Supplementary Materials to

# Partial Derivatives Estimation for Underlying Functional-Valued Process in a Unified Framework

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Some technical Lemmas needed for our main results are stated and proved below.

**Lemma 1.** *Let* $Z_{iml} = X_i(S_{iml}, T_{im})$ *or* $\varepsilon_{iml}$ *for* $i = 1, \cdots, n$, $m = 1, \cdots, M_i$, $l = 1, \cdots, L_{im}$. *Suppose for some* $\lambda \in (2, \infty)$ *that*

$$E\left(\sup_{s\in\mathcal{S},t\in\mathcal{T}} |X_i(S_{iml}, T_{im})|^\lambda\right) < \infty \text{ and } E|\varepsilon|^\lambda < \infty, \tag{1}$$

*Define*

$$D_n((s_1, s_2), (t_1, t_2)) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{M_i}\sum_{m=1}^{M_i}\frac{1}{L_{im}}\sum_{l=1}^{L_{im}} Z_{iml} I(S_{iml} \in [s_1 \wedge s_2, s_1 \vee s_2]) I(T_{im} \in [t_1 \wedge t_2, t_1 \vee t_2]),$$

*and* $D((s_1, s_2), (t_1, t_2)) = \mathrm{E}[D_n((s, s+u), (t, t+v))]$. *Let* $c_n$, $c_n'$ *be any positive sequences tending to 0 and* $\beta_n = c_n c_n'(\gamma_{n21} + \gamma_{n20} c_n + \gamma_{n11} c_n' + c_n c_n')$, *then if* $\beta_n^{-1}(\log n/n)^{1-2/\lambda} = o(1)$, *we have*

$$\sup_{\substack{s\in[0,a]\\ t\in[0,b]}}\ \sup_{\substack{|u|\leqslant c_n\\ |v|\leqslant c_n'}} |D_n((s, s+u), (t, t+v)) - D((s, s+u), (t, t+v))| = O_{a.s.}(n^{-1/2}(\beta_n \log n)^{1/2}).$$

**Proof.** We can treat the positive and negative parts of $Z_{iml}$ separately, and will assume below that $Z_{iml}$ is nonnegative. Define equally-spaced grids $\mathcal{B}_n = \{u_k\}$ and $\mathcal{B}_n' = \{v_k'\}$, where $u_k = kc_n$, for $k = 0, \cdots, [\frac{a}{c_n}]$, and $u_{[\frac{a}{c_n}]} = a$, and $v_{k'} = k'c_n'$, for $k' = 0, \cdots, [\frac{b}{c_n'}]$, and $v_{[\frac{b}{c_n'}]} = b$. For any $s \in [0, a]$, $t \in [0, b]$ and $|u| \leqslant c_n$, $|v| \leqslant c_n'$, let $u_k$ be a grid point that is within $c_n$ of both $s$ and $s + u$, which exists; let $v_{k'}$ be a grid point that is within $c_n'$ of both $t$ and $t + v$, which exists. Note that

$$\begin{aligned}
& |D_n((s, s+u), (t, t+v)) - D((s, s+u), (t, t+v))| \\
\leqslant\ & |D_n((u_k, s+u), (v_{k'}, t+v)) - D((u_k, s+u), (v_{k'}, t+v))| \\
& + |D_n((u_k, s), (v_{k'}, t)) - D((u_k, s), (v_{k'}, t))| + |D_n((u_k, s+u), (v_{k'}, t)) - D((u_k, s+u), (v_{k'}, t))| \\
& + |D_n((u_k, s), (v_{k'}, t+v)) - D((u_k, s), (v_{k'}, t+v))|,
\end{aligned}$$

then we have

$$\begin{aligned}
&\sup_{s\in[0,a],t\in[0,b]} |D_n((s,s+u),(t,t+v)) - D((s,s+u),(t,t+v))| \\
\leqslant\ & 4 \sup_{s\in\mathcal{B}_n,t\in\mathcal{B}'_n} V_n(s,t,c_n,c'_n), \qquad (2)
\end{aligned}$$

where $V_n(s,t,c_n,c'_n) = \sup_{|u|\leqslant c_n,|v|\leqslant c'_n} |D_n((s,s+u),(t,t+v)) - D((s,s+u),(t,t+v))|$.
Let $\alpha_n = n^{-1/2}(\beta_n \log n)^{1/2}$ and $\rho_n = \beta_n/\alpha_n$, and define $D_n^*((s,s+u),(t,t+v))$, $D^*((s,s+u),(t,t+v))$ and $V_n^*(s,t,c_n,c'_n)$ in the same way as $D_n((s,s+u),(t,t+v))$, $D((s,s+u),(t,t+v))$ and $V_n(s,t,c_n,c'_n)$, except with $Z_{iml}I(Z_{iml}\leqslant \rho_n)$ replacing $Z_{iml}$. Then

$$\sup_{s\in\mathcal{B}_n,t\in\mathcal{B}'_n} V_n(s,t,c_n,c'_n) \leqslant \sup_{s\in\mathcal{B}_n,t\in\mathcal{B}'_n} V_n^*(s,t,c_n,c'_n) + A_{n1} + A_{n2}, \qquad (3)$$

where

$$\begin{aligned}
A_{n1} &= \sup_{\substack{s\in[0,a]\\ t\in[0,b]}} \sup_{\substack{|u|\leqslant c_n\\ |v|\leqslant c'_n}} \left|D_n((s,s+u),(t,t+v)) - D_n^*((s,s+u),(t,t+v))\right|, \\
A_{n2} &= \sup_{\substack{s\in[0,a]\\ t\in[0,b]}} \sup_{\substack{|u|\leqslant c_n\\ |v|\leqslant c'_n}} \left|D((s,s+u),(t,t+v)) - D^*((s,s+u),(t,t+v))\right|.
\end{aligned}$$

We first consider $A_{n1}$ and $A_{n2}$. For all $s,t$ and $u,v$, by Markov's inequality,

$$\begin{aligned}
&(D_n((s,s+u),(t,t+v)) - D_n^*((s,s+u),(t,t+v))) \\
\leqslant\ & \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{1}{M_i}\sum_{m=1}^{M_i}\left[\frac{1}{L_{im}}\sum_{l=1}^{L_{im}} Z_{iml}I(Z_{iml}>\rho_n)\right]\right\} \\
\leqslant\ & \frac{\rho_n^{1-\lambda}}{n}\sum_{i=1}^{n}\left\{\frac{1}{M_i}\sum_{m=1}^{M_i}\left[\frac{1}{L_{im}}\sum_{l=1}^{L_{im}} Z_{iml}^{\lambda}I(Z_{iml}>\rho_n)\right]\right\}.
\end{aligned}$$

Consider the case $Z_{iml} = X_i(S_{iml},T_{im})$, the other case being simpler. It follows that

$$\frac{1}{M_i}\sum_{m=1}^{M_i}\left[\frac{1}{L_{im}}\sum_{l=1}^{L_{im}} Z_{iml}^{\lambda}I(Z_{iml}>\rho_n)\right] \leqslant \tilde{Z}_i, \qquad \text{where } \tilde{Z}_i = \sup_{\substack{s\in[0,a]\\ t\in[0,b]}} |X_i(S_{iml},T_{im})|^{\lambda},$$

then we have

$$\begin{aligned}
&\alpha_n^{-1}\left(D_n((s,s+u),(t,t+v)) - D_n^*((s,s+u),(t,t+v))\right) \\
\leqslant\ & \alpha_n^{-1}\rho_n^{1-\lambda}\frac{1}{n}\sum_{i=1}^{n}\tilde{Z}_i.
\end{aligned}$$

By the SLLN, $n^{-1}\sum_{i=1}^n \tilde{Z}_i \xrightarrow{a.s.} \mathrm{E}(\sup_{\substack{s\in[0,a]\\ t\in[0,b]}} |X_i(S_{iml},T_{im})|^\lambda) < \infty$. Since $\alpha_n^{-1}\rho_n^{1-\lambda} = o(1)$, it follows that

$$\alpha_n^{-1}A_{n1} \xrightarrow{a.s.} 0, \quad \text{and} \quad \alpha_n^{-1}A_{n2} = 0,$$

which yields to that

$$A_{n1} + A_{n2} = o_{a.s.}(\alpha_n). \tag{4}$$

We next focus on $\sup_{s\in\mathcal{B}_n, t\in\mathcal{B}'_n} V_n^*(s,t,c_n,c'_n)$. For fixed $s\in\mathcal{B}_n$ and $t\in\mathcal{B}'_n$, we perform a further partition. Define $\omega_n = [\rho^{1/2}c_n/\alpha_n^{1/2}+1]$ and $\omega'_n = [\rho^{1/2}c'_n/\alpha_n^{1/2}+1]$, and $u_r = rc_n/\omega_n$ and $v_{r'} = r'c'_n/\omega'_n$, for $r = -\omega_n, -\omega_n+1, \cdots, \omega_n$, and $r' = -\omega'_n, -\omega'_n+1, \cdots, \omega'_n$. Note that $D_n^*((s,s+u),(t,t+v))$ is monotone in both $|u|$ and $|v|$ since $Z_{iml}\geqslant 0$. Suppose that $0\leqslant u_r \leqslant u \leqslant u_{r+1}$, and $0\leqslant v_{r'}\leqslant v\leqslant v_{r'+1}$, then

$$D_n^*((s,s+u_r),(t,t+v_{r'})) \leqslant D_n^*((s,s+u),(t,t+v)) \leqslant D_n^*((s,s+u_{r+1}),(t,t+v_{r'+1})),$$

and

$$D^*((s,s+u_r),(t,t+v_{r'})) \leqslant D^*((s,s+u),(t,t+v)) \leqslant D^*((s,s+u_{r+1}),(t,t+v_{r'+1})).$$

Thus we have

$$\begin{aligned}
& D_n^*((s,s+u_r),(t,t+v_{r'})) - D^*((s,s+u_r),(t,t+v_{r'})) \\
& \qquad + D^*((s,s+u_r),(t,t+v_{r'})) - D^*((s,s+u_{r+1}),(t,t+v_{r'+1})) \\
\leqslant\ & D_n^*((s,s+u),(t,t+v)) - D^*((s,s+u),(t,t+v)) \\
\leqslant\ & D_n^*((s,s+u_{r+1}),(t,t+v_{r'+1})) - D^*((s,s+u_{r+1}),(t,t+v_{r'+1})) \\
& \qquad + D^*((s,s+u_{r+1}),(t,t+v_{r'+1})) - D^*((s,s+u_r),(t,t+v_{r'})),
\end{aligned}$$

from which we obtain that

$$\begin{aligned}
& |D_n^*((s,s+u),(t,t+v)) - D^*((s,s+u),(t,t+v))| \\
\leqslant\ & \max(\kappa_{nrr'}(s,t), \kappa_{n(r+1)(r'+1)}(s,t)) + D^*((s+u_r,s+u_{r+1}),(t+v_{r'},t+v_{r'+1})),
\end{aligned}$$

where $\kappa_{nrr'}(s,t) = |D_n^*((s,s+u_r),(t,t+v_{r'})) - D^*((s,s+u_r),(t,t+v_{r'}))|$. The same hold if $0\leqslant u_r\leqslant u\leqslant u_{r+1}$, $v_{r'}\leqslant v\leqslant v_{r'+1}\leqslant 0$ or $u_r\leqslant u\leqslant u_{r+1}\leqslant 0$, $0\leqslant v_{r'}\leqslant v\leqslant v_{r'+1}$ or

$u_r \leqslant u \leqslant u_{r+1} \leqslant 0$, $v_{r'} \leqslant v \leqslant v_{r'+1} \leqslant 0$. Thus,

$$|V_n^*(s,t,c_n,c_n')| \leqslant \max_{\substack{-\omega_n \leqslant r \leqslant \omega_n \\ -\omega_n' \leqslant r' \leqslant \omega_n'}} \kappa_{nrr'}(s,t) + \max_{\substack{-\omega_n \leqslant r \leqslant \omega_n \\ -\omega_n' \leqslant r' \leqslant \omega_n'}} D^*((s+u_r, s+u_{r+1}),(t+v_{r'}, t+v_{r'+1})).$$

For all $r$, $r'$

$$\begin{aligned}
& D^*((s+u_r, s+u_{r+1}),(t+v_{r'}, t+v_{r'+1})) \\
\leqslant\ & \rho_n \Pr\{s+u_r \leqslant S \leqslant s+u_{r+1}, t+v_{r'} \leqslant T \leqslant t+v_{r'+1}\} \\
\leqslant\ & \rho_n \Pr\{s+u_r \leqslant S \leqslant s+u_{r+1}\} \Pr\{t+v_{r'} \leqslant T \leqslant t+v_{r'+1}\} \\
\leqslant\ & M_S M_T \rho_n (\frac{c_n}{\omega_n})(\frac{c_n'}{\omega_n'}) \\
\leqslant\ & M_S M_T \rho_n \alpha_n.
\end{aligned}$$

Hence, for any $B > 0$,

$$\Pr\{V_n^*(s,t,c_n,c_n') \geqslant B\alpha_n\} \leqslant \Pr\{\max_{\substack{-\omega_n \leqslant r \leqslant \omega_n \\ -\omega_n' \leqslant r' \leqslant \omega_n'}} \kappa_{nrr'}(s,t) \geqslant (B - M_S M_T)\alpha_n\}. \tag{5}$$

Suppose $u_r \geqslant 0$, and $v_{r'} \geqslant 0$(other situations hold in the same way), let

$$W_{im}(s,t) = \frac{1}{L_{im}} \sum_{l=1}^{L_{im}} Z_{iml} I(Z_{iml} \leqslant \rho_n) I(S_{iml} \in (s, s+u_r]),$$

and

$$\tilde{W}_i(s,t) = \frac{1}{M_i} \sum_{m=1}^{M_i} W_{im} I(T_{im} \in (t, t+v_{r'}]),$$

so that $\kappa_{nrr'}(s,t) = \frac{1}{n} |\sum_{i=1}^n \{\tilde{W}_i(s,t) - \mathrm{E}(\tilde{W}_i(s,t))\}|$. Obviously, $|\tilde{W}_i(s,t) - \mathrm{E}(\tilde{W}_i(s,t))| \leqslant \rho_n$, we next focus on the variance of $\tilde{W}_i(s,t)$. Note that

$$\begin{aligned}
\mathrm{E}\tilde{W}_i^2(s,t) &= \frac{1}{M_i^2} \Bigg\{ \sum_{m=1}^{M_i} \mathrm{E}\Big[W_{im}^2 I(T_{im} \in (t, t+v_{r'}])\Big] \\
&\quad + \sum_{m \neq m'}^{M_i} \mathrm{E}\Big[W_{im} W_{im'} I(T_{im} \in (t, t+v_{r'}]) I(T_{im'} \in (t, t+v_{r'}])\Big] \Bigg\},
\end{aligned} \tag{6}$$

we consider $\mathrm{E}[W_{im}^2 I(T_{im} \in (t, t+v_{r'}])]$ and $\mathrm{E}[W_{im} W_{im'} I(T_{im} \in (t, t+v_{r'}]) I(T_{im'} \in (t, t+v_{r'}])]$ separately. Since,

$$\begin{aligned}
& \mathrm{E}\left[Z_{iml}^2 I(Z_{iml} \leqslant \rho_n) I(S_{iml} \in (s, s+u_r]) | T_{im}\right] \\
\leqslant\ & \mathrm{E}\left\{\mathrm{E}\left[Z_{iml}^2 | S_{iml}, T_{im}\right] I(S_{iml} \in (s, s+u_r]) | T_{im}\right\} \\
\leqslant\ & C \Pr\{s < S_{iml} \leqslant s+u_r | T_{im}\},
\end{aligned}$$

and for $l \neq l'$,

$$
\begin{aligned}
& \mathrm{E}\left[Z_{iml} Z_{iml'} I(Z_{iml} \leqslant \rho_n) I(Z_{iml'} \leqslant \rho_n) I(S_{iml} \in (s, s+u_r]) I(S_{iml'} \in (s, s+u_r]) | T_{im}\right] \\
\leqslant\ & \mathrm{E}\left\{\mathrm{E}\left[Z_{iml} Z_{iml'} | S_{iml}, S_{iml'}, T_{im}\right] I(S_{iml} \in (s, s+u_r]) I(S_{iml'} \in (s, s+u_r]) | T_{im}\right\} \\
\leqslant\ & C \Pr\{s < S_{iml} \leqslant s+u_r, s < S_{iml'} \leqslant s+u_r | T_{im}\},
\end{aligned}
$$

we have

$$
\begin{aligned}
& \mathrm{E}\left[W_{im}^2 | T_{im}\right] \\
=\ & \frac{1}{L_{im}^2}\Bigg\{\sum_{l=1}^{L_{im}} \mathrm{E}\left[Z_{iml}^2 I(Z_{iml} \leqslant \rho_n) I(S_{iml} \in (s, s+u_r]) | T_{im}\right] \\
& + \sum_{l \neq l'}^{L_{im}} \mathrm{E}\left[Z_{iml} Z_{iml'} I(Z_{iml} \leqslant \rho_n) I(Z_{iml'} \leqslant \rho_n) I(S_{iml} \in (s, s+u_r]) I(S_{iml'} \in (s, s+u_r]) | T_{im}\right]\Bigg\} \\
\leqslant\ & \frac{C}{L_{im}^2}\Bigg(\sum_{l=1}^{L_{im}} \Pr\{s < S_{iml} \leqslant s+u_r | T_{im}\} \\
& \qquad + \sum_{l \neq l'}^{L_{im}} \Pr\{s < S_{iml} \leqslant s+u_r, s < S_{iml'} \leqslant s+u_r | T_{im}\}\Bigg). \qquad (7)
\end{aligned}
$$

It follows from Condition (6) that

$$
\begin{aligned}
& \mathrm{E}\left[W_{im}^2 I(T_{im} \in (t, t+v_{r'}])\right] \\
=\ & \mathrm{E}\Big\{\mathrm{E}\left[W_{im}^2 | T_{im}\right] I(T_{im} \in (t, t+v_{r'}])\Big\} \\
\leqslant\ & C L_{im}^{-2}\Bigg(\sum_{l=1}^{L_{im}} \Pr\{s < S_{iml} \leqslant s+u_r, t < T_{im} \leqslant t+v_{r'}\} \\
& \quad + \sum_{l \neq l'}^{L_{im}} \Pr\{s < S_{iml} \leqslant s+u_r, s < S_{iml'} \leqslant s+u_r, t < T_{im} \leqslant t+v_{r'}\}\Bigg) \\
\leqslant\ & C L_{im}^{-2} \Pr\{t < T_{im} \leqslant t+v_{r'}\}\Bigg(\sum_{l=1}^{L_{im}} \Pr\{s < S_{iml} \leqslant s+u_r\} \\
& \qquad + \sum_{l \neq l'}^{L_{im}} \Pr\{s < S_{iml} \leqslant s+u_r\} \Pr\{s < S_{iml'} \leqslant s+u_r\}\Bigg) \\
\leqslant\ & C M_T v_{r'} L_{im}^{-2}(L_{im} M_S u_r + L_{im}^* M_S^2 u_r^2) \\
=\ & C M_T M_S v_{r'} u_r L_{im}^{-1}\left[1 + (L_{im} - 1) M_S u_r\right]. \qquad (8)
\end{aligned}
$$

On the other hand, by the Cauchy-Schwarz inequality,

$$
\begin{aligned}
& \mathrm{E}\left[W_{im}W_{im'}I(T_{im}\in(t,t+v_{r'}])I(T_{im'}\in(t,t+v_{r'}])\right] \\
= & \ \mathrm{E}\left\{\mathrm{E}\left[W_{im}W_{im'}|T_{im},T_{im'}\right]I(T_{im}\in(t,t+v_{r'}])I(T_{im'}\in(t,t+v_{r'}])\right\} \\
\leqslant & \ \frac{1}{2}\Big\{\mathrm{E}\left[W_{im}^2 I(T_{im}\in(t,t+v_{r'}])I(T_{im'}\in(t,t+v_{r'}])\right] \\
& \qquad + \mathrm{E}\left[W_{im'}^2 I(T_{im}\in(t,t+v_{r'}])I(T_{im'}\in(t,t+v_{r'}])\right]\Big\} \\
\leqslant & \ \frac{1}{2}CM_TM_Sv_{r'}^2u_r\left\{L_{im}^{-1}\left[1+(L_{im}-1)M_Su_r\right]+L_{im'}^{-1}\left[1+(L_{im'}-1)M_Su_r\right]\right\}. \qquad (9)
\end{aligned}
$$

By combining (8), (9) and (6), we have

$$
\mathrm{E}\tilde{W}_i^2(s,t)\le CM_SM_Tv_{r'}u_r\left[\frac{1}{M_i^2}\sum_{m=1}^{M_i}\frac{1}{L_{im}}+\frac{M_Su_r}{M_i}+\frac{v_{r'}}{M_i}\sum_{m=1}^{M_i}\frac{1}{L_{im}}+v_{r'}M_Su_r\right],
$$

from which we conclude that for some constant $C>0$,

$$
\begin{aligned}
\sum_{i=1}^n \mathrm{Var}(\tilde{W}_i(s,t)) \ &\leqslant \ \sum_{i=1}^n \mathrm{E}\tilde{W}_i^2(s,t) \\
&\leqslant \ Cc_nc_n'\sum_{i=1}^n\left[\frac{1}{M_i^2}\sum_{m=1}^{M_i}\frac{1}{L_{im}}+\frac{c_n}{M_i}+\frac{c_n'}{M_i}\sum_{m=1}^{M_i}\frac{1}{L_{im}}+c_n'c_n\right] \\
&= \ nC\beta_n. \qquad (10)
\end{aligned}
$$

Then, by Bernstein's inequality,

$$
\begin{aligned}
& \Pr\{\kappa_{nrr'}(s,t)\geqslant(B-M_SM_T)\alpha_n\} \\
\leqslant & \ \exp\{-\frac{(B-M_SM_T)^2\alpha_n^2n^2}{2nC\beta_n+\frac{2}{3}(B-M_SM_T)n\alpha_n\rho_n}\} \\
\leqslant & \ n^{\{-\frac{(B-M_SM_T)^2}{2C+\frac{2}{3}(B-M_SM_T)}\}}.
\end{aligned}
$$

Thus, by (5) and Boole's inequality,

$$
\begin{aligned}
& \Pr\{\sup_{\substack{s\in\mathcal{B}_n\\ t\in\mathcal{B}_n'}} V_n^*(s,t,c_n,c_n')\geqslant B\alpha_n\} \\
\leqslant & \ \left(\left[\frac{1}{c_n}\right]+1\right)\left(\left[\frac{1}{c_n'}\right]+1\right)\left(2\left[\frac{\rho_n^{1/2}c_n}{\alpha_n^{1/2}}+1\right]+1\right)\left(2\left[\frac{\rho_n^{1/2}c_n'}{\alpha_n^{1/2}}+1\right]+1\right)n^{-C} \\
\leqslant & \ C^*\rho_n^{-1}\alpha_nn^{\{-\frac{(B-M_SM_T)^2}{2C+\frac{2}{3}(B-M_SM_T)}\}} \qquad (11)
\end{aligned}
$$

for some finite constant $C^* > 0$. Note that $\rho_n^{-1}\alpha_n = n/\log n$, and hence $C^*\rho_n^{-1}\alpha_n n^{\{-\frac{(B-M_S M_T)^2}{2C+\frac{2}{3}(B-M_S M_T)}\}}$ is summable in $n$ if we select $B$ large enough. Thus by the Borel-Cantelli lemma,

$$\sup_{\substack{s\in\mathcal{B}_n \\ t\in\mathcal{B}'_n}} V_n^*(s,t,c_n,c'_n) = O_{a.s.}(\alpha_n),$$

which together with (2), (3) and (4) proves Lemma 1. □

**Lemma 2.** *Let $Z_{iml}$ be as in Lemma 1 and assume that (1) holds. For bandwidths $h_1$, $h_2$, and nonnegative integers $p$, $q$, let*

$$D_{npq}(s,t) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{M_i}\sum_{m=1}^{M_i}\frac{1}{L_{im}}\sum_{l=1}^{L_{im}} Z_{iml}K_{h_1}(S_{iml}-s)K_{h_2}(T_{im}-t)(\frac{S_{iml}-s}{h_1})^p(\frac{T_{im}-t}{h_2})^q.$$

*Let $\beta_n = h_1h_2(\gamma_{n21} + 2\gamma_{n20}h_1 + 2\gamma_{n11}h_2 + 4h_1h_2)$, assume that $h_1 \to 0$, $h_2 \to 0$, and $\beta_n^{-1}(\log n/n)^{1-2/\lambda} = o(1)$, then we have*

$$\sqrt{nh_1^2h_2^2/(\beta_n\log n)}\sup_{\substack{s\in[0,a] \\ t\in[0,b]}} |D_{npq}(s,t) - \mathrm{E}[D_{npq}(s,t)]| = O_{a.s.}(1). \tag{12}$$

**Proof.** Note that $K_{(p)}(s) = s^pK(s)$ is a bounded variation for nonnegative integer $p$ and any bandwidth $h$, then we can write $K_{(p)} = K_{(p)}^{(1)} - K_{(p)}^{(2)}$, where both $K_{(p)}^{(1)}$ and $K_{(p)}^{(2)}$ are increasing functions. Without loss of generality, we assume that $K_{(p)}^{(1)}(-1) = K_{(p)}^{(2)}(-1) = 0$. Let $K_{h,p}(s) = (s/h)^pK_h(s)$ , we further write

$$\begin{aligned}
D_{npq}(s,t) &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{M_i}\sum_{m=1}^{M_i}\frac{1}{L_{im}}\sum_{l=1}^{L_{im}} Z_{iml}K_{h_1,p}(S_{iml}-s)K_{h_2,q}(T_{im}-t) \\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{M_i}\sum_{m=1}^{M_i}\frac{1}{L_{im}}\sum_{l=1}^{L_{im}} Z_{iml}I(-h_1 \leqslant S_{iml}-s \leqslant h_1)I(-h_2 \leqslant T_{im}-t \leqslant h_2) \\
&\qquad \int_{-h_1}^{S_{iml}-s} dK_{h_1,p}(u)\int_{-h_2}^{T_{im}-t} dK_{h_2,q}(v) \\
&= \int_{-h_2}^{h_2}\int_{-h_1}^{h_1}\frac{1}{n}\left\{\sum_{i=1}^{n}\frac{1}{M_i}\sum_{m=1}^{M_i}\frac{1}{L_{im}}\sum_{l=1}^{L_{im}} Z_{iml}I(u \leqslant S_{iml}-s \leqslant h_1)I(v \leqslant T_{im}-t \leqslant h_2)\right\} \\
&\qquad dK_{h_1,p}(u)dK_{h_2,q}(v) \\
&= \int_{-h_2}^{h_2}\int_{-h_1}^{h_1} D_n((s+u,s+h_1),(t+v,t+h_2))dK_{h_1,p}(u)dK_{h_2,q}(v),
\end{aligned}$$

where $D_n((s+u,s+h_1),(t+v,t+h_2))$ is as defined in Lemma 1. By letting $c_n = 2h_1$ and $c'_n = 2h_2$, we apply Lemma 1 to $D_n$, which clearly satisfies assumption of Lemma 1, then we have

$$
\begin{aligned}
& \sup_{s\in[0,a],t\in[0,b]} |D_{npq}(s,t)-\mathrm{E}[D_{npq}(s,t)]| \\
\leqslant\ & \sup_{s\in[0,a],t\in[0,b]} V_n(s,t,2h_1,2h_2)\int_{-h_2}^{h_2}|dK_{h_2,q}(v)|\int_{-h_1}^{h_1}|dK_{h_1,p}(u)| \\
\leqslant\ & h_1^{-1}h_2^{-1}[K_{(p)}^{(1)}(1)+K_{(p)}^{(2)}(1)][K_{(q)}^{(1)}(1)+K_{(q)}^{(2)}(1)]\sup_{s\in[0,a],t\in[0,b]} V_n(s,t,2h_1,2h_2)
\end{aligned}
$$

By Lemma 1, we have

$$
\sup_{s\in[0,a],t\in[0,b]} V_n(s,t,2h_1,2h_2)=O_{a.s.}(n^{-1/2}(\beta_n\log n)^{1/2}), \tag{13}
$$

which yields to (12). Lemma 2 holds. □

**Lemma 3.** *Let* $Z_{imll'}=X_i(S_{iml},T_{im})X_i(S_{iml'},T_{im})$, $X_i(S_{iml},T_{im})\varepsilon_{iml'}$ *or* $\varepsilon_{iml}\varepsilon_{iml'}$ *for* $i=1,\cdots,n$, $m=1,\cdots,M_i$, $l,l'=1,\cdots,L_{im}$. *Suppose for some* $\lambda\in(2,\infty)$ *that*

$$
E\left(\sup_{s\in\mathcal{S},t\in\mathcal{T}}|X_i(S_{iml},T_{im})|^{2\lambda}\right)<\infty \text{ and } E|\varepsilon|^{2\lambda}<\infty, \tag{14}
$$

*Define*

$$
\begin{aligned}
& Q_n((s_1,s_2),(u_1,u_2),(t_1,t_2)) \\
=\ & \frac{1}{n}\sum_{i=1}^{n}\frac{1}{M_i}\sum_{m=1}^{M_i}\frac{1}{L_{im}^*}\sum_{l\neq l'}^{L_{im}} Z_{imll'} \\
& \qquad I(S_{iml}\in[s_1\wedge s_2,s_1\vee s_2])I(S_{iml'}\in[u_1\wedge u_2,u_1\vee u_2])I(T_{im}\in[t_1\wedge t_2,t_1\vee t_2]),
\end{aligned}
$$

*and* $Q((s_1,s_2),(u_1,u_2),(t_1,t_2))=\mathrm{E}[Q_n((s_1,s_2),(u_1,u_2),(t_1,t_2))]$. *Let* $c_n$, $c_n'$ *and* $c_n''$ *be any positive sequences tending to 0 and* $\beta_n=c_nc_n'c_n''(\gamma_{n22}+\gamma_{n20}c_nc_n'+\gamma_{n12}c_n''+c_nc_n'c_n'')$, *then if* $\beta_n^{-1}(\log n/n)^{1-2/\lambda}=o(1)$, *we have*

$$
\begin{aligned}
& \sup_{\substack{s,u\in[0,a]\\ t\in[0,b]}}\ \sup_{\substack{|v_1|\leqslant c_n\\ |v_2|\leqslant c_n'\\ |v_3|\leqslant c_n''}} |Q_n((s,s+v_1),(u,u+v_2),(t,t+v_3))-Q((s,s+v_1),(u,u+v_2),(t,t+v_3))| \\
=\ & O_{a.s.}(n^{-1/2}(\beta_n\log n)^{1/2}).
\end{aligned}
$$

**Proof.** The proof is similar to that of Lemma 1, and so we only outline the main differences. Let $\alpha_n$ and $\rho_n$ be as in Lemma 1. Let $\mathcal{B}_n=\{v_{k_1}\}$ be equally spaced grid on $[0,a]$ with $v_{k_1}=k_1c_n$, for $k_1=0,\cdots,[\frac{a}{c_n}]$, and $v_{[\frac{a}{c_n}]}=a$; let $\mathcal{B}_n'=\{v_{k_2}'\}$ be equally-spaced grid on

$[0,a]$ with $v'_{k_2} = k_2 c'_n$, for $k_2 = 0, \cdots, [\frac{a}{c'_n}]$, and $v'_{[\frac{a}{c'_n}]} = a$; let $\mathcal{B}''_n = \{v''_{k_3}\}$ be equally-spaced grid on $[0,b]$ with $v'_{k_3} = k_3 c''_n$, for $k_3 = 0, \cdots, [\frac{b}{c''_n}]$, and $v''_{[\frac{b}{c''_n}]} = b$. Define

$$\begin{aligned} & V_n(s,u,t,c_n,c'_n,c''_n) \\ = & \sup_{\substack{|v_1| \leqslant c_n \\ |v_2| \leqslant c'_n \\ |v_3| \leqslant c''_n}} \left| Q_n((s,s+v_1),(u,u+v_2),(t,t+v_3)) - Q((s,s+v_1),(u,u+v_2),(t,t+v_3)) \right|, \end{aligned}$$

then we have

$$\sup_{\substack{s,u \in [0,a] \\ t \in [0,b]}} V_n(s,u,t,c_n,c'_n) \leqslant 8 \sup_{\substack{s \in \mathcal{B}_n \\ u \in \mathcal{B}'_n \\ t \in \mathcal{B}''_n}} V_n(s,u,t,c_n,c'_n,c''_n) \tag{15}$$

Now define $Q^*_n$, $Q^*$ and $V^*_n$ in the same way as $Q_n$, $Q$ and $V_n$ except with $Z_{imll'} I(Z_{imll'} \leq \rho_n)$ replacing $Z_{imll'}$. Then, we have

$$\sup_{\substack{s \in \mathcal{B}_n \\ u \in \mathcal{B}'_n \\ t \in \mathcal{B}''_n}} V_n(s,u,t,c_n,c'_n,c''_n) \leqslant \sup_{\substack{s \in \mathcal{B}_n \\ u \in \mathcal{B}'_n \\ t \in \mathcal{B}''_n}} V^*_n(s,u,t,c_n,c'_n,c''_n) + A_{n1} + A_{n2}, \tag{16}$$

where

$$\begin{aligned} A_{n1} &= \sup_{\substack{s \in \mathcal{B}_n \\ u \in \mathcal{B}'_n \\ t \in \mathcal{B}''_n}} \sup_{\substack{|v_1| \leqslant c_n \\ |v_2| \leqslant c'_n \\ |v_3| \leqslant c''_n}} \Big| Q_n((s,s+v_1),(u,u+v_2),(t,t+v_3)) \\ & \qquad - Q^*_n((s,s+v_1),(u,u+v_2),(t,t+v_3)) \Big|, \\ A_{n2} &= \sup_{\substack{s \in \mathcal{B}_n \\ u \in \mathcal{B}'_n \\ t \in \mathcal{B}''_n}} \sup_{\substack{|v_1| \leqslant c_n \\ |v_2| \leqslant c'_n \\ |v_3| \leqslant c''_n}} \Big| Q((s,s+v_1),(u,u+v_2),(t,t+v_3)) \\ & \qquad - Q^*((s,s+v_1),(u,u+v_2),(t,t+v_3)) \Big|. \end{aligned}$$

By the similar argument in the proof of Lemma 1, we can show that

$$A_{n1} = o_{a.s.}(\alpha_n) \quad \text{and} \quad A_{n2} = o_{a.s.}(\alpha_n) \tag{17}$$

Now we consider $V^*_n(s,u,t,c_n,c'_n,c''_n)$. Similarly to the proof of Lemma 1, we create a further partition. Let $\omega_n = [\rho^{1/3} c_n / \alpha_n^{1/3} + 1]$, $\omega'_n = [\rho^{1/3} c'_n / \alpha_n^{1/3} + 1]$ and $\omega''_n = [\rho^{1/3} c''_n / \alpha_n^{1/3} + 1]$. Define $v_{r_1} = r_1 c_n / \omega_n$, $v'_{r_2} = r_2 c'_n / \omega'_n$, and $v''_{r_3} = r_3 c''_n / \omega''_n$, for $r_1 = -\omega_n, -\omega_n + 1, \cdots, \omega_n$, $r_2 =$

$-\omega'_n, -\omega'_n+1, \cdots, \omega'_n$, and $r_3 = -\omega''_n, -\omega''_n+1, \cdots, \omega''_n$. Then for fixed $(s,u,t) \in [0,a]^2 \times [0,b]$,

$$\begin{aligned}|V_n^*(s,u,t,c_n,c'_n,c''_n)| &\leqslant \max_{\substack{-\omega_n \leqslant r_1 \leqslant \omega_n \\ -\omega'_n \leqslant r_2 \leqslant \omega'_n \\ -\omega''_n \leqslant r_3 \leqslant \omega''_n}} \kappa_{nr_1r_2r_3}(s,u,t) \\ &\quad + \max_{\substack{-\omega_n \leqslant r_1 \leqslant \omega_n \\ -\omega'_n \leqslant r_2 \leqslant \omega'_n \\ -\omega''_n \leqslant r_3 \leqslant \omega''_n}} Q^*((s+v_{r_1}, s+v_{r_1+1}), (u+v'_{r_2}, u+v'_{r_2+1}), (t+v''_{r_3}, t+v''_{r_3+1})),\end{aligned}$$

where

$$\kappa_{nr_1r_2r_3}(s,u,t) = |Q_n^*((s,s+v_{r_1}),(u,u+v'_{r_1}),(t,t+v''_{r_3})) - Q^*((s,s+v_{r_1}),(u,u+v'_{r_1}),(t,t+v''_{r_3}))|.$$

For all $r_1, r_2$ and $r_3$, it's easy to show that

$$\begin{aligned}&Q^*((s+v_{r_1}, s+v_{r_1+1}), (u+v'_{r_2}, u+v'_{r_2+1}), (t+v''_{r_3}, t+v''_{r_3+1})) \\ \leqslant\ & M_S^2 M_T \alpha_n.\end{aligned}$$

Thus, for any $B > 0$,

$$\Pr\{V_n^*(s,u,t,c_n,c'_n,c''_n) \geqslant B\alpha_n\} \leqslant \Pr\{\max_{\substack{-\omega_n \leqslant r_1 \leqslant \omega_n \\ -\omega'_n \leqslant r_2 \leqslant \omega'_n \\ -\omega''_n \leqslant r_3 \leqslant \omega''_n}} \kappa_{nr_1r_2r_3}(s,u,t) \geqslant (B - M_S^2 M_T)\alpha_n\}. \quad (18)$$

Now let

$$W_{im}(s,u,t) = \frac{1}{L_{im}^*} \sum_{l \neq l'}^{L_{im}} Z_{imll'} I(Z_{imll'} \leqslant \rho_n) I(S_{iml} \in (s, s+v_{r_1}]) I(S_{iml'} \in (u, u+v'_{r_2}]),$$

and

$$\tilde{W}_i(s,u,t) = \frac{1}{M_i} \sum_{m=1}^{M_i} W_{im} I(T_{im} \in (t, t+v''_{r_3}]),$$

the by the similar algebra in the proof of Lemma 1, we

$$\begin{aligned}\sum_{i=1}^n \mathrm{Var}(\tilde{W}_i(s,u,t)) &\leqslant \sum_{i=1}^n \mathrm{E}\tilde{W}_i^2(s,u,t) \\ &\leqslant C c_n c'_n c''_n \sum_{i=1}^n \left[ \frac{1}{M_i^2} \sum_{m=1}^{M_i} \frac{1}{L_{im}^2} + \frac{c_n c'_n}{M_i} + \frac{c''_n}{M_i} \sum_{m=1}^{M_i} \frac{1}{L_{im}^2} + c_n c'_n c''_n \right] \\ &= nC\beta_n. \quad (19)\end{aligned}$$

The rest of the proof completely mirrors that of Lemma 1 and is omitted. □

**Lemma 4.** *Let $Z_{imll'}$ be as in Lemma 3 and assume that (14) holds. For bandwidths $h_3$, $h_4$ and $h_5$, and nonnegative integers $p$, $q$, $r$, let*

$$Q_{npqr}(s,u,t) \quad = \quad \frac{1}{n}\sum_{i=1}^{n}\frac{1}{M_i}\sum_{m=1}^{M_i}\frac{1}{L_{im}^*}\sum_{l\neq l'}^{L_{im}} Z_{imll'}K_{h_3,p}(S_{iml}-s)K_{h_4,q}(S_{iml'}-u)K_{h_5,r}(T_{im}-t)$$

*where $K_{h,p}(\cdot) = (\cdot/h)^p K_h(\cdot)$. Let $\beta_n = h_3h_4h_5(\gamma_{n22}+4\gamma_{n20}h_3h_4+2\gamma_{n12}h_5+8h_3h_4h_5)$, assume that $h_3 \to 0$, $h_4 \to 0$, $h_5 \to 0$, and $\beta_n^{-1}(\log n/n)^{1-2/\lambda} = o(1)$, then we have*

$$\sqrt{nh_3^2h_4^2h_5^2/(\beta_n \log n)} \sup_{\substack{s,u\in[0,a]\\ t\in[0,b]}} |Q_{npqr}(s,u,t) - \mathrm{E}[Q_{npqr}(s,u,t)]| = O_{a.s.}(1). \tag{20}$$

**Proof.** Obviously, we can write

$$\begin{aligned} Q_{npqr}(s,u,t) \quad &= \quad \int_{-h_5}^{h_5}\int_{-h_4}^{h_4}\int_{-h_3}^{h_3} Q_n((s+v_1,s+h_3),(u+v_2,u+h_4)(t+v_3,t+h_5)) \\ &\qquad dK_{h_3,p}(v_1)dK_{h_4,q}(v_2)dK_{h_5,r}(v_3), \end{aligned}$$

where $Q_n$ is as defined in Lemma 3. By letting $c_n = 2h_3$, $c_n' = 2h_4$, $c_n'' = 2h_5$, and applying Lemma 3, we can obtain that

$$\begin{aligned} &\sup_{s,u\in[0,a],t\in[0,b]} |Q_{npqr}(s,u,t) - \mathrm{E}[Q_{npqr}(s,u,t)]| \\ \leqslant \quad &\sup_{s,u\in[0,a],t\in[0,b]} V_n(s,u,t,2h_3,2h_4,2h_5)\int_{-h_5}^{h_5}|dK_{h_5,r}(v)|\int_{-h_4}^{h_4}|dK_{h_4,q}(v)||\int_{-h_3}^{h_3}|dK_{h_3,p}(v)| \\ \leqslant \quad &h_3^{-1}h_4^{-1}h_5^{-1}[K_{(p)}^{(1)}(1)+K_{(p)}^{(2)}(1)][K_{(q)}^{(1)}(1)+K_{(q)}^{(2)}(1)][K_{(r)}^{(1)}(1)+K_{(r)}^{(2)}(1)] \\ &\qquad \sup_{s,u\in[0,a],t\in[0,b]} V_n(s,u,t,2h_3,2h_4,2h_5) \\ = \quad &O_{a.s.}(\sqrt{\beta_n \log n/nh_3^2h_4^2h_5^2}). \end{aligned}$$

Thus (20) follows. □

**Lemma 5.** *Let $Z_{iml}$ be as in Lemma 1 and assume that (1) holds. For bandwidths $h_1$, and nonnegative integers $p$, let*

$$\bar{D}_n = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{M_i}\sum_{m=1}^{M_i}\frac{1}{L_{im}}\sum_{l=1}^{L_{im}} Z_{iml},$$

$$\tilde{D}_{np}(s) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{M_i}\sum_{m=1}^{M_i}\frac{1}{L_{im}}\sum_{l=1}^{L_{im}} Z_{iml}K_{h_1}(S_{iml}-s)(\frac{S_{iml}-s}{h_1})^p.$$

Then we have

$$\sqrt{n/(\log n)}\left|\bar{D}_n - \mathrm{E}\bar{D}_n\right| = O_{a.s.}(1). \tag{21}$$

Let $\beta_n = h_1(\gamma_{n21}+2\gamma_{n20}+2\gamma_{n11}h_2+4h_2)$, assume that $h_1 \to 0$, and $\beta_n^{-1}(\log n/n)^{1-2/\lambda} = o(1)$, then we have

$$\sqrt{nh_1^2/(\beta_n \log n)} \sup_{s\in[0,a]}\left|\tilde{D}_{np}(s) - \mathrm{E}[\tilde{D}_{np}(s)]\right| = O_{a.s.}(1). \tag{22}$$

**Proof.** (21) is obtained directly by the law of large numbers. Actually, if we take the second kernel function on $D_{npq}(s,t)$ defined in Lemma 2 as the uniform density function on [0,b], then $\tilde{D}_{np}(s)$ is identical to $D_{np0}(s,t)$ with $h_2 = b$. Since $b$ is finite, then following the same proofs of Lemma 1 and Lemma 2, we can obtain (22).

**Lemma 6.** *Let $Z_{imll'}$ be as in Lemma 3 and assume that (14) holds. For bandwidths $h_3$ and $h_4$, and nonnegative integers $p$ and $q$, let*

$$\begin{aligned}
\tilde{Q}_{npq}(s,u) &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{M_i}\sum_{m=1}^{M_i}\frac{1}{L_{im}^*}\sum_{l\neq l'}^{L_{im}} Z_{imll'}K_{h_3,p}(S_{iml}-s)K_{h_4,q}(S_{iml'}-u),\\
\check{Q}_{np}(s) &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{M_i}\sum_{m=1}^{M_i}\frac{1}{L_{im}^*}\sum_{l\neq l'}^{L_{im}} Z_{imll'}K_{h_3,p}(S_{iml}-s),\\
\bar{Q}_n &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{M_i}\sum_{m=1}^{M_i}\frac{1}{L_{im}^*}\sum_{l\neq l'}^{L_{im}} Z_{imll'}K_{h_3}(S_{iml}-S_{iml'}+uh_3),
\end{aligned}$$

*where $K_{h,p}(\cdot) = (\cdot/h)^p K_h(\cdot)$.*

*Let $\beta_n = h_3h_4(\gamma_{n22} + 4\gamma_{n20}h_3h_4 + 2\gamma_{n12} + 8h_3h_4)$, assume that $h_3 \to 0$, $h_4 \to 0$, and $\beta_n^{-1}(\log n/n)^{1-2/\lambda} = o(1)$, then we have*

$$\sqrt{nh_3^2h_4^2/(\beta_n \log n)} \sup_{s,u\in[0,a]}\left|\tilde{Q}_{npq}(s,u) - \mathrm{E}[\tilde{Q}_{npq}(s,u)]\right| = O_{a.s.}(1). \tag{23}$$

*Let $\tilde{\beta}_n = h_3(\gamma_{n22}+4\gamma_{n20}h_3+2\gamma_{n12}+8h_3)$, assume that $h_3 \to 0$, and $\tilde{\beta}_n^{-1}(\log n/n)^{1-2/\lambda} = o(1)$, then we have*

$$\sqrt{nh_3^2/(\tilde{\beta}_n \log n)} \sup_{s,u\in[0,a]}\left|\check{Q}_{np}(s) - \mathrm{E}[\tilde{Q}_{np}(s)]\right| = O_{a.s.}(1). \tag{24}$$

*and*

$$\sqrt{nh_3^2/(\tilde{\beta}_n \log n)}\left|\bar{Q}_n - \mathrm{E}\bar{Q}_n\right| = O_{a.s.}(1). \tag{25}$$

The proofs of (23) and (24) are similar to proofs of Lemma 5 and is omitted here. A slightly modified version of Lemma 1 leads to the "one-dimensional" rate of $\bar{Q}_n$ as (25).

The following Lemma 7 is needed for the proof of Theorem 3 and Theorem 4.

**Lemma 7.** *For any bounded measurable function $\psi$ on $[0, a]$,*

$$\sup_{u\in[0,a]} |(\Delta\psi)(u)| = O_{a.s.}(\delta_{n2}(h_4, 1, 1) + \gamma_{2d}(h_3, h_4, h_5) + \delta_{n1}(h_1, 1) + \gamma_{1d}(h_1, h_2)) \tag{26}$$

Proof.

$$\begin{aligned}
(\Delta\psi)(u) &= \int_0^a (\hat{G}_{\mathcal{S}}(s,u) - G_{\mathcal{S}}(s,u))\psi(s)ds \\
&= \int_0^a \int_0^b [\hat{D}(s,u,t) - D(s,u,t)]dt\psi(s)ds \\
&\qquad - \int_0^a \int_0^b [\hat{\mu}(s,t)\hat{\mu}(u,t) - \mu(s,t)\mu(u,t)]dt\psi(s)ds \\
&\equiv B_{n1} - B_{n2}.
\end{aligned}$$

Similarly to equation (36) in Appendix A.3, we have

$$\begin{aligned}
&[\hat{D}(s,u,t) - D(s,u,t)]\psi(s) \\
= \ &\left\{ \frac{\psi(s)}{f_2(s,u,t)}(\omega_{11}, \cdots, \omega_{1,d+3}) + O_{a.s}(\delta_{n2}(h_3,h_4,h_5) + h_3 + h_4 + h_5) \right\} \mathbf{R}_n^*(s,u,t).
\end{aligned} \tag{27}$$

Note that

$$\begin{aligned}
&\sup_{s,u\in[0,a],t\in[0,b]} \left[\frac{-\psi(s)}{f_2(s,u,t)}\right] \int_0^a \int_0^b R_{npqr}^*(s,u,t)dtds \\
&\qquad \leqslant \int_0^a \int_0^b \frac{\psi(s)}{f_2(s,u,t)} R_{npqr}^*(s,u,t)dtds \\
&\qquad\qquad \leqslant \sup_{s,u\in[0,a],t\in[0,b]} \left[\frac{\psi(s)}{f_2(s,u,t)}\right] \int_0^a \int_0^b R_{npqr}^*(s,u,t)dtds,
\end{aligned}$$

then $B_{n1}$ has the same uniform convergence rate of $\int_0^a \int_0^b \mathbf{R}_n^*(s,u,t)dtds$.

According to the expression (31) in Appendix A.3, of $R^*_{npqr}(s,u,t)$, it can be shown that

$$\begin{aligned}
&\int_0^a \int_0^b R^*_{npqr}(s,u,t)dtds \\
= \;& \Bigg\{ \frac{1}{n}\sum_{i=1}^{n}\frac{1}{M_i}\sum_{m=1}^{M_i}\frac{1}{L^*_{im}}\sum_{l\neq l'}^{L_{im}}\Big[\int_0^a K_{h_3,p}(S_{iml}-s)ds\int_0^b K_{h_5,r}(T_{im}-t)dt \\
&\qquad K_{h_4,q}(S_{iml'}-u)\left[Y_{iml}Y_{iml'}-D(S_{iml},S_{iml'},T_{im})\right]\Big] + O_{a.s}(\gamma_{2d}(h_3,h_4,h_5))\Bigg\} \\
= \;& \Bigg\{ \frac{\nu_p\nu_q}{n}\sum_{i=1}^{n}\frac{1}{M_i}\sum_{m=1}^{M_i}\frac{1}{L^*_{im}}\sum_{l\neq l'}^{L_{im}} K_{h_4,q}(S_{iml'}-u)\left[Y_{iml}Y_{iml'}-D(S_{iml},S_{iml'},T_{im})\right]\Big] \\
&\qquad + O_{a.s}(\gamma_{2d}(h_3,h_4,h_5))\Bigg\}
\end{aligned}$$

Thus by Lemma 6, we can obtain that uniformly for $u \in [0,a]$,

$$\int_0^a \int_0^b R^*_{npqr}(s,u,t)dtds = O_{a.s.}(\delta_{n2}(h_4,1,1)+\gamma_{2d}(h_3,h_4,h_5)).$$

which is also the rate of $B_{n1}$. Next, we write

$$\begin{aligned}
B_{n2} &= \int_0^a \int_0^b \{\hat{\mu}(s,t)\hat{\mu}(u,t)-\mu(s,t)\mu(u,t)\}dt\psi(s)ds \\
&\leqslant \int_0^a \int_0^b \hat{\mu}(s,t)\{\hat{\mu}(u,t)-\mu(u,t)\}dt\psi(s)ds + \int_0^a \int_0^b \{\hat{\mu}(s,t)-\mu(s,t)\}\mu(u,t)dt\psi(s)ds.
\end{aligned}$$

Similarly to the derivation to equation (36) in Appendix A.3, we have $B_{n2} = O_{a.s}(\delta_{n1}(h_1,1)+\gamma_{1d}(h_1,h_2))$. Thus, we can obtain that uniformly for $u \in [0,a]$,

$$|(\Delta\psi)(u)| = O_{a.s.}(\delta_{n2}(h_4,1,1)+\gamma_{2d}(h_3,h_4,h_5)+\delta_{n1}(h_1,1)+\gamma_{1d}(h_1,h_2)) \tag{28}$$

□