Hindawi
Journal of Applied Mathematics
Volume 2020, Article ID 7290403, 5 pages
https://doi.org/10.1155/2020/7290403

Research Article

Explicit Expression for Arbitrary Positive Powers of Special Tridiagonal Matrices

Mohammad Beiranvand and Mojtaba Ghasemi Kamalvand

Department of Mathematics, Lorestan University, Khorramabad, Iran

Correspondence should be addressed to Mojtaba Ghasemi Kamalvand; ghasemi.m@lu.ac.ir

Received 18 May 2020; Revised 23 July 2020; Accepted 27 July 2020; Published 12 September 2020

Academic Editor: Igor Andrianov

Tridiagonal matrices appear frequently in mathematical models. In this paper, we derive the positive integer powers of special tridiagonal matrices of arbitrary order.

1. Introduction

Recently, computing the integer powers of tridiagonal matrices has been a very popular problem. Tridiagonal matrices are used in different areas of science and engineering, for instance, the solution of difference systems [1], numerical solution of PDEs [2], telecommunication system analysis [3, 4], texture modeling [5], and image processing and coding [6]. In these areas, the computation of the powers of these matrices is necessary. Therefore, there are a lot of studies dealing with the powers of these matrices using the well-known expression $A^m = UJ^mU^{-1}$ where J is Jordan's form of the matrix A and U is the transforming matrix. We need the eigenvalues and eigenvectors of these matrices to calculate A^m.

Rimas [7–13], investigated positive integer powers of certain tridiagonal matrices, Oteles and Akbulak [14, 15] and Gutierrez [16] generalized some papers of Rimas, and Wang [17] derived the entries of positive integer powers of complex persymmetric anti-tridiagonal matrices with constant anti-diagonals. Some authors also investigated integer powers of certain tridiagonal matrices.

In this paper, we consider the n-th order tridiagonal matrix A of the following type

$$A = \begin{bmatrix} -\alpha + b & c & & & \\ a & b & c & 0 & \\ & \ddots & \ddots & \ddots & \\ & & a & b & c \\ & 0 & & a & -\beta + b \end{bmatrix}, \qquad (1)$$

where a, b, c, α, and β are the numbers in the complex C. There are many mathematical models that are involved in this form [18].

2. Main Results

According to the following lemmas, Wen-Chyuan Yueh obtains eigenvalue and corresponding eigenvectors for matrix (1), in special cases. In this section, we firstly find the transforming matrices and their invers for the matrix

(1). Secondly, we present a general expression for the entries of A^m for $m \in \mathbb{N}$.

Lemma 1 (see [19]). *Suppose $\alpha = 0$, $\beta = \sqrt{ac} \neq 0$, and $\rho = \sqrt{a/c}$, then the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of A are given by*

$$\lambda_k = b + 2\sqrt{ac} \cos \frac{2k\pi}{2n+1}, \quad k = 1, 2, \cdots, n, \tag{2}$$

the corresponding eigenvectors $u^{(k)} = (u_1^{(k)}, u_2^{(k)}, \cdots, u_n^{(k)})^T$, $k = 1, 2, \cdots, n$, are given by

$$u_j^{(k)} = \rho^{j-1} \sin \frac{2kj\pi}{2n+1}, \quad j = 1, 2, \cdots, n. \tag{3}$$

In this paper, we need the following theorem.

Theorem 2. *(canonical Jordan's form [20]). Let A be any square matrix. Then, there exists a nonsingular matrix U which transforms A into a block diagonal matrix J such that*

$$U^{-1}AU = J = \text{diag}\,(J_{n1}(\lambda_1), J_{n2}(\lambda_2), \cdots J_{nl}(\lambda_l)), \quad n_1 + n_2 + \cdots + n_l = n. \tag{4}$$

which is called the canonical Jordan's form, λ_j being the eigenvalues of A and $J_k \in C^{k\times k}$ a Jordan block of the form

$$J_1(\lambda) = [\lambda], J_2(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \cdots, J_k(\lambda) = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & 0 & \\ & & \ddots & \ddots & \\ 0 & & & \lambda & 1 \end{bmatrix}. \tag{5}$$

Since all the eigenvalues λ_k for $k = 1, 2, \cdots, n$ are distinct ($\cos\theta$ is a strictly decreasing function of θ on $(0,\pi)$, and $a \neq 0 \neq c$), columns of the transforming matrix U are the eigenvectors of the matrix (1). Also, all eigenvalues λ_k correspond to the single Jordan cell $J_1(\lambda_k)$ in the matrix J, then we write down Jordan's form of the matrix A as

$$J = \text{diag}\,(\lambda_1, \lambda_2, \cdots, \lambda_n). \tag{6}$$

From (3), we can write the column transforming matrix U as

$$u^{(k)} = \left[\sin \frac{2k\pi}{2n+1}, \rho^1 \sin \frac{4k\pi}{2n+1}, \cdots, \rho^{n-1} \sin \frac{2kn\pi}{2n+1}\right]^T, \tag{7}$$

for $k = 1, 2, \cdots, n$.

Hence,

$$U = \left[u^{(1)}, u^{(2)}, u^{(3)}, \cdots, u^{(n)}\right]. \tag{8}$$

Let

$$D = \text{diag}\,(1, \rho^1, \rho^2, \rho^3, \cdots, \rho^{n-1}), \quad \tilde{U} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \cdots, \tilde{u}_n), \tag{9}$$

in which

$$\tilde{u}_k = \left[\sin \frac{2k\pi}{2n+1}, \sin \frac{4k\pi}{2n+1}, \cdots, \sin \frac{2kn\pi}{2n+1}\right]^T, \tag{10}$$

for $k = 1, 2, \cdots, n$.

Therefore, we have

$$U = D\tilde{U}, \tag{11}$$

$$U^{-1} = \tilde{U}^{-1} D^{-1}. \tag{12}$$

So an explicit expression for $\tilde{U}^{-1}$ will suffice.

Theorem 3. *Suppose $\tilde{U}$ is defined as above, then*

$$\tilde{U}^{-1} = \frac{4}{2n+1} \tilde{U}. \tag{13}$$

Proof. We show that

$$\tilde{u}_i^T \tilde{u}_j = \begin{cases} \dfrac{2n+1}{4}, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{14}$$

From (10), equals $\sin\alpha \sin\beta = (1/2)(\cos(\alpha - \beta) - \cos(\alpha + \beta))$ and

$$\sum_{k=1}^{n} \cos k\theta = \frac{\sin(n + 1/2)\theta}{2 \sin(1/2)} - \frac{1}{2}, \tag{15}$$

follows

$$\tilde{u}_i^T \tilde{u}_j = \sum_{k=1}^{n} \sin \frac{2ki\pi}{2n+1} \sin \frac{2kj\pi}{2n+1} = \frac{1}{2} \sum_{k=1}^{n} \left(\cos \frac{2k(i-j)\pi}{2n+1} - \cos \frac{2k(i+j)\pi}{2n+1}\right). \tag{16}$$

If $i = j$,

$$\begin{aligned} \tilde{u}_i^T \tilde{u}_j &= \frac{n}{2} - \frac{1}{2}\left(\frac{\sin(n + 1/2)(4i\pi/(2n+1))}{2 \sin((4i\pi/(2n+1))/2)} - \frac{1}{2}\right) \\ &= \frac{n}{2} - \frac{1}{2}\left(\frac{\sin 2i\pi}{2 \sin(2i\pi/(2n+1))} - \frac{1}{2}\right) \\ &= \frac{n}{2} - \frac{1}{2}\left(0 - \frac{1}{2}\right) = \frac{2n+1}{4}. \end{aligned} \tag{17}$$

If $i \neq j$,

$$\begin{aligned}\tilde{u}_i^T \tilde{u}_j &= \frac{1}{2}\sum_{k=1}^{n}\left(\cos\frac{2k(i-j)\pi}{2n+1} - \cos\frac{2k(i+j)\pi}{2n+1}\right)\\ &= \frac{1}{2}\left(\frac{\sin(n+1/2)(2(i-j)\pi/(2n+1))}{2\sin((2(i-j)\pi/(2n+1))/2)} - \frac{\sin(n+1/2)(2(i+j)\pi/(2n+1))}{2\sin((2(i+j)\pi/(2n+1))/2)}\right)\\ &= \frac{1}{4}\left(\frac{\sin(i-j)\pi}{\sin((i-j)\pi/(2n+1))} - \frac{\sin(i+j)\pi}{\sin((i+j)\pi/(2n+1))}\right).\end{aligned} \tag{18}$$

If i and j are even or odd, then $i-j$ and $i+j$ are even; therefore, $\sin(i-j)\pi = \sin(i+j)\pi = 0$, so we have

$$\tilde{u}_i^T \tilde{u}_j = 0. \tag{19}$$

If one of i or j is even and the other is odd, then $i-j$ and $i+j$ are odd; therefore, we have

$$\sin\left((i-j)\pi - \frac{(i-j)\pi}{2n+1}\right) = \sin\frac{(i-j)\pi}{2n+1}, \tag{20}$$

$$\sin\left((i+j)\pi - \frac{(i+j)\pi}{2n+1}\right) = \sin\frac{(i+j)\pi}{2n+1}. \tag{21}$$

From (18), (20), and (21) follows

$$\tilde{u}_i^T \tilde{u}_j = 0. \tag{22}$$

For derivation of the formula for the entries of A^m, from (4), (6), (10), (12), and (13), we can conclude

$$\begin{aligned}A^m &= UJ^mU^{-1} = UJ^m\tilde{U}^{-1}D^{-1}\\ &= \frac{4}{2n+1}U\mathrm{diad}(\lambda_1^m, \lambda_2^m, \cdots, \lambda_n^m)\\ &\quad\cdot \tilde{U}^{-1}\,\mathrm{diag}\left(1, \left(\frac{a}{c}\right)^{-1/2}, \left(\frac{a}{c}\right)^{-2/2}, \cdots, \left(\frac{a}{c}\right)^{-(n-1)/2}\right).\end{aligned} \tag{23}$$

By substituting U and $\tilde{U}^{-1}$ in the latter equation and doing the necessary computation follows

$$[A^m]_{i,j} = \frac{4}{2n+1}\left(\frac{a}{c}\right)^{(i-j)/2}\sum_{k=1}^{n}\lambda_k^m \sin\frac{2ik\pi}{2n+1}\sin\frac{2jk\pi}{2n+1}, \tag{24}$$

for $i, j = 1, 2, \cdots, n$, where $\lambda_k = b + 2\sqrt{ac}\cos(2k\pi/(2n+1))$, $k = 1, 2, \cdots, n$.

Lemma 4 (see [19]). *Suppose $\alpha = -\sqrt{ac} \neq 0$ and $\beta = 0$ and $\rho = \sqrt{ac}$, then the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of A_n are given by*

$$\lambda_k = b + 2\sqrt{ac}\cos\frac{(2k-1)\pi}{2n+1}, \quad k = 1, 2, \cdots n. \tag{25}$$

The corresponding eigenvectors, $v^{(k)} = (v_1^{(k)}, v_2^{(k)}, \cdots, v_n^{(k)})^T$, $k = 1, 2, \cdots, n$, are given by

$$v_j^{(k)} = \rho^{j-1}\cos\frac{(2k-1)(2j-1)}{2(2n+1)}, \quad j = 1, 2, \cdots, n. \tag{26}$$

From (26) for $k = 1, 2, \cdots, n$, we can write the columns transforming matrix V as

$$v^{(k)} = \left[\cos\frac{(2k-1)\pi}{2(2n+1)}, \rho^1\cos\frac{3(2k-1)\pi}{2(2n+1)}, \cdots, \rho^{n-1}\cos\frac{(2n-1)(2k-1)\pi}{2(2n+1)}\right]^T \tag{27}$$

Hence,

$$V = \left[v^{(1)}, v^{(2)}, \cdots, v^{(n)}\right]. \tag{28}$$

Let

$$\begin{aligned}D &= \mathrm{diag}\left(1, \rho^1, \rho^2, \cdots, \rho^{n-1}\right),\\ \tilde{V} &= (\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_n),\end{aligned} \tag{29}$$

in which

$$\tilde{v}_k = \left[\cos\frac{(2k-1)\pi}{2(2n+1)}, \cos\frac{3(2k-1)\pi}{2(2n+1)}, \cdots, \cos\frac{(2n-1)(2k-1)\pi}{2(2n+1)}\right]^T, \tag{30}$$

for $k = 1, 2, \cdots, n$.

Therefore, we have

$$\begin{aligned}V &= D\tilde{V},\\ V^{-1} &= \tilde{V}^{-1}D^{-1}.\end{aligned} \tag{31}$$

So an explicit expression for $\tilde{V}^{-1}$ will suffice.

Theorem 5. *Suppose $\tilde{V}$ is defined as above, then*

$$\tilde{V}^{-1} = \frac{4}{2n+1}\tilde{V}. \tag{32}$$

Proof. We show that

$$\tilde{v}_i^T \tilde{v}_j = \begin{cases}\dfrac{2n+1}{4}, & \text{if } i = j,\\ 0, & \text{if } i \neq j.\end{cases} \tag{33}$$

From (30) equals $\cos\alpha\cos\beta = (1/2)(\cos(\alpha-\beta)+\cos(\alpha+\beta))$ and

$$\sum_{k=1}^{n} \cos(2k-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}, \tag{34}$$

follows

$$\begin{aligned}\tilde{v}_i^T\tilde{v}_j &= \sum_{k=1}^{n}\cos\frac{(2k-1)(2i-1)\pi}{2(2n+1)}\cos\frac{(2k-1)(2j-1)\pi}{2(2n+1)}\\ &= \frac{1}{2}\sum_{k=1}^{n}\left(\cos\left(\frac{(2k-1)(2i-1)\pi}{2(2n+1)}-\frac{(2k-1)(2j-1)\pi}{2(2n+1)}\right)\right.\\ &\quad\left.+\cos\left(\frac{(2k-1)(2i-1)\pi}{2(2n+1)}+\frac{(2k-1)(2j-1)\pi}{2(2n+1)}\right)\right)\\ &= \frac{1}{2}\sum_{k=1}^{n}\left(\cos\left(\frac{2(2k-1)(i-j)\pi}{2(2n+1)}\right)\right.\\ &\quad\left.+\cos\left(\frac{2(2k-1)(i+j-1)\pi}{2(2n+1)}\right)\right).\end{aligned} \tag{35}$$

If $i=j$,

$$\begin{aligned}\tilde{v}_i^T\tilde{v}_j &= \frac{n}{2}+\frac{1}{2}\sum_{k=1}^{n}\cos\left(\frac{2(2k-1)(2i-1)\pi}{2(2n+1)}\right)\\ &= \frac{n}{2}+\frac{1}{2}\frac{\sin 2n((2i-1)\pi/(2n+1))}{2\sin((2i-1)\pi/(2n+1))}\\ &= \frac{n}{2}+\frac{1}{2}\frac{\sin(2n/(2n+1))(2i-1)\pi}{2\sin((2i-1)\pi/(2n+1))}\\ &= \frac{n}{2}+\frac{1}{2}\frac{\sin(2i-1)\pi-(2i-1)\pi/(2n+1)}{2\sin((2i-1)\pi/(2n+1))}\frac{n}{2}\\ &\quad+\frac{1}{2}\frac{\sin((2i-1)\pi/(2n+1))}{2\sin((2i-1)\pi/(2n+1))}\\ &= \frac{2n+1}{4}.\end{aligned} \tag{36}$$

If $i\neq j$,

$$\begin{aligned}\tilde{v}_i^T\tilde{v}_j &= \frac{1}{2}\sum_{k=1}^{n}\left(\cos\frac{(2k-1)(i-j)\pi}{2n+1}+\cos\frac{(2k-1)(i+j-1)\pi}{2n+1}\right)\\ &= \frac{1}{2}\left(\frac{\sin 2n((2k-1)(i-j)\pi/(2n+1))}{2\sin((2k-1)(i-j)\pi/(2n+1))}\right.\\ &\quad\left.+\frac{\sin 2n((2k-1)(i+j-1)\pi/(2n+1))}{2\sin((2k-1)(i+j-1)\pi/(2n+1))}\right)\\ &= \frac{1}{2}\left(\frac{\sin 2n((i-j)\pi/(2n+1))}{2\sin((i-j)\pi/(2n+1))}\right.\\ &\quad\left.+\frac{\sin 2n((i+j-1)\pi/(2n+1))}{2\sin((i+j-1)\pi/(2n+1))}\right)\\ &= \frac{1}{2}\left(\frac{\sin(1-1/(2n+1))(i-j)\pi}{2\sin((i-j)\pi/(2n+1))}\right.\\ &\quad\left.+\left(\frac{\sin(1-1/(2n+1))(i+j-1)\pi}{2\sin((i+j-1)\pi/(2n+1))}\right)\right).\end{aligned} \tag{37}$$

If i and j are even or odd, then $i-j$ is even and $i+j-1$ is odd; therefore,

$$\begin{aligned}\sin\left(1-\frac{1}{2n+1}\right)(i-j)\pi &= -\sin\frac{(i-j)\pi}{2n+1},\\ \sin\left(1-\frac{1}{2n+1}\right)(i+j-1)\pi &= -\sin\frac{(i+j-1)\pi}{2n+1}.\end{aligned} \tag{38}$$

So we have

$$\tilde{v}_i^T\tilde{v}_j = 0. \tag{39}$$

If one of i or j is even and the other is odd, then $i-j$ and $i+j-1$ are even; therefore, we have

$$\sin\left(1-\frac{1}{2n+1}\right)(i-j)\pi = -\sin\frac{(i-j)\pi}{2n+1} \tag{40}$$

and

$$\sin\left(1-\frac{1}{2n+1}\right)(i+j-1)\pi = -\sin\frac{(i+j-1)\pi}{2n+1}. \tag{41}$$

Therefore,

$$\tilde{v}_i^T\tilde{v}_j = 0. \tag{42}$$

Similar for (24), we can conclude

$$\begin{aligned}[A^m]_{i,j} &= \frac{4}{2n+1}\left(\frac{a}{c}\right)^{(i-j)/2}\\ &\quad\cdot\sum_{k=1}^{n}\lambda_k^m\cos\frac{(2k-1)(2i-1)\pi}{2(2n+1)}\sin\frac{(2k-1)(2j-1)\pi}{2(2n+1)},\end{aligned} \tag{43}$$

for $i,j=1,2,\cdots,n$.

Lemma 6 (see [18]). *Suppose $\alpha=-\beta=\sqrt{ac}\neq 0$ and $\rho=\sqrt{ac}$, then the eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_n$ of A_n are given by*

$$\lambda_k = b+2\sqrt{ac}\cos\frac{(2k-1)\pi}{2n},\quad k=1,2,\cdots,n. \tag{44}$$

The corresponding eigenvectors are given by

$$\begin{aligned}u^{(k)} &= \left(u_1^{(k)},u_2^{(k)},\cdots,u_n^{(k)}\right)^T,\quad k=1,2,\cdots,n,\\ u_j^{(k)} &= \rho^{j-1}\sin\frac{(2k-1)(2j-1)\pi}{4n},\quad j=1,2,\cdots,n.\end{aligned} \tag{45}$$

In the case $\alpha = -\beta = -\sqrt{ac} \neq 0$, the eigenvalues are given by (44) and the corresponding eigenvectors by

$$v^{(k)} = \left(v_1^{(k)}, v_2^{(k)}, \cdots, v_n^{(k)}\right)^T, \quad k = 1, 2, \cdots, n,$$
$$v_j^{(k)} = \rho^{j-1} \cos \frac{(2k-1)(2j-1)\pi}{4n}, \quad j = 1, 2, \cdots, n. \tag{46}$$

In these cases which are similar to previous cases, we showed that integer powers of the matrix A for $i, j = 1, 2, \cdots, n$, respectively,

$$[A^m]_{i,j} = \frac{2}{n}\left(\frac{a}{c}\right)^{(i-j)/2} \cdot \sum_{k=1}^{n} \lambda_k^m \sin \frac{(2k-1)(2i-1)\pi}{4n} \sin \frac{(2k-1)(2j-1)\pi}{4n}, \tag{47}$$

where $\lambda_k = b + 2\sqrt{ac} \cos((2k-1)\pi/2n)$, $k = 1, 2, \cdots, n$, and

$$[A^m]_{i,j} = \frac{2}{n}\left(\frac{a}{c}\right)^{(i-j)/2} \cdot \sum_{k=1}^{n} \lambda_k^m \cos \frac{(2k-1)(2i-1)\pi}{4n} \cos \frac{(2k-1)(2j-1)\pi}{4n}, \tag{48}$$

where $\lambda_k = b + 2\sqrt{ac} \cos((2k-1)\pi/2n)$, $k = 1, 2, \cdots, n$.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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