\mathcal{T}_h : h > 0\}$ be a family of partitions of Ω into parallelepiped K obtained by uniform division having each side length h .

For the construction of the nonconforming MFE space \mathbf{V}_h , we require that

$$\begin{aligned} (1) \mathbf{V}_h &= \left\{ \mathbf{v} \in (L^2(\Omega))^3 \mid \mathbf{v}|_K \in H(\operatorname{div}, K), \quad \forall K \in \mathcal{T}_h \right\}, \\ (2) \mathbf{V}_h &\not\subseteq \mathbf{V} = H(\operatorname{div}, \Omega). \end{aligned} \quad (7)$$

That is, a function in the space \mathbf{V}_h is locally in $H(\operatorname{div}, K)$ but not in $H(\operatorname{div}, \Omega)$ over the triangulation \mathcal{T}_h . A lack of continuity of normal components across interelement boundaries gives rise to a nonconforming approximation. But, we still require some local conformity.

For any domain D in \mathbb{R}^2 or \mathbb{R}^3 , let $P_\ell(D)$ be the space of polynomials of total degree ℓ and $Q_{\ell,m}(D)$ or $Q_{\ell,m,n}(D)$ be the space of polynomials of degree less or equal to ℓ, m, n , respectively, in each variable.

3.1. Patch Conditions. First, we recall a well-known type of "patch condition": let f be the common face of two adjacent elements K_i and K_j .

$$\begin{aligned} (H1) \int_f q \left(\mathbf{v}_h|_{K_i} \cdot \mathbf{n} - \mathbf{v}_h|_{K_j} \cdot \mathbf{n} \right) dA &= 0, \\ \forall q \in P_\ell(f) \text{ for some } \ell &= 1, 2, \dots, \end{aligned} \quad (8)$$

where \mathbf{n} is an outer unit normal vector to each element. For example, Hiptmair used this condition together with the assumption of continuous interpolation and showed a suboptimal error estimate [10]. However, for the 3D case (parallelepiped), we see that such condition is not enough to guarantee the existence of continuous interpolation which is necessary to derive an approximation. With hypothesis (H1) holding for $q \in P_\ell(f)$ is not enough to determine a continuous interpolation over the whole domain Ω . In fact, if such an interpolation exists, then (H1) holds for $q \in Q_{\ell,\ell}(f)$, which would imply the conformity of the space \mathbf{V}_h . Thus, to study a nonconforming MFE, we need a stronger patch condition but not strong enough to make the \mathbf{V}_h space fully conforming.

So, we suggest new patch conditions (H2): for all vertical faces $f_V := \partial K_i \cap \partial K_j$ ($i \neq j$) and all $\mathbf{v}_h \in \mathbf{V}_h$, we have

$$(H2(a)) \int_{f_V} q \left(\mathbf{v}_h|_{K_i} \cdot \mathbf{n} - \mathbf{v}_h|_{K_j} \cdot \mathbf{n} \right) dA = 0, \quad \forall q \in Q_{\ell,\ell}(f_V). \quad (9)$$

For horizontal faces $f_H = \partial K_i \cap \partial K_j$ and all $\mathbf{v}_h \in \mathbf{V}_h$, we have

$$\begin{aligned} (H2(b)) \int_{f_H} q \left(\mathbf{v}_h|_{K_i} \cdot \mathbf{n} - \mathbf{v}_h|_{K_j} \cdot \mathbf{n} \right) dA &= 0, \\ \forall q \in Q_{\ell,\ell}(f_H) \setminus \{x^k y^k\}. \end{aligned} \quad (10)$$

This means that the moments up to order ℓ of the discrete velocity are continuous across horizontal interelement boundaries with respect to $Q_{\ell,\ell}(f_V)$, and the moments

across vertical interelement boundaries are continuous up to $Q_{\ell,\ell}(f_H) \setminus \{x^k y^k\}$ only.

Now, we introduce a new nonconforming MFE. We denote by $Q_{\ell,m,n}^*$ the set of all polynomials of $Q_{\ell,m,n}$ except those having the form $x^\ell y^m z^p$ for $p = 1, \dots, n$.

Definition 1. Let $\mathbf{V}_h(K)$ be the subspace of

$$\begin{aligned} & Q_{k+1,k,k}^*(K) \times Q_{k,k+1,k}^*(K) \\ & \times \left(Q_{k,k,k+1}^*(K) \oplus \{x^{k+1} y^i, x^i y^{k+1}, i = 0, \dots, k-1\} \right), \end{aligned} \quad (11)$$

where the elements $(x^{k+1} y^k, 0, 0)$ and $(0, x^k y^{k+1}, 0)$ are replaced by the single element $(x^{k+1} y^k, -x^k y^{k+1}, 0)$.

We note that this element is similar to [11], but the number of DOFs is reduced by 2 on each element.

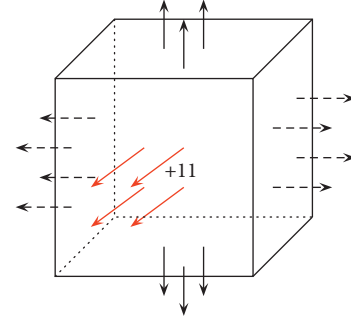


FIGURE 1: DOFs when $k = 1, 4$ for vertical faces and 3 for horizontal faces.

Then, the dimension of $\mathbf{V}_h(K)$ is $2\{(k+2)(k+1)^2 - k\} + \{(k+1)^2(k+2) - (k+1) + 2k\} - 1 = 3k^3 + 12k^2 + 14k + 4$. For $k = 1$, we have $\mathbf{u} = (u_1, u_2, u_3)$, where

$$\begin{aligned} u_1 &= P_1(x, y, z) + a_1 xy + a_2 yz + a_3 zx + a_4 xyz + a_5 x^2 + a_6 x^2 z + dx^2 y, \\ u_2 &= P_1(x, y, z) + b_1 xy + b_2 yz + b_3 zx + b_4 xyz + b_5 y^2 + b_6 y^2 z - dx y^2, \\ u_3 &= P_1(x, y, z) + c_1 xy + c_2 yz + c_3 zx + c_4 z^2 + c_5 xz^2 + c_6 yz^2 + c_7 x^2 + c_8 y^2. \end{aligned} \quad (12)$$

This has 33 unknowns in each element, in which 3 less than the RTN space [5] (see Figure 1).

To define the degrees of freedom, we need an auxiliary space. Let $\Psi_h(K)$ be the subspace consisting of element type $\phi = (\phi_1, \phi_2, 0)$:

$$Q_{k-1,k,k}^*(K) \times Q_{k,k-1,k}^*(K) \times \{0\}, \quad (13)$$

where the elements $(x^{k-1} y^k, 0, 0)$ and $(0, x^k y^{k-1}, 0)$ are replaced by the single element $(x^{k-1} y^k, -x^k y^{k-1}, 0)$.

For any $\mathbf{u}_h = (u_1, u_2, u_3) \in \mathbf{V}_h(K)$, the degrees of freedom are given on face f with unit normal \mathbf{n} and in the interior of K as follows:

$$\int_{f_H} \mathbf{u}_h \cdot \mathbf{n} q dA, \quad \forall q \in Q_{k,k}(f_H) \setminus \{x^k y^k\}, \text{ for each horizontal face } f_H, \quad (14)$$

$$\int_{f_V} \mathbf{u}_h \cdot \mathbf{n} q dA, \quad \forall q \in Q_{k,k}(f_V), \text{ for each vertical face } f_V, \quad (15)$$

$$\int_K u_3 \phi_3 dx, \quad \forall \phi_3 \in Q_{k,k,k-1}^* \oplus \text{span}\{x^{k+1} y^i, x^i y^{k+1}, i = 0, \dots, k-1\}, \quad (16)$$

$$\int_K (u_1 \phi_1 + u_2 \phi_2) dx, \quad \forall (\phi_1, \phi_2, 0) \in \Psi_h(K). \quad (17)$$

Then, the number of conditions is $2\{(k+1)^2 - 1\} + 4(k+1)^2 + \{k(k+1)^2 - (k-1) + 2k\} + 2\{k(k+1)^2 - k\} - 1$. We start our analysis of this element by showing that the element is unisolvent.

Theorem 1. A vector function $\mathbf{u}_h = (u_1, u_2, u_3) \in \mathbf{V}_h(K)$ is uniquely determined by the degrees of freedom (14)–(17).

Proof. We first note that the dimension of $\mathbf{V}_h(K)$ is $3k^3 + 12k^2 + 14k + 4$, and this is also the number of degrees of freedom. Hence, it suffices to prove that if all the conditions are zero, then $\mathbf{u}_h = 0$. Since $\mathbf{u}_h \cdot \mathbf{n} \in Q_{k,k}(f_V)$ on the vertical faces, (15) implies $\mathbf{u}_h \cdot \mathbf{n} = 0$ for each vertical face. Then, we have

$$\begin{aligned} u_1 &= x(1-x)v_1, \\ u_2 &= y(1-y)v_2, \\ \mathbf{v} &= (v_1, v_2, 0) \in \Psi_h(K). \end{aligned} \quad (18)$$

Choosing $\phi = \mathbf{v}$ in (17) shows that $\mathbf{v} = 0$. Hence, $u_1 = u_2 = 0$.

Now, consider $u_3 \in Q_{k,k,k+1}^*(K) \oplus \text{span}\{x^{k+1} y^i, x^i y^{k+1}, i = 0, \dots, k-1\}$. We note that

$$u_3 = s + r_1 z^k + r_2 z^{k+1}, \quad (19)$$

for some $s \in Q_{k,k,k-1}^*(K) \oplus \text{span}\{x^{k+1} y^i, x^i y^{k+1}, i = 0, \dots, k-1\}$ and $r_1, r_2 \in Q_{k,k}(f_H) \setminus \{x^k y^k\}$. Then, the degrees of freedom (16) implies

$$u_3 = r'_1 z^k + r'_2 z^{k+1}, \quad \forall r'_1, r'_2 \in Q_{k,k}(f_H) \setminus \{x^k y^k\}. \quad (20)$$

From (14), we prove that $u_3 = 0$, and we are done. \square

Definition 2. For the scalar variable, we define

$$W_h(K) = Q_{k,k,k}^* \setminus \{x^k y^k\}. \quad (21)$$

The dimension of $W_h(K)$ is $\{(k+1)^3 - k\} - 1 = k^3 + 3k^2 + 2k$. Also, we know that

$$\begin{aligned} (1) \dim \mathbf{V}_h(K) &= W_h(K), \\ (2) W_h &\subset W. \end{aligned} \quad (22)$$

Since $\mathbf{V}_h \not\subset \mathbf{V}$, we cannot guarantee that the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ make sense for functions of \mathbf{V}_h . So, we define extensions to the larger space $\mathbf{X}_h = \mathbf{V}_h + H(\operatorname{div}, \Omega)$. Let

$$\begin{aligned} a_h(\mathbf{u}, \mathbf{v}) &= \sum_i \int_{K_i} \kappa^{-1} \mathbf{u} \mathbf{v} \, dx, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{X}_h, \\ b_h(\mathbf{v}, p) &= - \sum_i \int_{K_i} p \operatorname{div} \mathbf{v} \, dx, \quad \forall p \in W_h, \forall \mathbf{v} \in \mathbf{X}_h. \end{aligned} \quad (23)$$

We equip \mathbf{X}_h with a norm which is an extension of $\|\cdot\|_{H(\operatorname{div}, \Omega)}$:

$$\|\mathbf{u}\|_{\mathbf{X}_h}^2 = \sum_i \|\mathbf{u}\|_{H(\operatorname{div}, K_i)}^2, \quad \forall \mathbf{u} \in \mathbf{X}_h. \quad (24)$$

$$\begin{aligned} \int_{f_H} (\mathbf{u} - \Pi_h \mathbf{u}) \cdot \mathbf{n} q \, dA &= 0, \quad \forall q \in Q_{k,k}(f_H) \setminus \{x^k y^k\}, \\ \int_{f_V} (\mathbf{u} - \Pi_h \mathbf{u}) \cdot \mathbf{n} q \, dA &= 0, \quad \forall q \in Q_{k,k}(f_V), \\ \int_K (u_3 - \Pi_h u_3) \phi_3 \, dx &= 0, \quad \forall \phi_3 \in Q_{k,k,k-1}^* \oplus \operatorname{span}\{x^{k+1} y^i, x^i y^{k+1}, i = 0, \dots, k-1\}, \\ \int_K \{(u_1 - \Pi_h u_1) \phi_1 + (u_2 - \Pi_h u_2) \phi_2\} \, dx &= 0, \quad \forall (\phi_1, \phi_2, 0) \in \Psi_h(K). \end{aligned} \quad (27)$$

Then, we have the following lemma.

Lemma 1. *If $\Pi_h \mathbf{u}$ is the interpolation of \mathbf{u} , then we have*

$$\int_K \operatorname{div}(\mathbf{u} - \Pi_h \mathbf{u}) q \, dx = 0, \quad \forall \mathbf{u} \in \mathbf{V}_h(K), q \in W_h(K). \quad (28)$$

Proof. First, let $q \in W_h(K)$. Then, we know that $q|_{f_H} \in Q_{k,k}(f_H) \setminus \{x^k y^k\}$ for each horizontal face f_H and $q|_{f_V} \in Q_{k,k}(f_V)$ for each vertical face f_V . Also, we see that

$$\partial_x q \in Q_{k-1,k,k}^* \setminus \{x^{k-1} y^k\}, \quad (29)$$

$$\partial_y q \in Q_{k,k-1,k}^* \setminus \{x^k y^{k-1}\}, \quad (30)$$

$$\partial_z q \in Q_{k,k,k-1}^* \setminus \{x^k y^k\}. \quad (31)$$

Since $(\partial_x q, \partial_y q, 0) \in \Psi_k(K)$ and $\partial_z q \in Q_{k,k,k-1}^* \oplus \{x^{k+1} y^i, x^i y^{k+1}, i = 0, \dots, k-1\}$, by the definition of Π_h and that $q \in W_h(K)$ and by (29)–(31), we have

For W_h , we equip L^2 -norm. Now, we have the MFE problem corresponding to (6): find $(\mathbf{u}_h, p_h) \in \mathbf{X}_h \times W_h$ such that

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) &= 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b_h(\mathbf{u}_h, q_h) &= -(f, q_h), \quad \forall q_h \in W_h. \end{aligned} \quad (25)$$

It is easy to show that the bilinear forms $a_h(\cdot, \cdot)$ and $b_h(\cdot, \cdot)$ satisfy the following estimate:

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) &\leq c_1 \|\mathbf{u}_h\|_{\mathbf{X}_h} \|\mathbf{v}_h\|_{\mathbf{X}_h}, \\ b_h(\mathbf{v}_h, p_h) &\leq c_2 \|p_h\|_{W_h} \|\mathbf{v}_h\|_{\mathbf{X}_h}, \end{aligned} \quad (26)$$

for some positive constants c_1 and c_2 .

Given a typical box element K , its boundary ∂K consists of six planar pieces which we distinguish into two kinds: the horizontal plane by f_H and the vertical plane f_V . We define an interpolation operator $\Pi_h: \mathbf{H}^{k+1}(K) \rightarrow \mathbf{V}_h(K)$ by

$$\begin{aligned} \int_K (\operatorname{div} \Pi_h \mathbf{u}) q \, dx &= \int_{\partial K} \Pi_h \mathbf{u} \cdot \mathbf{n} q \, dA - \int_K \Pi_h \mathbf{u} \cdot \nabla q \, dx \\ &= \left\{ \int_{\partial f_H} \Pi_h \mathbf{u} \cdot \mathbf{n} q \, dA + \int_{\partial f_V} \Pi_h \mathbf{u} \cdot \mathbf{n} q \, dA \right\} \\ &\quad - \left\{ \int_K \Pi_h u_3 \partial_z q \, dx + \int_K (\Pi_h u_1 \partial_x q + \Pi_h u_2 \partial_y q) \, dx \right\} \\ &= \left\{ \int_{\partial f_H} \mathbf{u} \cdot \mathbf{n} q \, dA + \int_{\partial f_V} \mathbf{u} \cdot \mathbf{n} q \, dA \right\} \\ &\quad - \left\{ \int_K u_3 \partial_z q \, dx + \int_K (u_1 \partial_x q + u_2 \partial_y q) \, dx \right\} \\ &= \int_{\partial K} \mathbf{u} \cdot \mathbf{n} q \, dA - \int_K \mathbf{u} \cdot \nabla q \, dx \\ &= \int_K \operatorname{div} \mathbf{u} q \, dx. \end{aligned} \quad (32)$$

Let B be an operator from X_h to W' defined by $B(\mathbf{v}, q) = b(\mathbf{v}, q)$, for all $\mathbf{v} \in X_h$ and $q \in W$. Similarly, we define $B_h: \mathbf{V}_h \rightarrow W'_h$ by $B_h(\mathbf{v}_h, q_h) = b_h(\mathbf{v}_h, q_h)$ for all $\mathbf{v}_h \in \mathbf{V}_h$ and $q_h \in W_h$.

To confirm the stability of discrete problem (25), we let

$$\begin{aligned}
N(B_h) &= \{\mathbf{v}_h \in \mathbf{V}_h \mid b_h(\mathbf{v}_h, q_h) = 0, \quad \forall q_h \in W_h\}, \\
N(B) &= \{\mathbf{v} \in \mathbf{X}_h \mid \operatorname{divv}|_K = 0, \quad \forall K \in \mathcal{T}_h\}, \\
N(B_h^*) &= \{p_h \in W_h \mid b_h(\mathbf{v}_h, p_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h\}.
\end{aligned} \tag{33}$$

By the definitions and the fact $\operatorname{div} \mathbf{V}_h = W_h$, we see that $N(B_h) \subset N(B)$, which leads to the following lemma. \square

Lemma 2. *We have*

$$\sup_{\mathbf{v}_h \in N(B_h)} \frac{a_h(\mathbf{v}_h, \mathbf{w}_h)}{\|\mathbf{v}_h\|_{\mathbf{X}_h}} \geq \alpha \|\mathbf{w}_h\|_{\mathbf{X}_h}, \quad \forall \mathbf{w}_h \in N(B_h), \tag{34}$$

for some positive constant α independent of h .

By Lemma 1, the following inf-sup condition can be shown by standard technique [12].

Lemma 3. *We have*

$$\sup_{\mathbf{v}_h \in \mathbf{X}_h} \frac{b_h(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{\mathbf{X}_h}} \geq \beta \|q_h\|_{W_h/N(B_h^*)}, \quad \forall q_h \in W_h. \tag{35}$$

Theorem 2. *Problem (25) has a unique solution.*

Proof. The result follows from Lemmas 2 and 3 and Babuška-Brezzi theory. \square

4. Error Estimates

To prove convergence of our method, we need to estimate the approximation error and the consistency error. Since \mathbf{V}_h and W_h have polynomials of degree k , we can easily obtain the following approximation error of order k :

$$\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{V}_h} + \inf_{q_h \in W_h} \|p - q_h\|_{W_h} \leq Ch^{k+1} \|p\|_{H^{k+2}(\Omega)}. \tag{36}$$

However, to prove the consistency error is more difficult. We postpone it until later.

To obtain error estimate, we need the following theorem which is essentially given in [10], but we include it for the completeness.

Theorem 3. *Let $(\mathbf{u}, p) \in \mathbf{V} \times W$ be the solution of (6) and $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ be the discrete solution of (25). Then, we obtain the following error estimates:*

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{X}_h} + \|p - p_h\|_{\theta_{W_h}} \\
&\leq C \left(\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{V}_h} + \inf_{q_h \in W_h} \|p - q_h\|_{W_h} \right. \\
&\quad \left. + \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{|a_h(\mathbf{u}, \mathbf{v}_h) + b_h(\mathbf{v}_h, p)|}{\|\mathbf{v}_h\|_{\mathbf{X}_h}} \right). \tag{37}
\end{aligned}$$

Proof. For an arbitrary $\mathbf{v}_h \in \mathbf{V}_h$, we let $\mathbf{w}_h = \mathbf{\Pi}_h(\mathbf{u} - \mathbf{v}_h)$ so that

$$b_h(\mathbf{w}_h, q_h) = b_h(\mathbf{u} - \mathbf{v}_h, q_h), \quad \forall q_h \in W_h. \tag{38}$$

Then, $\mathbf{x}_h = \mathbf{v}_h + \mathbf{w}_h$ satisfies

$$b_h(\mathbf{x}_h, q_h) = b_h(\mathbf{u}, q_h), \quad \forall q_h \in W_h. \tag{39}$$

Hence, $\mathbf{u}_h - \mathbf{x}_h \in N(B_h)$, and we have the following inequality from the coercivity (Lemma 2) of $a_h(\cdot, \cdot)$ on $N(B_h)$ and (25):

$$\begin{aligned}
\alpha \|\mathbf{u}_h - \mathbf{x}_h\|_{\mathbf{X}_h} &\leq \sup_{\mathbf{v}_h \in N(B_h)} \frac{1}{\|\mathbf{v}_h\|_{\mathbf{X}_h}} |a_h(\mathbf{v}_h, \mathbf{u}_h - \mathbf{x}_h)| \\
&\leq \sup_{\mathbf{v}_h \in N(B_h)} \frac{1}{\|\mathbf{v}_h\|_{\mathbf{X}_h}} |a_h(\mathbf{v}_h, \mathbf{u}_h - \mathbf{u}) + a_h(\mathbf{v}_h, \mathbf{u} - \mathbf{x}_h)| \\
&\leq \sup_{\mathbf{v}_h \in N(B_h)} \frac{1}{\|\mathbf{v}_h\|_{\mathbf{X}_h}} |a_h(\mathbf{v}_h, \mathbf{u} - \mathbf{x}_h) - b_h(\mathbf{v}_h, q_h - p) \\
&\quad - \{a_h(\mathbf{u}, \mathbf{v}_h) + b_h(\mathbf{v}_h, p)\}|, \tag{40}
\end{aligned}$$

for any $q_h \in W_h$. We know that both bilinear forms are bounded. From this, we get

$$\begin{aligned} \alpha \|\mathbf{u}_h - \mathbf{x}_h\|_X &\leq c_1 \|\mathbf{u} - \mathbf{x}_h\|_X + c_2 \|p - q_h\|_W \\ &+ \sup_{\mathbf{v}_h \in N(B_h)} \frac{|a_h(\mathbf{u}, \mathbf{v}_h) + b_h(\mathbf{v}_h, p)|}{\|\mathbf{v}_h\|_X}, \\ \alpha \|\mathbf{u} - \mathbf{u}_h\|_{X_h} &\leq \alpha \left(\|\mathbf{u} - \mathbf{x}_h\|_{X_h} + \|\mathbf{u}_h - \mathbf{x}_h\|_{X_h} \right) \\ &= (\alpha + c_1) \|\mathbf{u} - \mathbf{x}_h\|_X + c_2 \|p - q_h\|_W \\ &+ \sup_{\mathbf{v}_h \in N(B_h)} \frac{|a_h(\mathbf{u}, \mathbf{v}_h) + b_h(\mathbf{v}_h, p)|}{\|\mathbf{v}_h\|_X}. \end{aligned} \quad (41)$$

Therefore,

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{X_h} &\leq \left(1 + \frac{c_1}{\alpha}\right) \|\mathbf{u} - \mathbf{x}_h\|_{X_h} + \frac{c_2}{\alpha} \|p - q_h\|_{X_h} \\ &+ \frac{1}{\alpha} \sup_{\mathbf{v}_h \in N(B_h)} \frac{|a_h(\mathbf{u}, \mathbf{v}_h) + b_h(\mathbf{v}_h, p)|}{\|\mathbf{v}_h\|_{X_h}}. \end{aligned} \quad (42)$$

From (35) and (38), we also have

$$\|\mathbf{u} - \mathbf{x}_h\|_{X_h} \leq \|\mathbf{u} - \mathbf{v}_h\|_{X_h} + \|\mathbf{w}_h\|_{X_h} \leq \left(1 + \frac{c_2}{\beta}\right) \|\mathbf{u} - \mathbf{v}_h\|_{X_h}. \quad (43)$$

By (42) and (43), we have the following estimate:

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{X_h} &\leq \left(1 + \frac{c_1}{\alpha}\right) \left(1 + \frac{c_2}{\beta}\right) \|\mathbf{u} - \mathbf{v}_h\|_{X_h} + \frac{c_2}{\alpha} \|p - q_h\|_{X_h} \\ &+ \frac{1}{\alpha} \sup_{\mathbf{v}_h \in N(B_h)} \frac{|a_h(\mathbf{u}, \mathbf{v}_h) + b_h(\mathbf{v}_h, p)|}{\|\mathbf{v}_h\|_{X_h}}. \end{aligned} \quad (44)$$

Using the first equation of (25), we know that

$$a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p - p_h) = a_h(\mathbf{u}, \mathbf{v}_h) + b_h(\mathbf{v}_h, p), \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \quad (45)$$

Then,

$$b_h(\mathbf{v}_h, p - p_h) = -a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + a_h(\mathbf{u}, \mathbf{v}_h) + b_h(\mathbf{v}_h, p), \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \quad (46)$$

From (35), we obtain

$$\|p - p_h\|_{W_h/N(B_h^*)} \leq \frac{1}{\beta} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{|b_h(\mathbf{v}_h, p - p_h)|}{\|\mathbf{v}_h\|_{X_h}} \leq \frac{1}{\beta} \left(c_3 \|\mathbf{u} - \mathbf{u}_h\|_{X_h} + \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{|a_h(\mathbf{u}, \mathbf{v}_h) + b_h(\mathbf{v}_h, p)|}{\|\mathbf{v}_h\|_{X_h}} \right). \quad (47)$$

Joining estimates (44) and (47), we obtain

$$\|\mathbf{u} - \mathbf{u}_h\|_{X_h} + \|p - p_h\|_{W_h} \leq C \left(\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{V}_h} + \inf_{q_h \in W_h} \|p - q_h\|_{W_h} + \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{|a_h(\mathbf{u}, \mathbf{v}_h) + b_h(p, \mathbf{v}_h)|}{\|\mathbf{v}_h\|_{X_h}} \right), \quad (48)$$

where C is a constant depending only on c_1, c_2, c_3, α , and β .

Now, we state an assumption regarding the consistency errors:

$$(A.1) \quad \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{|a_h(\mathbf{u}, \mathbf{v}_h) + b_h(\mathbf{v}_h, p)|}{\|\mathbf{v}_h\|_{X_h}} \quad (49)$$

$$\leq Ch^\ell \|p\|_{H^{k+2}(\Omega)}, \quad \ell = k \text{ or } \ell = k + 1.$$

We prove the error estimates. \square

Theorem 4. *Suppose the following regularity holds: the solution (\mathbf{u}, p) of (6) satisfies $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$, $p \in H^{k+2}(\Omega)$, and there exists some constant $C > 0$ such that*

$$\|\mathbf{u}\|_{k+1} + \|p\|_{k+2} \leq C \|f\|_{H^k(\Omega)}. \quad (50)$$

If (A.1) is satisfied, then the following estimates hold:

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{X_h} + \|p - p_h\|_{W_h} &\leq Ch^{\min\{k+1, \ell\}} \|p\|_{H^{k+2}(\Omega)} \\ &\leq Ch^{\min\{k+1, \ell\}} \|f\|_{H^k(\Omega)}. \end{aligned} \quad (51)$$

Proof. This follows directly from Theorem 3, Assumption (A.1), and (36).

Now, we prove (A.1) for the case $\ell = k$. The case $\ell = k + 1$ is left open. \square

Proof of (A.1). By the integration by parts, we see that

$$\begin{aligned} a_h(\mathbf{u}, \mathbf{v}_h) + b_h(p, \mathbf{v}_h) &= \sum_{K \in \mathcal{T}_h} \int_K \kappa^{-1} \mathbf{u} \mathbf{v}_h \, dx - \sum_{K \in \mathcal{T}_h} \int_K p \operatorname{div} \mathbf{v}_h \, dx \\ &= \sum_{K \in \mathcal{T}_h} \int_K \kappa^{-1} \mathbf{u} \mathbf{v}_h \, dx - \sum_{K \in \mathcal{T}_h} \int_{\partial K} p \mathbf{v}_h \cdot \mathbf{n} \, d\sigma + \sum_{K \in \mathcal{T}_h} \int_K \nabla p \cdot \mathbf{v}_h \, dx \\ &= - \sum_{K \in \mathcal{T}_h} \int_{\partial K} p \mathbf{v}_h \cdot \mathbf{n} \, d\sigma = - \sum_{K \in \mathcal{T}_h} \int_K \operatorname{div}(p \mathbf{v}_h) \, dx. \end{aligned} \quad (52)$$

Let $\mathbb{I}_h p$ be a continuous interpolation of p such that $\mathbb{I}_h p|_{f_V} \in Q_{k,k}$ and $\widehat{K}_h p|_{f_H} \in Q_{k,k} \setminus \{x^k y^k\}$. Then, by the patch condition (H2(a), H2(b)), we see

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \int_K \operatorname{div}(p \mathbf{v}_h) \, dx &= \sum_{K \in \mathcal{T}_h} \int_{\partial K} p \mathbf{v}_h \cdot \mathbf{n} \, d\sigma \\ &= \sum_{K \in \mathcal{T}_h} \int_{\partial K} (p - \mathbb{I}_h p) \mathbf{v}_h \cdot \mathbf{n} \, d\sigma \\ &= \sum_{K \in \mathcal{T}_h} \int_K \operatorname{div}[(p - \mathbb{I}_h p) \mathbf{v}_h] \, dx \\ &= \sum_{K \in \mathcal{T}_h} \int_K \nabla(p - \mathbb{I}_h p) \cdot \mathbf{v}_h \, dx \\ &\quad + \sum_{K \in \mathcal{T}_h} \int_K (p - \mathbb{I}_h p) \operatorname{div} \mathbf{v}_h \, dx \\ &= I + II. \end{aligned} \tag{53}$$

We bound I by the Cauchy–Schwarz inequality and interpolation property:

$$|I| \leq \|p - \mathbb{I}_h p\|_{L^2(\Omega)} \|\mathbf{v}_h\|_{H(\operatorname{div}, \Omega)} \leq Ch^{k+1} \|p\|_{H^{k+2}(\Omega)} \|\mathbf{v}_h\|_{H(\operatorname{div}, \Omega)}. \tag{54}$$

Similarly, we can bound II as

$$|II| \leq \|\nabla(p - \mathbb{I}_h p)\|_{L^2(\Omega)} \|\mathbf{v}_h\|_{L^2(\Omega)} \leq Ch^k \|p\|_{H^{k+2}(\Omega)} \|\mathbf{v}_h\|_{L^2(\Omega)}. \tag{55}$$

Combining inequalities (54) and (55), we have

$$|a_h(\mathbf{u}, \mathbf{v}_h) + b_h(p, \mathbf{v}_h)| \leq Ch^k \|p\|_{H^{k+2}(\Omega)} \|\mathbf{v}_h\|_{H(\operatorname{div}, \Omega)}. \tag{56}$$

By dividing $\|\mathbf{v}_h\|_{H(\operatorname{div}, \Omega)}$ and taking supremum over \mathbf{v}_h , we obtain (A.1). \square

5. Numerical Results

In this section, we report some results of the numerical experiment using our new element for $k = 1$. We remark that the optimal convergence rate expected with respective Theorem 4 is

$$\|\mathbf{u} - \mathbf{u}_h\|_{H(\operatorname{div}, \Omega)} + \|p - p_h\|_{L^2(\Omega)} \leq Ch^2, \tag{57}$$

when $p \in H^3(\Omega)$ and $\mathbf{u} \in H^2(\Omega)$. We solve problem (2) on the unit cube $\Omega = [0, 1]^3$. The numerical experiments are carried out on uniform triangulation \mathcal{T}_h by cubes having size $h = 2^{-m}$, $m = 1, 2, \dots, 5$. We consider two examples where the coefficients κ are discontinuous across some interfaces. In each example, the pressure (respectively, velocity) variable belongs to H^3 (respectively, H^2) on each subregion, and we report L^2 -errors for the pressure variable and $H(\operatorname{div})$ -error for the velocity variable in Tables 1 and 2. We see that we obtain optimal convergence for both examples.

Example 1. We choose $\kappa = 1$ for $0 < z < (1/2)$ and $\kappa = 100$ for $(1/2) \leq z < 1$. The function

TABLE 1: L^2 -error for pressure and $H(\operatorname{div})$ -error of velocity (Example 1).

Level	$\ p - p_h\ _{L^2}$	Order	$\ \mathbf{u} - \mathbf{u}_h\ _{H(\operatorname{div})}$	Order
1	3.652E-04		1.649E-02	
2	1.043E-04	1.807	4.626E-03	1.834
3	2.681E-05	1.960	1.188E-03	1.961
4	6.746E-06	1.991	2.990E-04	1.990
5	1.689E-06	1.998	7.488E-05	1.998

TABLE 2: L^2 -error for pressure and $H(\operatorname{div})$ -error of velocity (Example 2).

Level	$\ p - p_h\ _{L^2}$	Order	$\ \mathbf{u} - \mathbf{u}_h\ _{H(\operatorname{div})}$	Order
1	1.324E-05		2.608E-03	
2	5.113E-06	1.373	8.617E-04	1.597
3	1.469E-06	1.799	2.327E-04	1.889
4	3.802E-07	1.950	5.932E-05	1.972
5	9.587E-08	1.988	1.490E-05	1.993

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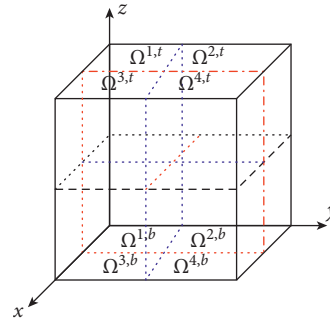


FIGURE 2: Subregions $\Omega^{i,b}$ and $\Omega^{i,t}$, $i = 1, 2, 3, 4$.

$$p(x, y, z) = \frac{1}{\kappa} x(1-x)y(1-y)z(z-0.5)(1-z) \tag{58}$$

is chosen as the exact solution. We report L^2 -errors for the pressure variable and $H(\operatorname{div})$ -error for the velocity variable in Table 1. We observe optimal convergence in both variables.

Example 2. We divide $\Omega = [0, 1]^3$ by eight octant $\Omega^{i,s}$, $i = 1, 2, 3, 4$ and $s = t, b$ (see Figure 2). Here, $\Omega^{i,b}$ are the four bottom regions ($z \leq 0.5$) and $\Omega^{i,t}$ are four top regions ($z > 0.5$).

The coefficient κ is given by

$$\kappa = \begin{cases} 1, & \text{for } x \in \Omega^{1,b}, \\ 10, & \text{for } x \in \Omega^{2,b}, \\ 100, & \text{for } x \in \Omega^{3,b}, \\ 1000, & \text{for } x \in \Omega^{4,b}, \\ 10, & \text{for } x \in \Omega^{1,t}, \\ 100, & \text{for } x \in \Omega^{2,t}, \\ 1000, & \text{for } x \in \Omega^{3,t}, \\ 1, & \text{for } x \in \Omega^{4,t}, \end{cases} \tag{59}$$

and $p = (1/\kappa)Lv$, where $L = (x - 0.5)(y - 0.5)(z - 0.5)$ and $v = x(1 - x)y(1 - y)z(1 - z)$. We report the errors in Table 2. Again, we observe optimal convergence in both variables.

6. Conclusion

MFEMs to solve for elliptic equations have been actively studied because approximating the Darcy velocity is as significant as approximating the displacement variable. Most of the MFEMs proposed so far were based on the conforming method in the sense that the space of all vector functions is contained in $H(\text{div})$. On the contrary, the nonconforming approach-based MFEMs were barely studied. In this paper, we suggest a new patch condition under which the nonconforming MFEM has optimal convergence and provides a framework for the convergence. Also, we introduce a new family of nonconforming MFE space which satisfies this condition and provide numerical experiments supporting our analysis.

Data Availability

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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