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Research Article

Some Common Fixed Point Theorems in Partially Ordered Sets

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The purpose of this paper is to prove some new fixed point theorem and common fixed point theorems of a commuting family of order-preserving mappings defined on an ordered set, which unify and generalize some relevant fixed point theorems.

1. Introduction and Preliminaries

The fixed points theory has experienced a great development in recent years, among enormous results which have enriched this theory, Tarski's theorem on ordered sets is "any application which preserves order on a complete lattice admits a fixed point and the set of its fixed points is a complete lattice."

In our previous work, we have introduced the concept of complete T-lattice to show that a complete lattice is a special case of a complete T-lattice. So, the reader may wonder if we can get a generalization of Tarski's theorem.

In one of our articles [1, 2], we proved that any application that preserves the order on a complete T-lattice has a fixed point. In this work, we give a structure of the set of these fixed points and some other results.

So, we begin by describing the relevant notation and terminology. Let $(E, \leq)$ be a partially ordered set and $M \subset E$ a nonempty subset. Recall that an upper (resp. lower) bound for M is an element $p \in E$ with $m \leq p$ (resp. $p \leq m$) for each $m \in M$; the least upper (resp. greatest-lower) bound of M will be denoted sup M (resp. inf M).

Next, let $(E, \leq)$ be a partially ordered set with the least element 0 and greatest element 1 and T be a given operator on $(E, \leq)$ reversing the order such that $x \leq Tx$ or $Tx \leq x$ for all $x \in E$. We consider the following subsets K_r and K_l of E, $K_r = \{x \in E, Tx \leq x\}$ and $K_l = \{x \in E, x \leq Tx\}$, so K_r and K_l are not empty (since $T1 \leq 1$ and $T1 \leq T^2 1$). In what follows, let us say that the set $(E, \leq)$ a complete T-lattice if every subset A of E admits the greatest lower (resp. least upper) bound as soon as $TA \prec A$ (resp. $A \prec TA$), where TA designates the image of A under the map T for $A \subset E$, and we denote by $A \prec B$ if for every $a \in A$, $b \in B$, we have $a \leq b$ for A and B be two subsets of E.

2. Complete Sub-T-Lattice and Fixed Point Theorems

2.1. Complete Sub-T-Lattice. Inspired by the success of the complete lattice subset concept introduced by Birkhoff [3] in 1940, we propose a similar concept in the partially ordered sets. The aim of the concept is to structure the set of fixed points of an increasing application on an ordered set.

Let $(E, \leq)$ be a partially ordered set and T be a given operator on $(E, \leq)$ reversing the order such that $x \leq Tx$ or $Tx \leq x$ for all $x \in E$.

Let us denote by $\leq_A$ the restriction of the ordered relation of E on A with A a subset of E.

Definition 1. We say that $(A, \leq_A)$ is a complete sub-T-lattice of E if $(A, \leq_A)$ admits a least and a greatest element and every party X of $A \inf_A X (\text{resp.} \sup_A X)$ exists as soon as $TX \prec X$ (resp. $X \prec TX$):

(1) Every finite subset A of the set $\mathbb{N}$ ordered by the usual order relation is a sub-T-lattice with $T(a) = 0_A$ for all $a \in A$ with 0_A is the least element of A

(2) We consider Figure 1: We take $E = \{a_0, a_1, a_2, a_3, a_4, a_5 ; b_0, b_1, b_2, b_3\}$and the following order relation $\leq$ defined by:$a_5 \leq a_i \leq a_0$for all$i \in \{1, 2, 3, 4\}$,$a_i \leq a_1$, and$a_i \leq a_2$for$i = 3, 4$. Moreover, we have $b_3 \leq b_2 \leq b_1 \leq b_0$ and $a_3 \leq b_2 \leq a_1 \leq b_0$ where $T(a_i) = a_5$ for $i = 0, 2$; $T(a_i) = b_2$ for $i = 1, 3$; and $T(a_i) = a_0$ for $i = 4, 5$. $T(b_i) = b_2$ for $i = 0, 1, 2$, $T(b_3) = b_0$ Let $A = \{a_0, a_1, a_4, a_5, a_5\}$, so $(A, \leq_A)$ is a complete sub-T-lattice

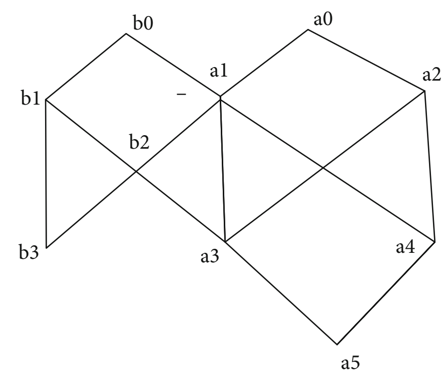

Figure 1: Example of a complete sub-T-lattice.

Recall. Let us note that $\inf_B X$ (resp. $\sup_B X$) is equal to the greatest-lower (resp. least-upper) bound of X in B for every party X of B.

Proposition 2. *Let $(A, \leq_A)$ be a complete sub-T-lattice with 0_A the least element and 1_A the greatest element. Then, for every party $X \subset K_r \cap A$ (resp. $X \subset K_l \cap A$) $sup_A X$ (resp. $inf_A X$) exists in K_r(resp. in K_l).*

Proof. Let X be a nonempty party of $K_r \cap A$. Let us define the set $M = \{y \in A : x \leq y \quad \text{for all} \quad x \in X\}$. We have $M \neq \emptyset$, since $1_A \in M$ because $K_r \cap A \neq \emptyset$ in this case $1_A \in K_r$, and for every $m_0, m_1 \in M$, we have $Tm_0 \leq Tx \leq x \leq m_0$; thus, $Tm_0 \leq Tx \leq x \leq m_1$ for all $x \in X$ which shows that $TM \prec M$. Consequently, $m_g = \inf_A M$ exists since $(A, \leq_A)$ is a complete sub-T-lattice. As we have $x \leq m_g$ for all $x \in X$, therefore $Tm_g \leq Tx \leq x \leq m_g$ which implies $m_g \in K_r$. If X was a party of $K_l \cap A$, the proof of existence of $\inf_E X$ in K_l would be symmetric.

By this proposition, we can easily prove that each chain $C = (c_i)$ of a complete sub-T-lattice has the least upper bound and the greatest lower bound; indeed, we assume that $C = (c_i)$ is an increasing chain if $C \cap K_r \neq \emptyset$; then, the sup $C \cap K_r =$ sup C exists in K_r by Proposition 2, but if $C \cap K_r = \emptyset$, we have $c_i \leq c_j \leq Tc_j$ for all $j \geq i$ which shows that $C \prec TC$ and since $C = (c_i)$ is a chain of a complete sub-T-lattice then sup C exists. If C is a decreasing sequence, the proof of existence of inf C is symmetric.

Recall that a chain M of E is maximal if there is an element a of E comparable to every element of M then $a \in M$.

In the following, we define $(\leftarrow, m] = \{x \in E, x \leq m\}$ and the $[m, \rightarrow) = \{x \in E, m \leq x\}$ for any $m \in E$ called intervals.

If $(E, \leq)$ is a complete T-lattice; then, $(E, \leq)$ is a complete sub-T-lattice, and we can show that each interval of E is a complete sub-T-lattice, indeed. We have for all $x \in [m, \rightarrow)$, $m \leq x \leq 1_E$ which implies that m is the least element and 1_E is the greatest element of $[m, \rightarrow)$.

Let $B \subset [m, \rightarrow)$ be a nonempty set such that $B \prec TB$. Since E is a complete lattice, so $b_0 = \sup_E B$ exists in E. We will only prove that $\sup_E B$ exists in $[m, \rightarrow)$. Then, we have for all $b \in B$, $m \leq b \leq b_0$ which gives $b_0 \in [m, \rightarrow)$. Consequently, $\sup_E B \in [m, \rightarrow)$. The proof for the existence of the greatest-lower if $TB \prec B$ follows identically.

In the same fashion, we can prove that $[a, b] = \{x \in X : a \leq x \leq b\}$ is a complete sub-T-lattice.

Proposition 3. *In a complete sub-T-lattice, any decreasing family of nonempty intervals has a nonempty intersection and it is an interval.*

Proof. Let $(A, \leq_A)$ be a complete sub-T-lattice and $([a_i, b_i])_{i \in I}$ a decreasing family of nonempty intervals in E, so we have $a_i \leq a_{i+1}$ and $b_{i+1} \leq b_i$. Consequently, $(a_i)_{i \in I}$ is an increasing sequence, and $(b_i)_{i \in I}$ is a decreasing sequence; furthermore, we have $a_i \leq b_j$ for all $i, j \in I$; indeed, $a_i \leq a_j \leq b_j$ if $j \geq i$ and $a_i \leq b_i \leq b_j$ if $i \geq j$. Therefore, since A is a complete T-lattice which shows that $\mathrm{Sup}_{i \in I} a_i = a$ and $\inf_{i \in I} b_i = b$ exist in A.

For all $i, j \in I$, we have $a_i \leq a \leq b \leq b_j$. Consequently, $\bigcap_{i \in I} [a_i, b_i] = [a, b]$.

2.2. Fixed Point Theorems. The theory of fixed points is concerned with the conditions which guarantee that a map$F : E \rightarrow E$ of a set E into itself admits one or more fixed points, that there are points$x \in E$for which$F(x) = x$.

Now, let $(E, \leq)$ be an ordered set and T be a given operator on $(E, \leq)$ reversing the order such that $x \leq Tx$ or $Tx \leq x$ for all $x \in E$.

We take a part A of E and a monotone maps $f : A \rightarrow A$; we note the set of fixed points of f by $\delta(f)$.

Theorem 4. *If $(A, \leq_A)$ is a complete sub-T-lattice with the least element 0_A and the greatest element 1_A, then, $\delta(f)$ is a nonempty complete sub-T-lattice.*

Indeed, we have $0_A \leq f(0_A)$; this shows that $\Delta = \{x \in A : x \leq f(x)\}$ is a nonempty set. Furthermore, we also have for any $x \in \Delta, f(x) \in \Delta$ and for any chain C of Δ, we put $s = \sup_A C$; for all $x \in C$, we have $x \leq s$ apply the mapping f, $x \leq f(x) \leq f(s)$ that gives $s \leq f(s)$ so $s \in \Delta$. Therefore, Δ satisfies the assumptions of Zorn's lemma. Hence, for every $x \in \Delta$, there exists a maximal element $x_m \in \Delta$ such that $x \leq x_m$. We claim that if x_m is a maximal element of Δ, then $x_m \leq f(x_m)$ which gives us $f(x_m) \leq f^2(x_m)$ which implies that $f(x_m) \in \Delta$. According to the maximality of x_m, we have $x_m = f(x_m)$ which proves that $\delta(f) \neq \emptyset$.

To show that $\delta(f)$ is a complete sub-T-lattice, let $S = \{x \in A : x \leq f(x) \leq x^* \quad \text{for all} \quad x^* \in \delta(f)\}$. It is clear that $S \neq \emptyset$ since $0_A \in S$. For any chain $(x_n)_n$ of S, $\sup_A(x_n)_n = c$ exists in A, and we have $x_n \leq x^*$ for all $x^* \in \delta$ which implies $x_n \leq c \leq x^*$. We apply f and we obtain $x_n \leq f(x_n) \leq f(c) \leq f(x^*) = x^*$; hence, $c \leq f(c) \leq x^*$ that proves $c \in S$. Consequently, we have the existence of a maximal element of S according to Zorn's lemma. We claim that if x_m is a maximal element of S, then $x_m \leq f(x_m) \leq x^*$ for all $x^* \in \delta$. In the same fashion, we apply f that gives $x_m \leq f(x_m) \leq f^2(x_m) \leq x^*$ for all $x^* \in \delta$ that implies $f(x_m) \in S$. It is easy from here to see that in fact we have $x_m = f(x_m)$ by the maximality of x_m and $x_m = f(x_m) \leq x^*$ for all $x^* \in \delta$ that shows that x_m is the least element of $\delta(f)$. The proof of existence of the greatest element of $\delta(f)$ is symmetrical.

Now, let $B \subset \delta(f)$ be a nonempty set such that $TB \prec B$. We want to find $\inf_{\delta(f)} B$, the greatest-lower bound of B in $\delta(f)$. Let us put $e = \inf_A B$. For all $b \in B$, $e \leq b$, as $f(e) \leq b = f(b)$. Thus $f(e) \leq e$. Therefore, the decreasing sequence $(f^n(e))_n$ tends to a fixed point of f which will be the greatest lower bounds (or the greatest-lower bound) of B in $\delta(f)$. The construction of $\sup_{\delta(f)} B$ is the least upper bound of B in $\delta(f)$ if $B \prec TB$ is symmetric. Thereafter, $(\delta(f), \leq)$ is a complete sub-T-lattice.

3. Common Fixed Points for Commuting Family of Order-Preserving Mappings

3.1. Main Results. In this section, we give a generalization of Tarski's theorem [4] that implies that any finite commuting family of order-preserving mappings has a common fixed point.

The existence of a common fixed point for a finite family of order-preserving applications is related to the intersection of a finite decreasing sequence of complete sub-T-lattices.

Theorem 5. *Let $(E, \leq)$ be a complete T-lattice. Then, any finite commuting family of order-preserving mappings (monotone mappings) $(T_i)_{i\in I}$, $T_i : E \to E$, has a common fixed point. Moreover, if we denote by Fix$((T_i))$ the set of the common fixed points, then Fix$((T_i))$ is a complete sub-T-lattice of E.*

Proof. We take $I = \{0, 2.., n\}$, where $n \in \mathbb{N}$. Let $(E, \leq)$ be a complete T-lattice so the set $\delta_0 = \delta(T_0)$ of fixed points of T_0 is a complete sub-T-lattice by Theorem 4. As $T_1 \circ T_0 = T_0 \circ T_1$, we have $T_0(T_1(x)) = T_1(T_0(x)) = T_1(T_0(x)) = T_1(x)$ that implies $T_1(x) \in \delta(T_0)$ for all $x \in \delta(T_0)$. Therefore we can restrict the maps T_1 to δ_0, and since δ_0 is a complete sub-T-lattice, the set $\delta_1 = \mathrm{Fix}(T_1/\delta_0) = \{x \in \delta(T_0), T_1(x) = x\}$ of fixed points of T_1 in $\delta_0 = \delta(T_0)$ is a complete sub-T-lattice. It is easy to see that the family $(\delta_i)_{i\in I}$ such that $\delta_i = \mathrm{Fix}(T_i/\delta_{i-1}) = \{x \in \delta_{i-1}, T_i(x) = x\}$ for all $i \in I$ with $\delta_0 = \delta(T_0)$ is a decreasing sequence.

Now, we consider the map

$$F : \delta_0 \times \delta_1 \times \cdots \times \delta_n \to \delta_0 \times \delta_1 \times \cdots \times \delta_n, \quad (x_0, x_1, .., x_n) \to (x_1, x_2, .., x_n, x_n). \tag{1}$$

As the map F is monotone and the set $\delta_0 \times \delta_1 \times \cdots \times \delta_n$ is a complete T-lattice, then there exists $(x_0, x_1, .., x_n) \in \delta_0 \times \delta_1 \times \cdots \times \delta_n$ such that $F(x_0, x_1, .., x_n) = (x_0, x_1, .., x_n)$ that means that $(x_0, x_1, .., x_n) = (x_1, x_2, .., x_n, x_n)$ that gives $x_0 = x_1 = \cdots = x_n$ and $x_0 \in Fix((T_i))$. Consequently, $\mathrm{Fix}((T_i)) \neq \emptyset$.

It remains therefore to show that $\mathrm{Fix}((T_i))$ is a complete sub-T-lattice. For this, it is easy to see that $\mathrm{Fix}((T_i)) = \delta_n$ which shows that $\mathrm{Fix}((T_i))$ is a complete T-lattice.

3.2. Common Fixed Points. In the following, we investigate the existence of a common fixed point of a commuting family of order-preserving mappings defined on a complete sub-T-lattice. The proof of our result follows the ideas of Baillon [5] developed in hyperconvex metric spaces and Abu-Sbeih and Khamsi [6] on the partially ordered sets. It is astonishing that we will develop their ideas in the case of the complete sub-T-trellis despite the difficulty of the demonstration which comes from the fact that in complete sub-T-trellis, we do not always have the existence of sup and inf of its parts.

We have the following result in partially ordered sets.

Let $(E, \leq)$ be a partially ordered set and T be a given operator on $(E, \leq)$ reversing the order such that $x \leq Tx$ or $Tx \leq x$ for all $x \in E$. Then,

Theorem 6. *For any decreasing family of nonempty complete sub-T-lattice subsets $(X_\beta)_{\beta\in\Gamma}$ of E, where Γ is a directed index set, we have $\bigcap_{\beta\in\Gamma} X_\beta$ is not empty and it is a complete sub-T-lattice.*

Proof. Consider the family

$$\mathscr{P} = \left\{ (A_\beta)_{\beta\in\Gamma} ; A_\beta \text{ is a nonempty interval in } X_\beta \text{ and } \cdot (A_\beta)_{\beta\in\Gamma} \text{ is a sequence decreasing} \right\}. \tag{2}$$

The set $\mathscr{P}$ is not empty since $(X_\beta)_{\beta\in\Gamma} \in \mathscr{P}$. If we order $\mathscr{P}$ by the inclusion relation, every chain of $\mathscr{P}$ has a lower bound since in a complete sub-T-lattice, any decreasing family of nonempty intervals has a nonempty intersection and it is an interval by Proposition 3. Therefore, $\mathscr{P}$ satisfies the assumptions of Zorn's lemma. Hence, for every $D \in \mathscr{P}$, there exists a minimal element $A \in \mathscr{P}$ such that $A \subset D$. We claim that if $(A_\beta)_{\beta\in\Gamma}$ is minimal, then each A_β is a singleton. Indeed, let us fix $\beta_0 \in \Gamma$. We know that $A_{\beta_0} = [m_{\beta_0}, M_{\beta_0}]$. We consider the new family

$$B_\beta = \begin{cases} A_\beta \text{ if } \beta \geq \beta_0 \text{ or } \beta \text{ not comparable to } \beta_0, \\ \left\{ x \in X_\beta, m_{\beta_0} \leq x \leq M_{\beta_0} \right\} \quad \text{if} \, \beta_0 \geq \beta \end{cases}. \tag{3}$$

Our assumptions on (X_β) and (A_β) imply that $(B_\beta) \in \mathscr{P}$. Moreover, we have $B_\beta \subset A_\beta$ for any $\beta \in \Gamma$. Since $(A_\beta)_{\beta\in\Gamma}$ is minimal, we get $B_\beta = A_\beta$ for any $\beta \in \Gamma$. In particular, we have

$$A_\beta = \left\{ x \in X_\beta \, ; m_{\beta_0} \le x \le M_{\beta_0} \right\} \text{ for } \beta \le \beta_0. \tag{4}$$

If $A_\beta = [m_\beta, M_\beta]$, then we must have $m_\beta = m_{\beta_0}$ and $M_\beta = M_{\beta_0}$. Therefore, we proved the existence of $m, M \in E$ such that $A_\beta = \{x \in X_\beta \, ; m \le x \le M\}$ for any $\beta \in \Gamma$.

It is easy from here to show that in fact we have $m = M$ by the minimality of $(A_\beta)_{\beta\in\Gamma}$, which proves our claim. Clearly, we have $m \in A_\beta$ for any $\beta \in \Gamma$ which implies $\varpi = \bigcap_{\beta\in\Gamma} X_\beta$ is not empty.

We will prove that ϖ is a complete sub-T-lattice. First, we start by proving the existence of the least and the greatest element of ϖ for that we consider the set:

$$\mathscr{K} = \left\{ (A_\beta)_{\beta\in\Gamma} \, ; \varpi \subset A_\beta \text{ is a interval in } X_\beta \text{ and } (A_\beta) \text{ is decreasing} \right\}. \tag{5}$$

For the same reason above, we have:

(1) $\mathscr{K}$ is not empty since $(X_\beta)_{\beta\in\Gamma} \in \mathscr{K}$

(2) $\mathscr{K}$ satisfies the assumptions of Zorn's lemma. Hence, for every $D \in \mathscr{K}$, there exists a minimal element $A \in \mathscr{K}$ such that $A \subset D$

(3) If $(A_\beta)_{\beta\in\Gamma}$ is a minimal element of $\mathscr{K}$, then $A_\beta = A_{\beta'}$ for all β, β' according to the minimality of $(A_\beta)_{\beta\in\Gamma}$ which gives us $A_\beta = [m_0, M_0] = \varpi$ for all β. Hence the result we are searching

Secondly, let $A \subset \varpi$ be nonempty such that $A \prec TA$. We will only prove that the sup A exists in ϖ. The proof for the existence of the infimum follows identically if $TA \prec A$. For any $\beta \in \Gamma$, we have $A \subset X_\beta$. Since X_β is a complete sub-T-lattice, then $a_\beta = \sup_{X_\beta} A$ exists in X_β and the family $(a_\beta)_{\beta\in\Gamma}$ is an increasing chain.

Now, we consider the set $X_\beta^* = [a_\beta, \rightarrow) \cap X_\beta$ for $\beta \in \Gamma$. Then, X_β^* is a nonempty complete sub-T-lattice of X_β. It is easy to see that the family $(X_\beta^*)_\beta$ is a decreasing chain of complete sub-T-lattice. Hence, $\bigcap_{\beta\in\Gamma} X_\beta^{å} = I$ is a not empty interval and $I \subset \varpi$. Obviously, we have $\sup_\varpi A = \inf I$ which completes the proof of Theorem 6.

As a consequence of this theorem, we obtain the following common fixed point result.

Theorem 7. *If $(E, \le)$ is a complete T-lattice, then any commuting family of order-preserving mappings $(T_i)_{i\in I}$, $T_i : E \rightarrow E$, has a common fixed point. Moreover, if we noted by $Fix((T_i))$ the set of the common point fixed points, then $Fix((T_i))$ is a complete sub-T-lattice of E.*

Proof. First, note that the fixed point theorem (Theorem2) implies that any finite commuting family of order-preserving mappings$T_1, T_2, .., T_n$, $T_i : E \rightarrow E$, has a common fixed point. Moreover, if we denote by$\mathrm{Fix}((T_i))$the set of the common fixed points, i.e., $Fix((Ti)) = \{x \in M \, ; T_i(x) = x \quad i = 1, .., n\}$, it is a complete sub-T-lattice. Let $\Gamma = \{\beta \, ; \beta \text{ is a finite nonempty subset of } I\}$. Clearly, Γ is downward directed (where the order on Γ is the set inclusion). For any $\beta \in \Gamma$, the set F_β of common fixed point set of the mappings T_i, $i \in \beta$, is a nonempty complete sub-T-lattice. Clearly, the family $(F_\beta)_{\beta\in\Gamma}$ is decreasing. The theorem above implies that $\bigcap_{\beta\in\Gamma} F_\beta$ is nonempty and it is a complete sub-T-lattice.

The commutativity assumption may be relaxed using a new concept discovered in [6] (see also [7, 8]). Of course, this new concept was initially defined in the metric setting; therefore, we will extend it to the case of partially ordered sets in the next work where we will arrive at a result similar to De Marr's result [9] without compactness assumption of the domain.

4. Conclusion

In this paper, we have extended some Tarski's theorems of the fixed point into ordered sets by new fixed point theorems. The original proof of fixed point for complete T-lattice [2] is beautiful and elegant but nonconstructive and somewhat uninformative. In [3], we have given a constructive proof that generalizes the Tarski's version results. In this paper, we have given a structure to the set of fixed points of an increasing application on an ordered set and we have investigated the existence of a common fixed point of a commuting family of order-preserving mappings defined on a complete sub-T-lattice. Our next work concerns the development of some fixed point theorems in hyperconvex metric spaces.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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